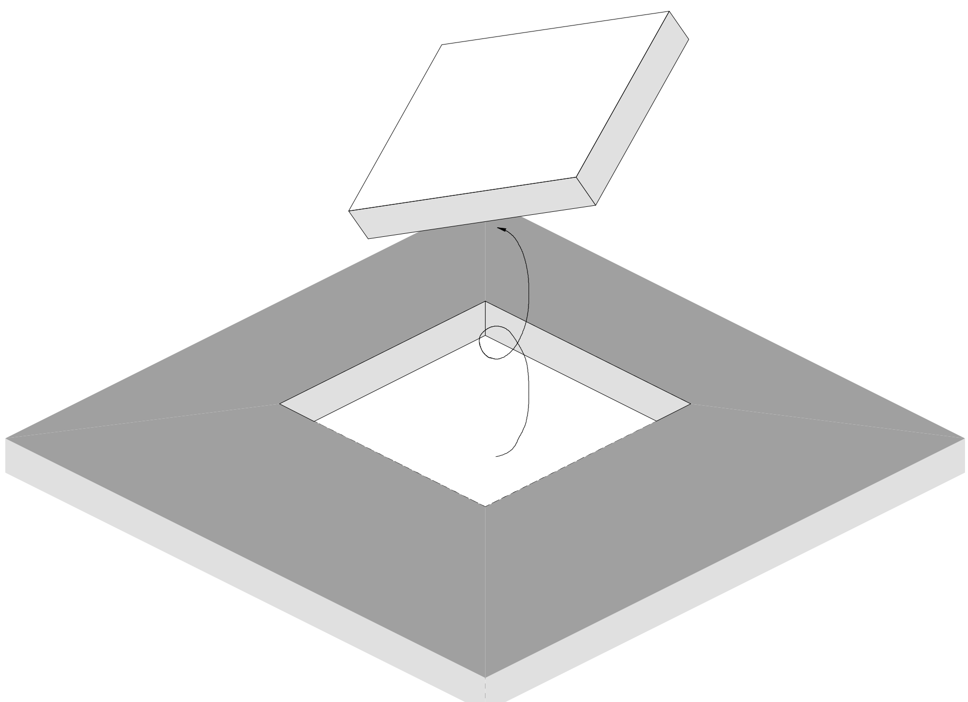
FLiP-CHIP™ ALGEBRA

INTRODUCTORY LEVEL

Frank Edge — Steven Kant

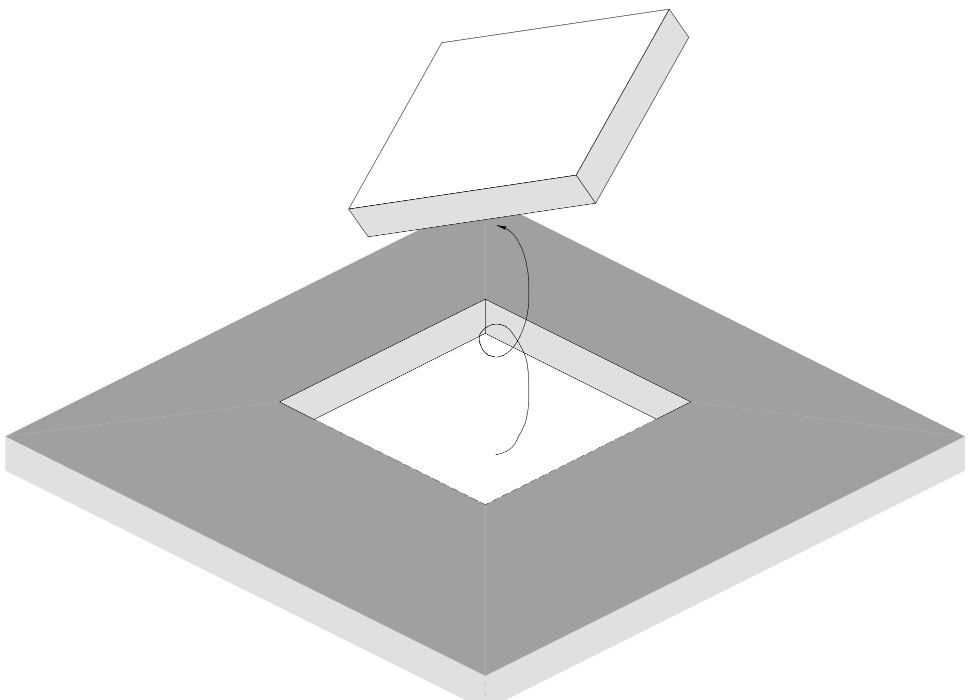


Third Edition

FLiP-CHIP™ ALGEBRA

INTRODUCTORY LEVEL

Frank Edge — Steven Kant



Second Edition

The authors would appreciate corrections and suggestions. For more information, write to Flip-Chip Enterprises, 103 Thomas Street NW, Olympia WA, 98502 or stevenrkant@zohomail.com.

Created and written by Frank Edge and Steven Kant. Designed and produced by Steven Kant.

Dedicated to the memory of Frank Edge

The authors gratefully acknowledge the assistance of Greg Falxa and Max Kant.

Table of Contents

Preface 1

Chapter 1: Introduction 3

1. Learning By Discovery 4

Chapter 2: Positive and Negative Numbers 11

1. Positive and Negative Numbers 12
2. Addition of Signed Numbers 17
3. Subtraction of Signed Numbers 21
4. Addition and Subtraction 30
5. Multiplication 33
6. Division 38
7. The Number Line 43

Chapter 3: Symbols and the Order of Operations { [( )] } 47

1. Rules of Language 48
2. Order of Operations 53
3. Parentheses 56
4. Division and Fractions 61
5. Absolute Value 67

Chapter 4: Multiplication and Division of Fractions 71

1. Multiplication of Fractions 72
2. Division of Fractions 77
3. Compound Fractions 85

Chapter 5: Properties 89

1. Properties of Addition and Multiplication 90
2. The Distributive Property 96
3. Identities and Inverses 104
4. Properties of Zero 109
5. Properties or Rules? (Optional) 111

Chapter 6: Expressions **115**

1. Simple Expressions 116
2. Multiples of x 120
3. Combining Similar Terms 124
4. Expressions and Parentheses 128
5. Expressions Containing Fractions 132
6. Properties of Expressions 135

Chapter 7: Equations **141**

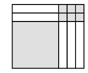
1. Introduction to Equations 142
2. The Equation Game 144
3. Equations using Unknowns 147
4. Equations with Multiples of Unknowns 152
5. Unknowns in More than One Term 156
6. Equations with Parentheses 162
7. Equations with Fractions or Decimals 165
8. Special Solutions 172

Chapter 8: Powers and Roots **175**

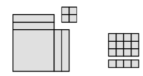
1. Introduction to Exponents 176
2. Squares 179
3. Cubes 182
4. Higher Powers 185
5. Other Exponents: Negative Numbers, Zero, and One 188
6. Properties of Powers 193
7. Simplifying Expressions 202
8. Roots and Radicals 209
9. Irrational Numbers 214
10. Properties of Roots 218

Chapter 9: Polynomials **221**

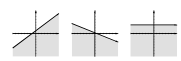
1. Using Unknowns 1, x, x^2 222
2. Adding and Subtracting Polynomials 230
3. Multiplying Polynomials 234
4. Special Products 243

Chapter 10: Factoring Polynomials  253

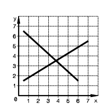
1. Introduction to Rectangles and Factoring 254
2. Positive Units, Negative Bars 261
3. Rectangles Having Negative Units 265
4. Factoring Trinomials with More than One x^2 272
5. Using the Grid 276
6. A Shortcut Method 280
7. Recognizing Special Products 289
8. Expressions Which Cannot be Factored 297

Chapter 11: Quadratic Equations  301

1. Introduction 302
2. The Zero Product Rule 305
3. Standard Form 312
4. Factoring Quadratic Equations 317
5. Completing the Square 325
6. Equations with More than One x^2 336
7. Imaginary Solutions 343
8. The Quadratic Formula 350

Chapter 12: Rules and Graphs  355

1. Related Numbers 356
2. Machines, Charts, and a Second Variable 363
3. Graphs and Coordinates 369
4. Graphs of Lines 377
5. Slopes and Intercepts 387
6. Graphing with Slopes and Intercepts 397
7. Graphing with Two Intercepts 404
8. Summary 409

Chapter 13: Systems of Equations  411

1. Equations and Solutions 412
2. Solving by Graphing 419
3. The Substitution Method 425
4. The Addition Method 439
5. Choosing a Method 448
6. Special Cases 452

Chapter 14: Rational Expressions $\frac{x}{x+3}$ 457

1. Introduction 458
2. Simplifying Rational Expressions 463
3. Division Using Chips 473
4. Long Division 484
5. Multiplication 492
6. Division 499
7. Addition 501
8. Summary 509

Appendices + 511

Appendix 1: Division of Fractions, Part 2 512
Appendix 2: Mixed Numbers 516
Appendix 3: The Function Game 520
Appendix 4: Functions and Maps 529
Appendix 5: Factoring By Grouping 534

Answers to Exercises ? A-1

Chapter 1: Introduction A–2
Chapter 2: Positive and Negative Numbers A–3
Chapter 3: Symbols and the Order of Operations A–5
Chapter 4: Multiplication and Division of Fractions A–7
Chapter 5: Properties A–8
Chapter 6: Expressions A–9
Chapter 7: Equations A–11
Chapter 8: Powers and Roots A–13
Chapter 9: Polynomials A–16
Chapter 10: Factoring Polynomials A–18
Chapter 11: Quadratic Equations A–20
Chapter 12: Rules and Graphs A–22
Chapter 13: Systems of Equations A–27
Chapter 14: Rational Expressions A–29
Appendices A–31

Preface

A Different Way of Learning

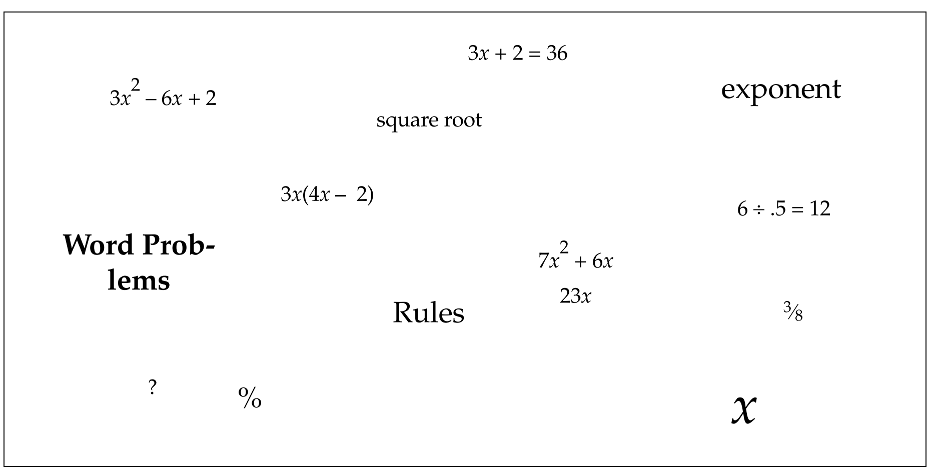
This is not a typical algebra textbook. There will be no rules or meaningless formulas to memorize, no lengthy practice of mysterious techniques.

As you use this book, you will *understand* algebra instead of *memorizing* it. This understanding will be a challenge but it will also be enjoyable; the knowledge will last your whole lifetime instead of a few weeks or years.

Understanding Instead of Memorizing

Pick a word that you know. Does the word represent something you have touched or experienced? If you look at a word like “banana” or “computer” you cannot help visualizing an object, for *language is easy to learn because it means something*.

Mathematics can be difficult to learn because it is often taught with no recognizable meaning. Do you visualize anything at all when you see $\mathbf{2x}$, $\mathbf{x^2}$, or $\frac{2}{3} \cdot \frac{3}{4}$? Do you know what these symbols mean? Because we may not know what the symbols stand for, learning mathematics can be like attempting to memorize a long nonsense poem—a poem that does not even rhyme. Memorizing mathematics is similar to attempting to learn how to read without knowing what the words mean.



Algebra is a language, and as with Spanish, English, or Hebrew, groups of symbols have specific meanings. In algebra each symbol can be represented by a physical object; each rule is something true about all these objects; each formula or technique is just a way of discovering something about a certain group of objects.

Mathematics and Symbols

Working with objects instead of abstract symbols makes many topics of algebra very easy; once the objects have been named using algebraic symbols, the symbol manipulations become as easy and obvious as the manipulations of the objects. Learning algebra takes on the quality of learning a clever board game.

Memorization is almost never necessary because the rules make such perfect sense. (For example: "When counting, all pieces of the same size and shape get counted together.") Exercises involve a process of discovery; it is usually obvious to each student when a successful answer is found. (For example: "Arrange all of these pieces to form a single rectangle.") *Flip-Chip Algebra* can be learned by virtually anyone who knows the multiplication tables (and it can help in learning those tables); once the algebra is learned it will not be easily confused or forgotten.

A note to people who have difficulty with mathematics

If you have not been successful in learning algebra when it has been taught with traditional methods, then this book is for you. If you have not enjoyed algebra in the past, it may be that you have either been unable or unmotivated to memorize a large amount of meaningless information.

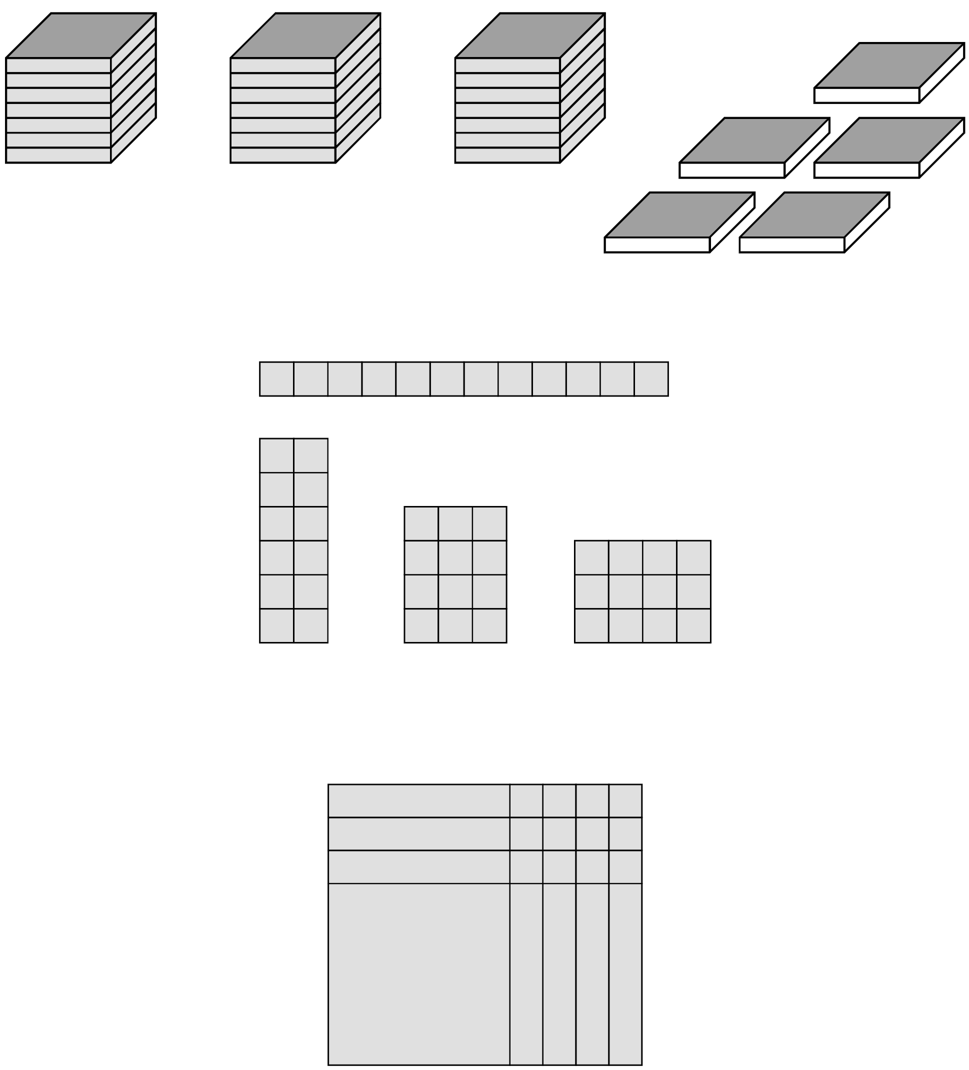
You *have* learned a great many other meaningful things in your life—reading, history, music—and many were not difficult. When we give algebra meaning, you will learn it in the same natural way.

This change may be difficult because we are asking you to give up the only crutch you may think you have—rote memory—and encouraging you to take the risk of believing that you will be able to understand the material without memorizing it. We know that you will be successful.

Steven Kant
Frank Edge

Chapter 1

Introduction



Learning By Discovery

Purpose of this chapter

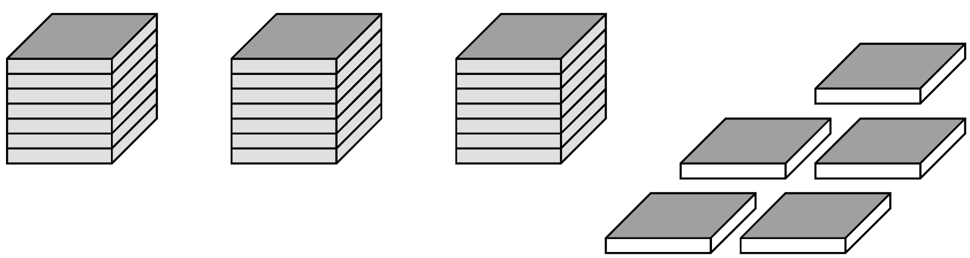
This chapter is a short introduction to the methods of the book. Three demonstrations will be presented. Each of these concepts will be covered in detail in a later chapter, so it is not important to try to memorize or practice these examples; instead you need only follow the demonstrations and enjoy the challenge of solving the problems. Your job is to understand, not to obey.

For the demonstrations, you will need the cardboard chips and a pencil and paper. As in the rest of the book, you will learn more if you follow along with the text by doing the work instead of merely reading or watching.

Demonstration 1: Solving Equations

Count out 26 of the small cardboard chips. Remove 5 chips and set them aside. Take the remaining chips and stack them up in three equal piles. *Do not count the piles; you can tell if they are equal by feeling that each pile is the same height.*

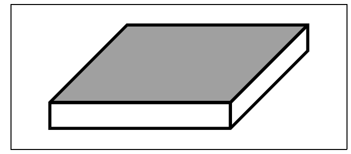
Here is what you should have:

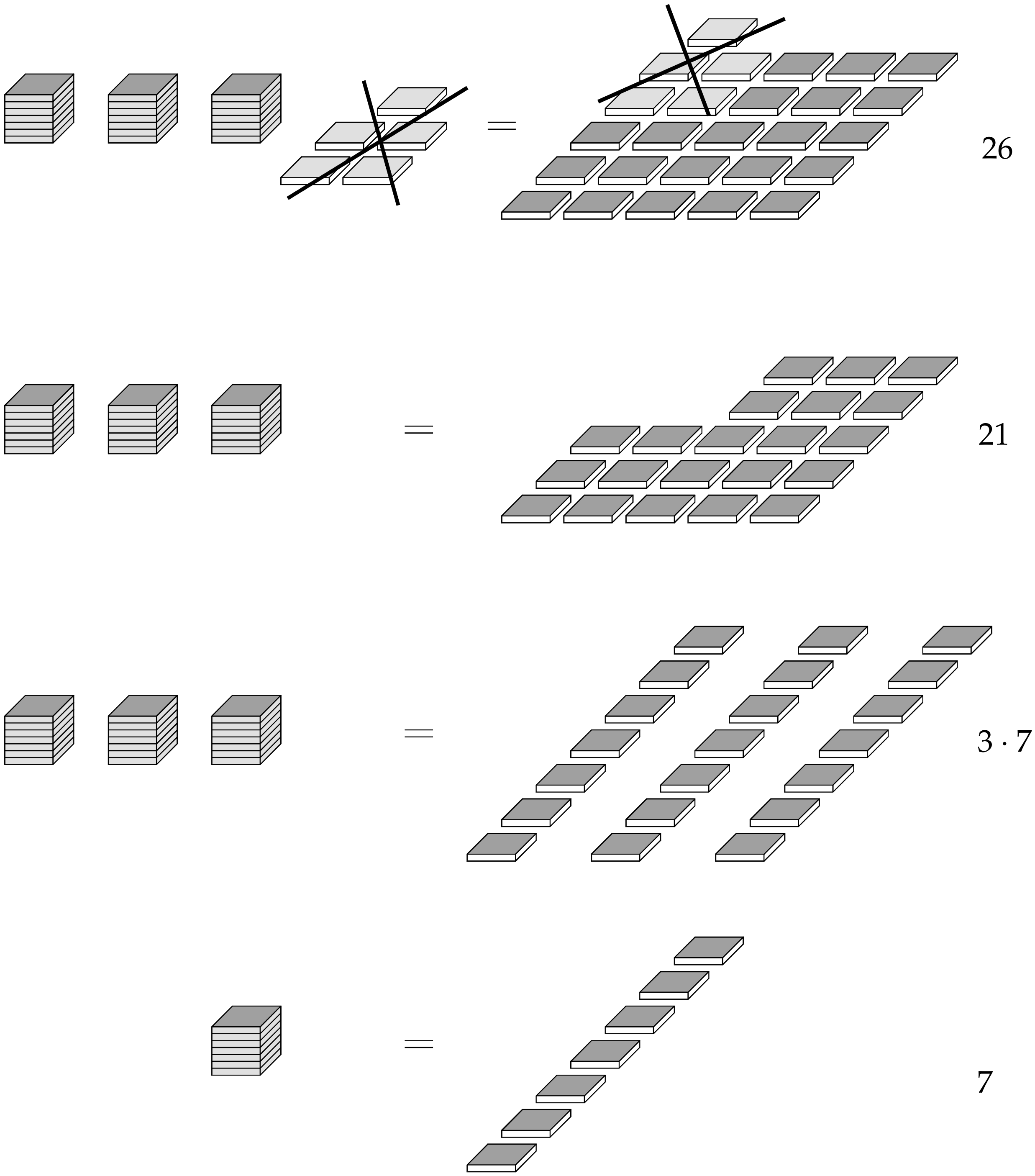


Place the three stacks and the five extra chips on a piece of paper and write an equals sign and the number 26. You have now made a statement that

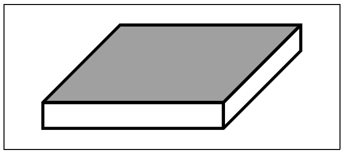
$$3 \text{ stacks} + 5 = 26$$

Now, *without counting the stacks*, can you figure out how many chips are in each stack? It is not usually difficult; most people do something like this:

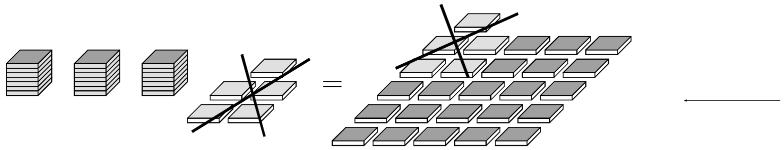




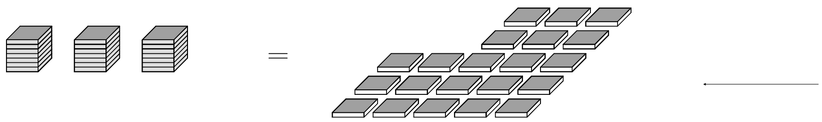
If the three stacks plus five are a total of 26 chips, then the three stacks alone must be 26 minus 5 or 21. If three equal stacks total 21, then each must be 21 divided by 3, or 7. Count a stack, and you will find that you were correct.



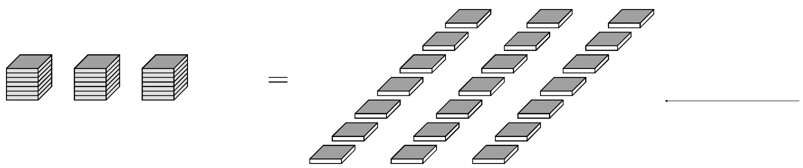
With algebra symbols, this is how you would do it:



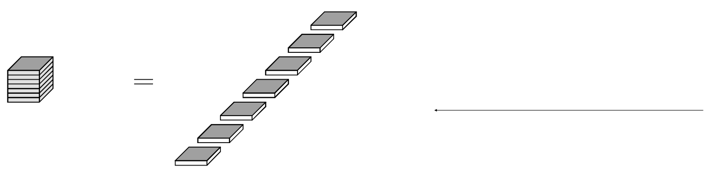
$$\begin{array}{rcr} 3x + 5 & = & 26 \\ -5 & & -5 \end{array}$$



$$3x = 21$$



$$\frac{3x}{3} = \frac{21}{3}$$



$$x = 7$$

As you can see, the ***x*** stands for the number of chips in a stack and **3*x*** stands for the number in three stacks. *In this book, the symbols you use will stand for something real; the algebra techniques will generally be shown as a movement of chips rather than just a manipulation of symbols.*

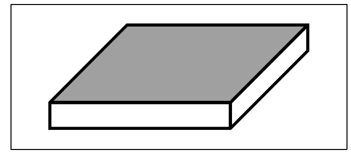
You have just solved a **linear equation in one variable**. As you progress through this book, you will find that most of algebra is this easy; you may also find, as you did here, that you know many of the concepts already.

Demonstration 2: Factoring

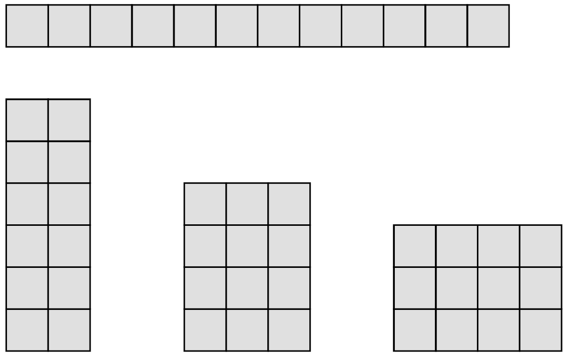
The mysterious art of factoring is usually thought to require lengthy practice and repetition. Here you will do it painlessly in a few minutes.

For this exercise, you will need:

- **1 large square**
- **7 long bars**
- **12 small squares**



First we will do a preparatory exercise. Take the 12 small squares and arrange them into a rectangle. There are several possibilities:



We call this **factoring**. When we take 12 and arrange it as 2 groups of 6, we say that

$$12 = 2 \cdot 6$$

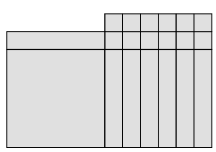
The other possibilities are:

$$12 = 3 \cdot 4$$
$$12 = 1 \cdot 12$$

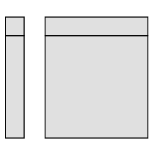
Factoring is making rectangles.

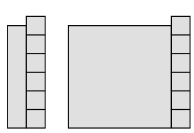
Now we will do the main exercise using all of the chips listed above. Your job is to rearrange these chips into a rectangle. Here are the rules of the game:

- **You must make a smooth rectangle. No holes or projecting chips are allowed. A square is considered to be a type of rectangle.**

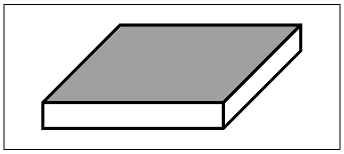
No 

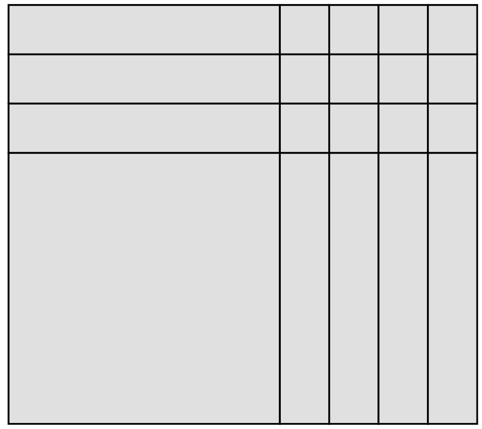
- **The small squares will only match with the end of the bars. They will not fit along the side of the large square or on the long side of the bars.**

Yes 

 No

Try it now. If you get stuck, keep moving the pieces around until you see the answer. Our solution is on the following page (yours may be slightly different):

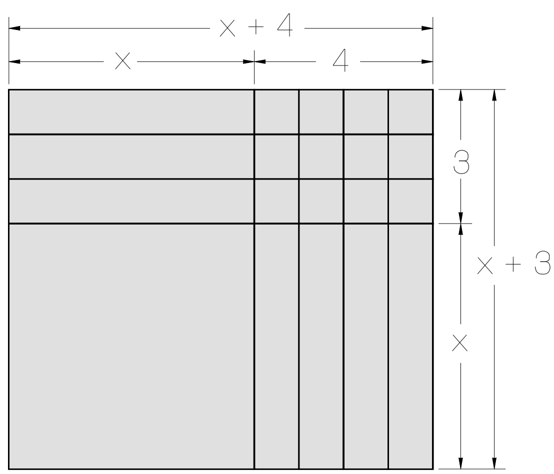




However your rectangle is formed, it will have a bar and three chips along one edge, and a bar and four chips along the other edge.

In the language of algebra, the small squares stand for **1**, the bars stand for x, and the large squares stand for x^2 (read as "x squared"). Because the finished rectangle has a length of one bar plus three chips and a width of one bar plus four chips, we say that:

$$x^2 + 7x + 12 = (x + 4)\cdot(x + 3)$$



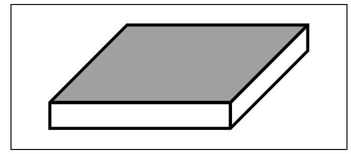
You have just **factored a quadratic expression**.

Demonstration 3: Dividing Fractions

Without writing anything down, can you quickly answer this question:

$$8 \div \frac{1}{4} = ?$$

Do not work out the answer using any rules.



Typical answers are 2, 12, 32, and "I don't know." If you can recall the rules of arithmetic, you would probably do it this way:

$$8 \div \frac{1}{4} = 8 \cdot \frac{4}{1} = \frac{32}{1} = 32$$

Invert (Why?) Multiply

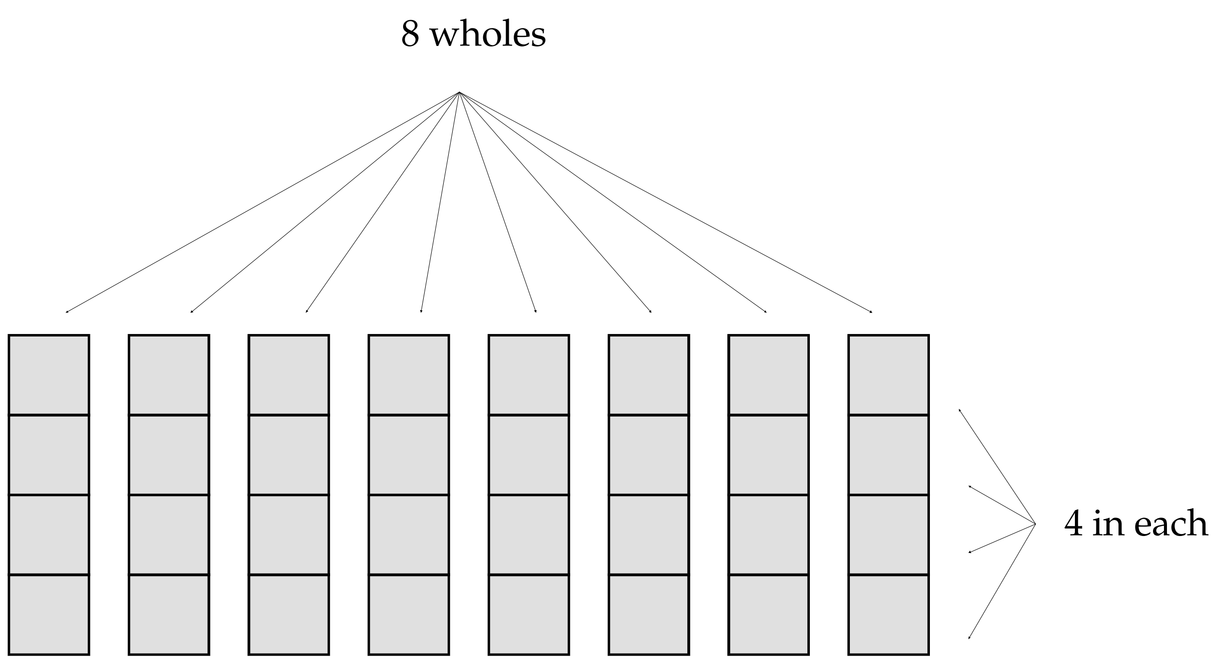
What does this mean? Let's pose this question another way:

How many quarters are in $8.00?

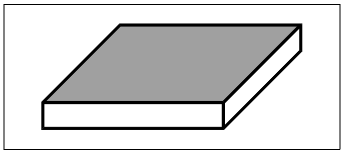
Since there are 4 quarters in 1 dollar, there are $4 \cdot 8$ or 32 quarters in 8 dollars.

Was this second problem much easier than the first? Yes, because it meant something very real. In fact, most people are able to do it even if they do not remember rules about dividing fractions.

Now we will do the problem with chips. If we decide that 4 chips together are one whole unit, then each chip is ¼. Set up 8 whole units like this:



You can see that there are 8 groups of 4 or $8 \cdot 4 = 32$ quarters.



Summary

Here are some of the important lessons of this chapter:

- **The symbols of arithmetic and algebra can stand for real objects.**
- **We already know many of the concepts of algebra.**
- **The best way to learn algebra is to understand the meaning of the symbols, techniques, and properties. Understanding algebra is more enjoyable and more efficient than memorizing a list of rules.**

Exercises

Use the chips to solve these problems:

Solving equations: Set up stacks and determine how many chips are in a stack. Remember that x is a stack.

1. $4x + 2 = 26$ (Use 26 chips)

2. $2x + 9 = 19$

3. $7x + 6 = 34$

Factoring: Make the chips into a rectangle.

4. $x^2 + 8x + 12$

5. $x^2 + 8x + 15$

6. $2x^2 + 5x + 2$ ($2x^2$ means two large squares)

Dividing fractions: Show the answer with the chips.

7. $6 \div \frac{1}{3}$

8. $4 \div \frac{1}{2}$

9. $4 \div \frac{2}{3}$

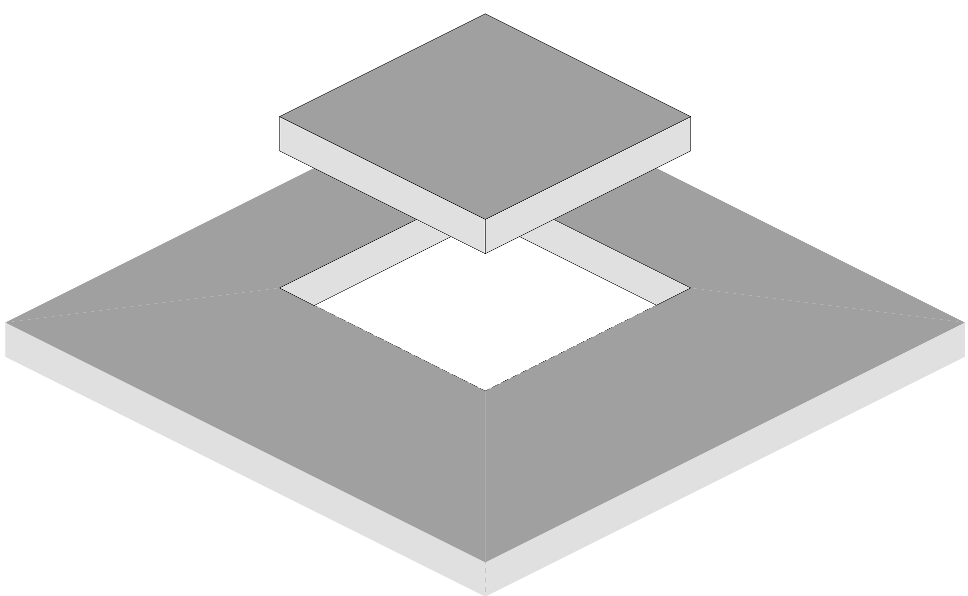
10. $12 \div \frac{3}{4}$

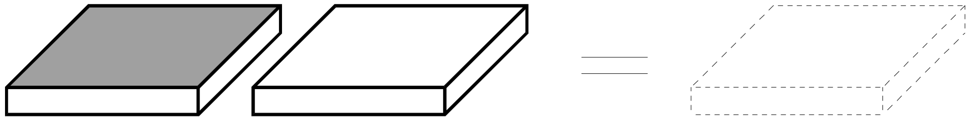
11. $3 \div \frac{1}{3}$

12. $2 \div \frac{2}{3}$

Chapter 2

Positive and Negative Numbers

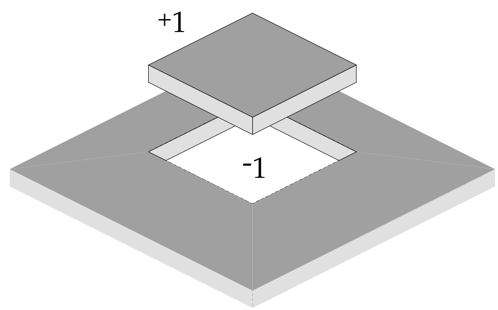




Section 1
Positive and Negative Numbers

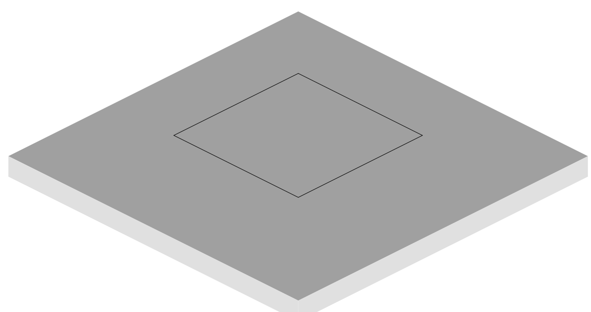
The Meaning of Positive and Negative Numbers

Imagine a slab with a square section removed:

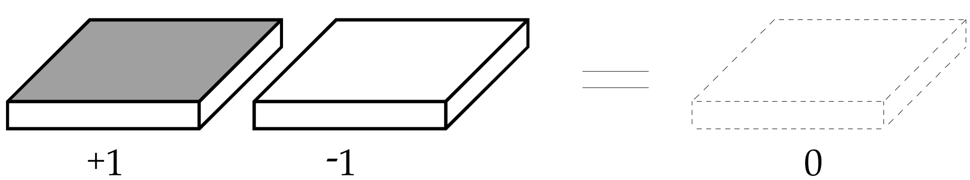


Positive one ($^+1$) is the square chip that is cut out of the slab. Negative one ($^-1$) is the hole that it came out of.

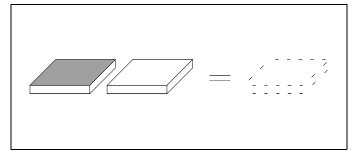
Add $^+1$ and $^-1$ back together and you fill in the hole; zero is your result:



For practical purposes, it is more convenient to use two chips of different colors to represent $^+1$ and $^-1$. When they are added together, they cancel each other out, leaving zero.



Signed Numbers and Flip-Chips™

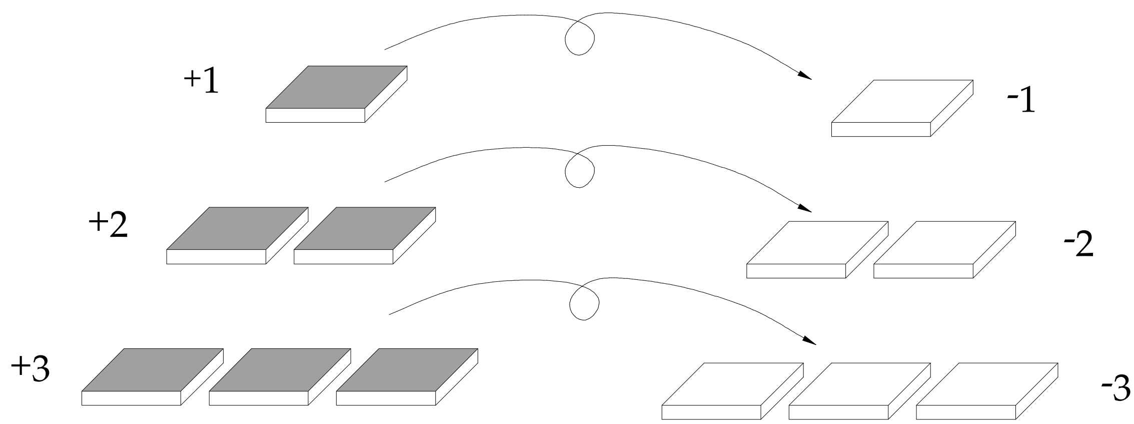


A number with a sign (+ or –) directly to its left (in front of the number when reading from left to right) is called a **signed number**. The positive (+) or negative (–) sign tells *what color* chips the number represents and the number tells *how many* of these chips are represented. Together, all of the positive and negative numbers are called **integers**.

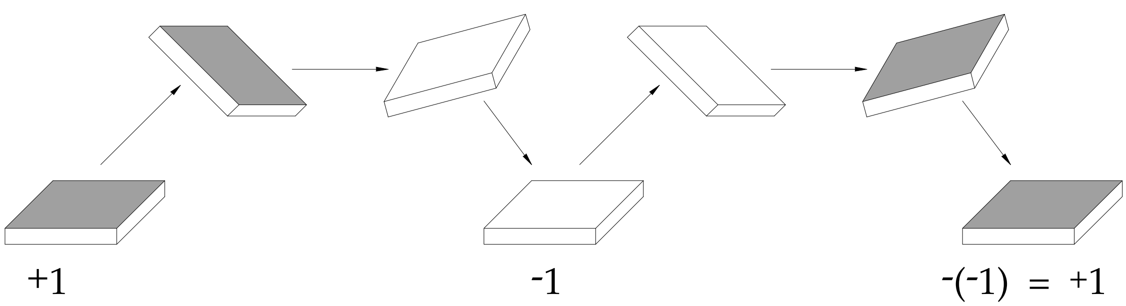
With a piece of material which has a different color on each side it is possible to make a **Flip-Chip**—a piece which represents $^{+}1$ with one side up, and $^{-}1$ with the other side up. *Flipping the chip changes the sign***!**

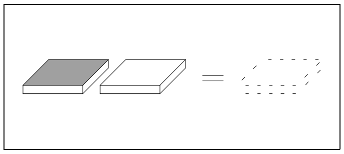
The chips we use are colored on one side and white on the other side, so we call the colored side **positive** or **plus** (+) and the white side **negative** or **minus** (–). This way we always know which side is which.

Flipping the chip changes the sign!



And a second negative sign flips the chip *again*!:





Double or Multiple Signs

A number may be shown having more than one sign in front (to the left) of it. These signs can be written in several ways; parentheses are often used to enclose the number and one sign:

$^{+}(^{-}3)$

$^{-}(^{-}2)$

$^{+}(^{+}5)$

$^{+}(^{-}4)$

Thinking of these numbers as chips, remember that each negative (–) sign in front of a number flips the chips one time, so two minus signs flip the chips twice, giving a positive (+) side up. We always begin with the colored (+) side up before we start flipping. Here is the result of four different combinations of signs:

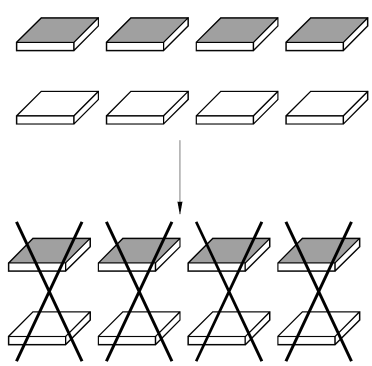
$$^{+}(^{+}3) = {}^{+}3$$
$$^{+}(^{-}3) = {}^{-}3$$
$$^{-}(^{+}3) = {}^{-}3$$
$$^{-}(^{-}3) = {}^{+}3$$

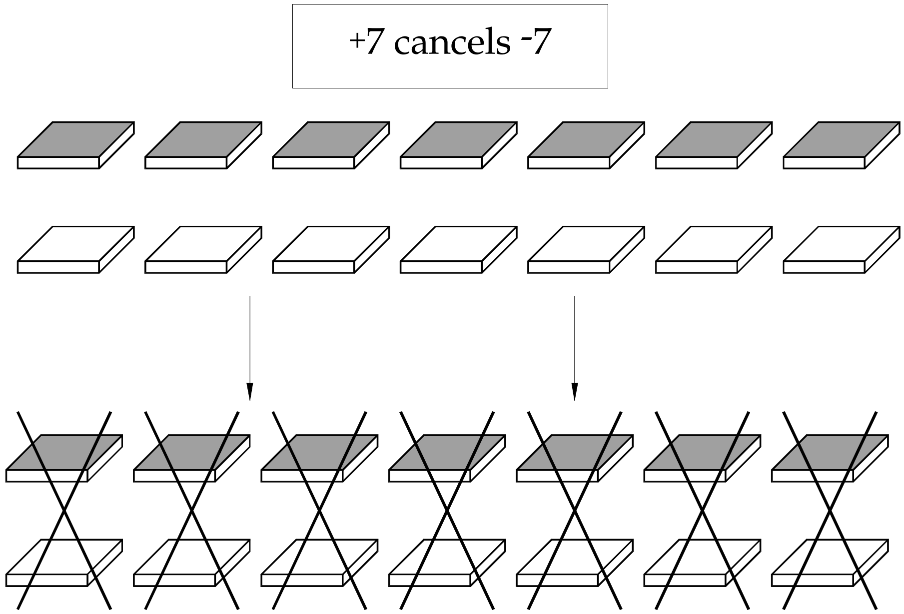
Each negative sign means to flip the chips once; each positive sign means to leave them alone. We always start with the colored (positive) side up.

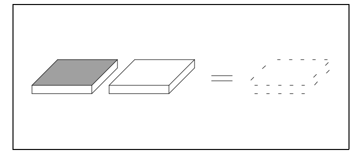
Cancelling of Positives and Negatives

The basic principle of grouping positive and negative chips together is that one positive chip grouped with one negative chip cancels to give zero. This means that if we put an equal number of positive and negative chips together, they will cancel to give zero:

$^{+}4$ cancels $^{-}4$

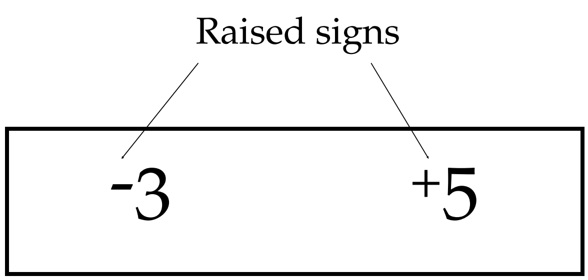




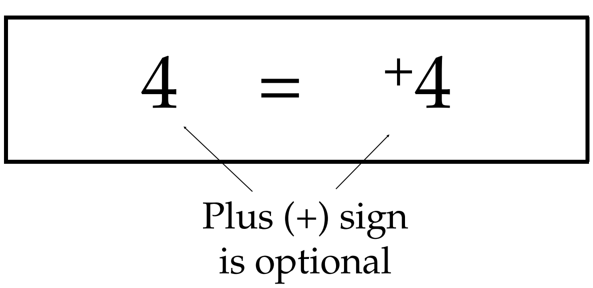


Symbols and Signs

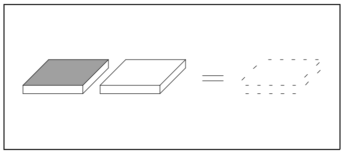
We have been using several symbols that may be unfamiliar. First we have been showing positive and negative numbers with small plus or minus signs that are on the left of the number and raised up slightly.



Positive numbers can be shown *with or without* the positive sign. The familiar number 4 and the new symbol $^{+}4$ have the same meaning:



Although a positive sign is optional, a negative number must always be shown with a minus sign so that we can tell that it is negative.

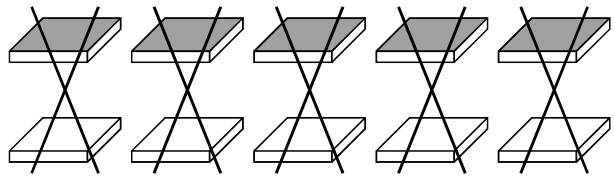


Exercises

Use the chips to illustrate the following results:

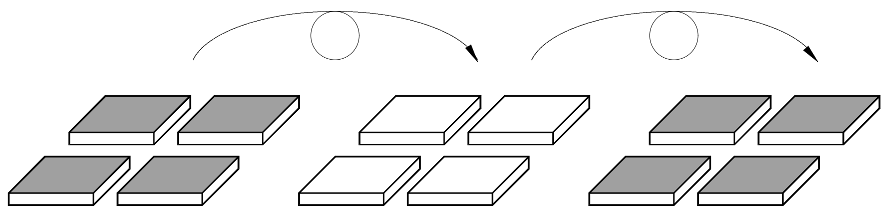
Example: 5 and $^{-}5$ cancel to 0

Solution:



Example: $^{-}(^{-}4) = {}^{+}4$

Solution:



Start with 4 Flip to $^{-}4$ Flip again, to $^{-}(^{-}4) = {}^{+}4$

1. $^{-}(^{-}7) = {}^{+}7$

2. $^{-}(+3) = {}^{-}3$

3. $^{-}(^{-}11) =$

4. $^{-}(+3) =$

5. $^{+}(^{-}9) =$

6. $^{-}(^{-}10) =$

7. $^{-}(3) = {}^{-}3$

8. 3 and $^{-}3$ cancel to 0

9. 6 and $^{-}6$ cancel to 0

10. $^{-}6$ and $^{-}(^{-}6)$ cancel to 0

11. $^{-}(11)$ and $^{+}11$ cancel to 0

12. $^{-}(^{-}17) =$

13. $^{+}(^{-}0) =$

14. $^{-}(^{-}0) =$

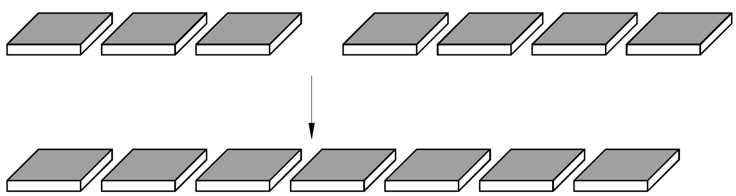
Section 2
Addition of Signed Numbers

The Meaning of Addition

In the past, adding two numbers meant that we took two amounts and combined them. Now that we have invented positive and negative numbers, addition will still have the same basic meaning, as long as we understand the idea that equal groups of positive and negative chips cancel each other out.

Adding Two Positives

If we are adding two positive numbers, we simply combine two groups of positive chips to give one larger group of all positive chips:



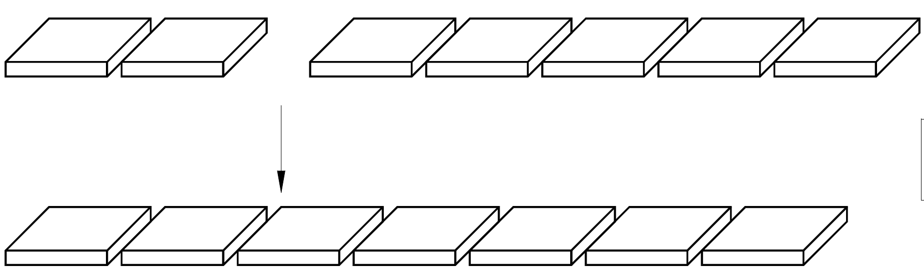
$$(^+3) + (^+4) = 7$$

Adding Negatives

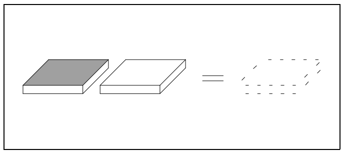
To add negative numbers, we combine the groups of negative chips. For example:

$$(^-2) + (^-5)$$

This expression means that we should take 2 negative chips and 5 negative chips and group them together. The result is clearly 7 negative chips:



$$(^-2) + (^-5) = {}^-7$$



As we can see from the last two examples, adding numbers with the same sign is very easy—we simply combine the chips and count the total number:

$$(^{+}6) + (^{+}3) \;=\; {}^{+}9$$

$$(^{+}12) + (^{+}3) \;=\; {}^{+}15$$

$$(^{-}3) + (^{-}5) \;=\; {}^{-}8$$

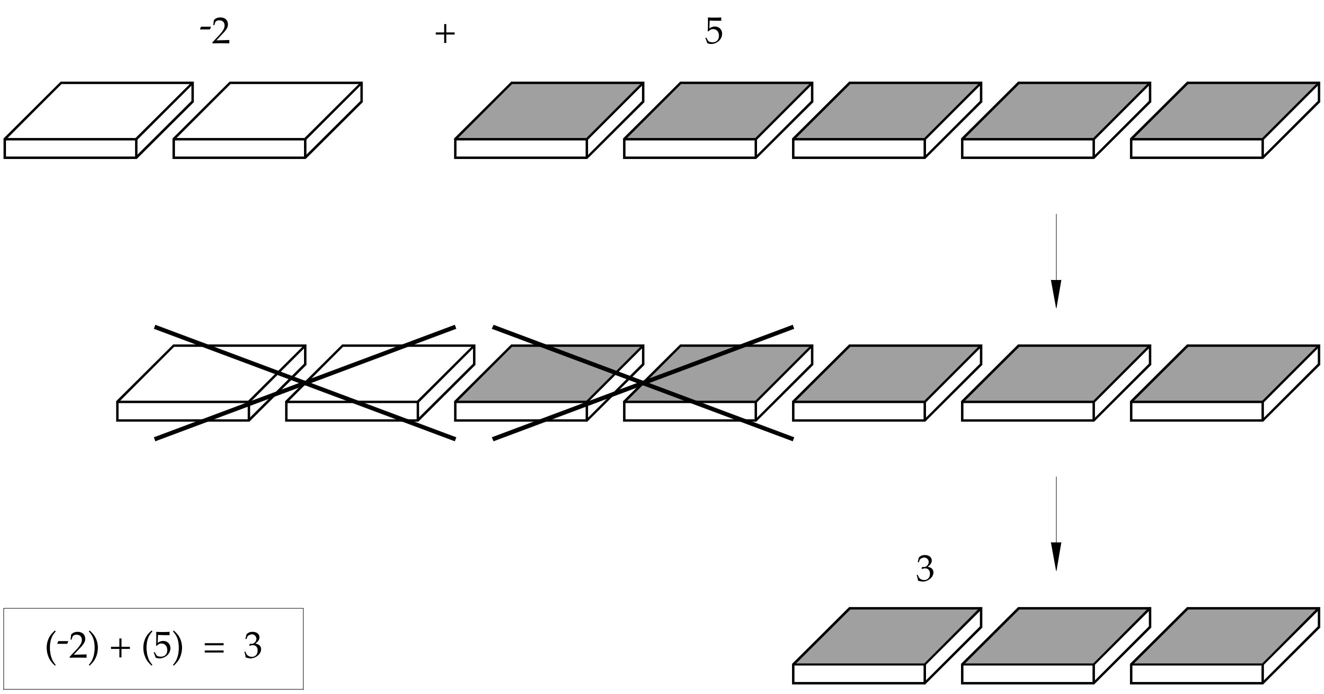
$$(^{-}6) + (^{-}4) \;=\; {}^{-}10$$

The parentheses shown above are not required but can be helpful. We use them to separate the number from the addition sign; if you leave them out, make sure to keep the negative signs raised and close to the numbers:

$${}^{-}6 + {}^{-}4 \;=\; {}^{-}10$$

Adding Negative and Positive Numbers

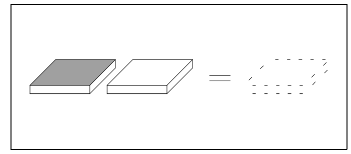
If we need to add a negative number and a positive number, we combine the two groups of chips and cancel out pairs of negatives and positives:

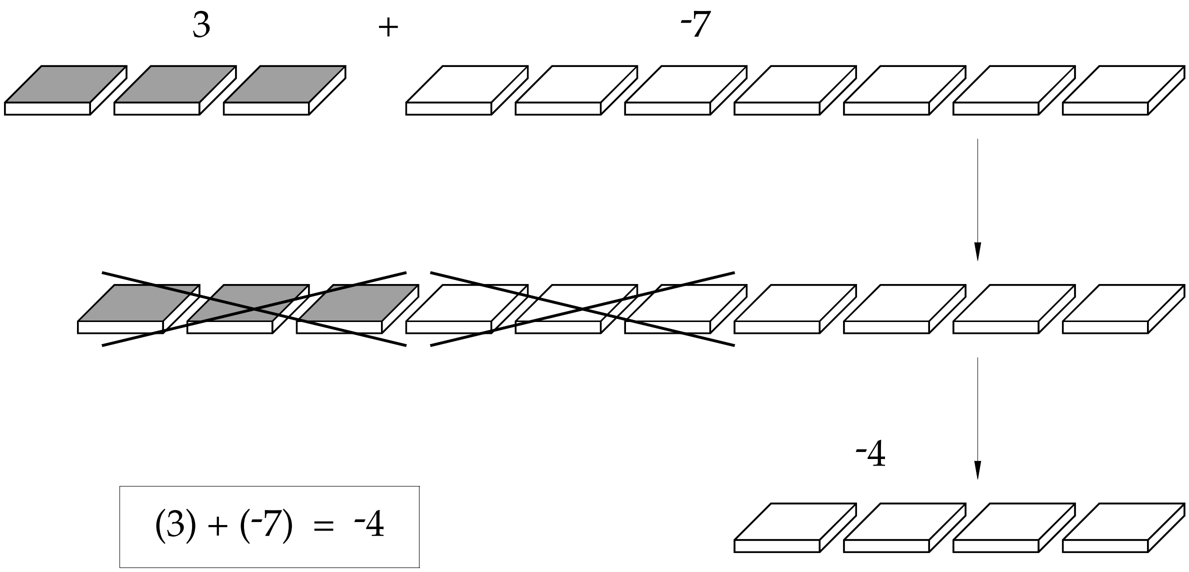


$$(^{-}2) + (5) \;=\; 3$$

Did you notice that there were more positives than negatives? Because of this, when the cancelling is done, we are left with positives.

Here is an example of adding a positive number and a negative number where there are more negative chips:



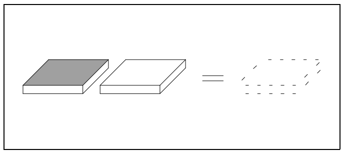


As you would expect, the positive chips cancelled out some of the negatives, but there are still negatives remaining.

Summary

To add two numbers, we combine the chips, cancelling if we have a mixed group of positives and negatives:

- **Adding two positives—Combine the groups of chips for a total of more positives.**
- **Adding two negatives—Combine the groups of chips for a total of more negatives.**
- **Adding a positive and a negative—Combine the groups of chips and let positive and negative chips cancel out in pairs. The chips which remain will have the same color (sign) as the larger original group.**



Exercises

Use your chips to set up and solve the following addition problems:

1. $(^{-}5) + (^{+}5) =$
2. $(^{+}3) + (^{+}11) =$
3. $(^{-}5) + (^{-}1) =$
4. $(^{-}3) + (^{-}3) =$
5. $(^{-}1) + (^{-}1) =$
6. $(^{+}8) + (^{+}4) =$
7. $(^{-}4) + (^{-}3) =$
8. $(^{-}3) + (^{-}4) =$
9. $(^{-}6) + (^{-}7) =$
10. $(^{-}12) + (^{-}1) =$
11. $(^{-}7) + (^{+}6) =$
12. $(^{+}7) + (^{-}6) =$
13. $(^{-}11) + (^{+}2) =$
14. $11 + (^{-}2) =$
15. $4 + {}^{-}5 =$
16. ${}^{-}4 + 5 =$
17. $1 + (^{-}2) =$
18. ${}^{-}1 + (^{-}3) =$
19. ${}^{-}1 + 3 =$
20. ${}^{-}2 + {}^{-}3 =$
21. $7 + {}^{-}5 =$
22. ${}^{-}3 + {}^{-}5 =$
23. ${}^{-}3 + 5 =$
24. $6 + 2 =$
25. $6 + (^{-}2) =$
26. ${}^{-}6 + 2 =$
27. ${}^{-}6 + (^{-}2) =$
28. $4 + 5 =$
29. $4 + {}^{-}5 =$
30. ${}^{-}4 + 5 =$

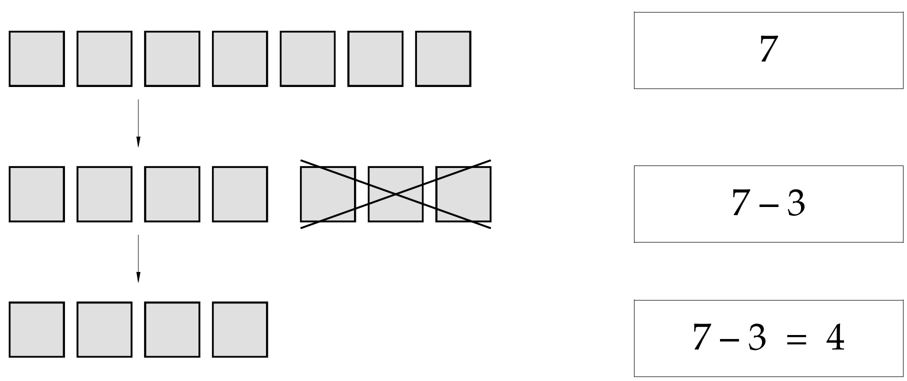
Section 3
Subtraction of Signed Numbers

The Meaning of Subtraction

We were able to easily extend our old idea of addition to cover signed numbers, but we will have to do a little more work to invent a new definition of subtraction. By subtraction, we have always meant the concept of taking away part of what we have. For example:

$$7 - 3$$

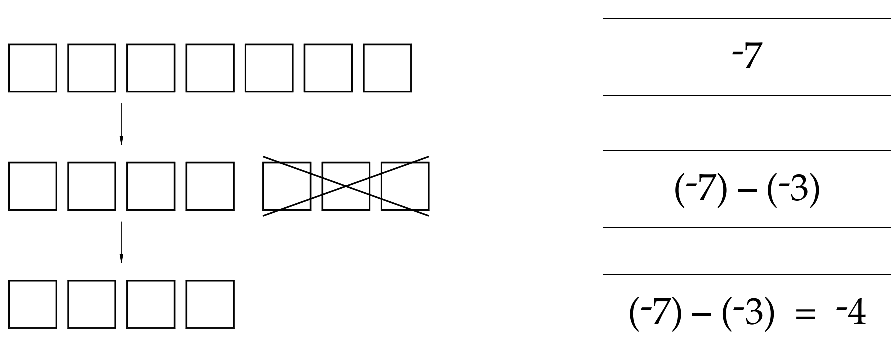
With the chips, this means that we start with 7 chips and then take away 3 chips. The result is 4:

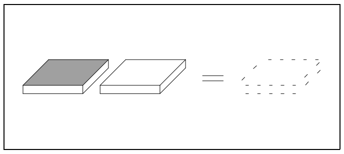


This standard idea of subtraction will also work well with the following example:

$$(^-7) - (^-3)$$

We start with 7 negative chips and take away 3 negative chips:





Although these examples work well with our idea of "taking away," subtraction is not always that easy. What if we are asked to subtract more chips than we start with?

$$5 - 7$$
$$3 - 18$$
$$9 - 10$$

$$(^-5) - (^-6)$$
$$(^-2) - (^-8)$$

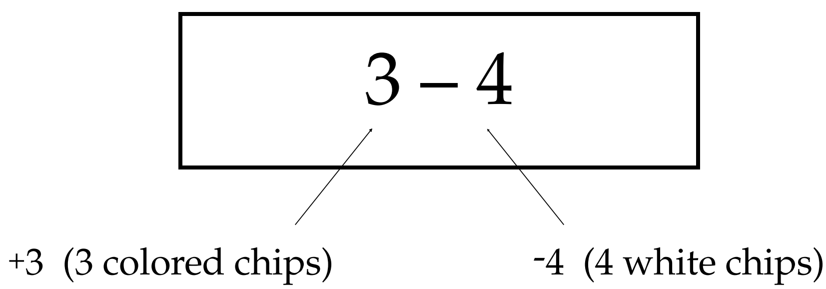
Our system of subtraction needs to make sense when given these types of problems. We also need to know what to do if we start with one color chips, but we are asked to take away or subtract the *other* color of chips:

$$3 - (^-2)$$
$$^-4 - 5$$
$$0 - (^-6)$$

The old idea of "take away" is clearly not going to work for subtraction of signed numbers.

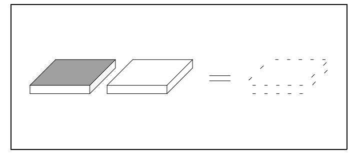
Subtraction of Signed Numbers: Method I

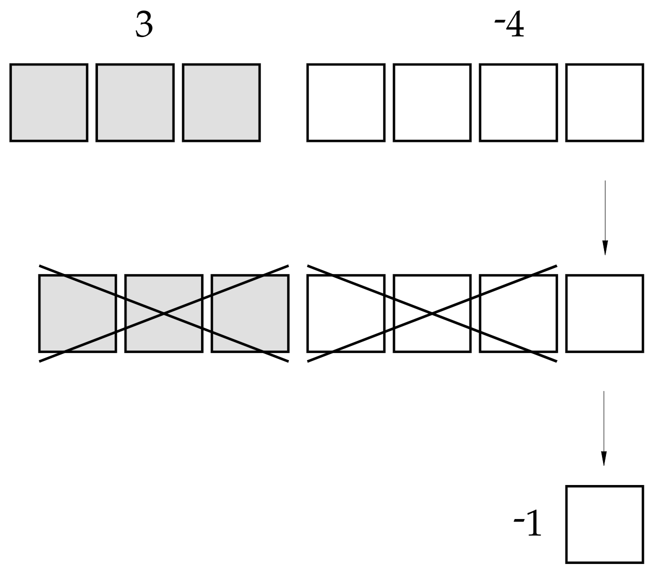
Our first new method of doing subtraction will be very simple—in a given expression, each number will tell us how many chips are in one group, and the sign in front (to the left) of each number will tell us what color chips are in that group. We will then add the groups together. If the chips are different colors, let them cancel in pairs.



Instead of subtraction, we think of the problem as adding groups of chips which are sometimes different colors. Look at each number and the sign to its left. Since 3 has no sign, it is positive; since 4 has a minus sign (–) it is negative. In this situation, *the subtraction sign is considered to be the same as a negative sign.*

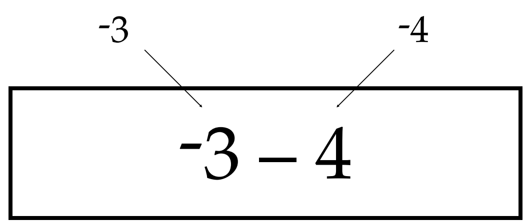
When we add 3 and $^{-}4$, the result is $^{-}1$:



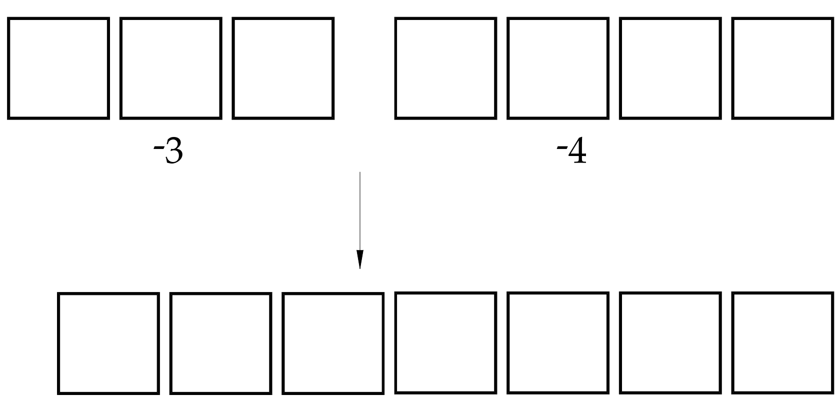


$3 - 4 = {}^{-}1$

Here is another example:

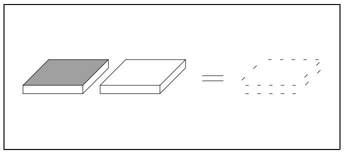


We think of the problem as starting with $^{-}3$ and adding $^{-}4$:



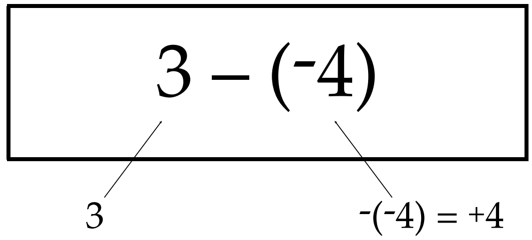
$^{-}3 - 4 = {}^{-}7$

The result is $^{-}7$.



Subtraction and Double Signs: Method I

If two signs appear next to each other with no number in between, think of them as double signs. Flip the chips for *each* negative or subtraction sign. If there are two negative signs, we flip the chips twice and the result is positive. We then add: For example:



This gives:

$$3 - (^{-}4) = 3 + 4 = 7$$

Summary: Method I

To add or subtract:

- **Identify each number as positive or negative by the sign in front of it. Choose the correct color chips for each group, then add the groups together.**
- **If there are double signs in front of any number, flip that group of chips the proper number of times, then add the groups together.**
- **Think of all addition and subtraction as *addition.***

Subtraction: Method II

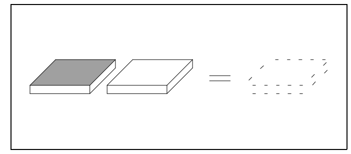
We will now look at another way of subtracting. Method II is very much like Method I; you should use whatever method is most comfortable. It is best to understand both methods—they are simply two different ways to illustrate the same idea.

First, let's look at some examples of adding and subtracting where two different problems have the same answer:

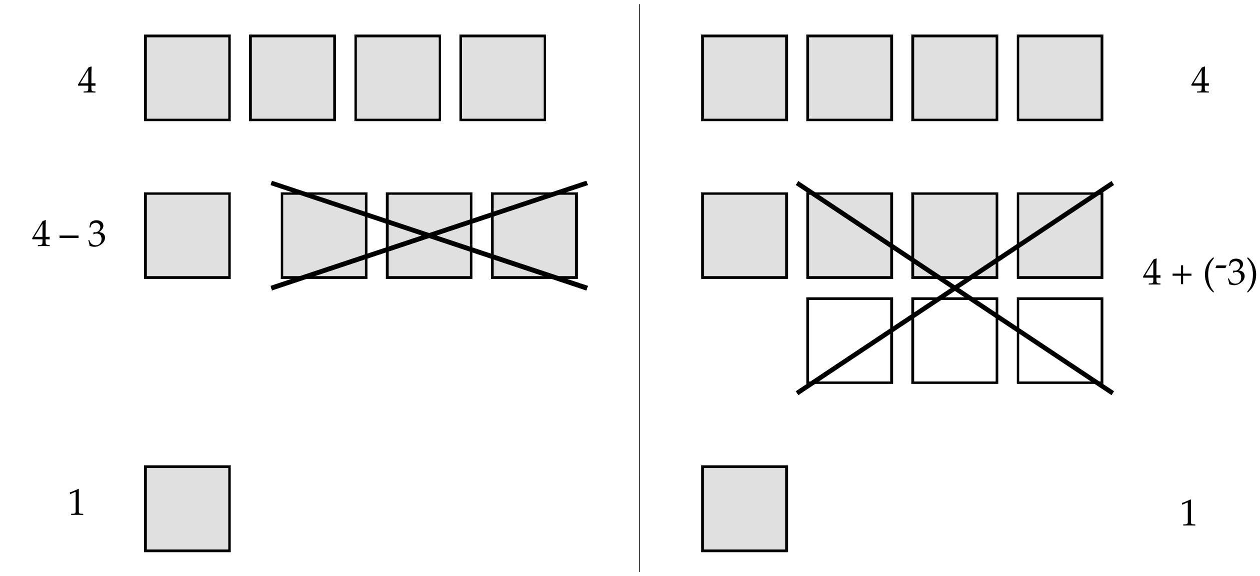
$$4 - 3 = 1$$

$$4 + (^{-}3) = 1$$

In the diagram below, you can see that *adding* $^{-}3$ is the same as s*ubtracting* 3:a



$$4 - 3 \quad = \quad 4 + (^{-}3)$$



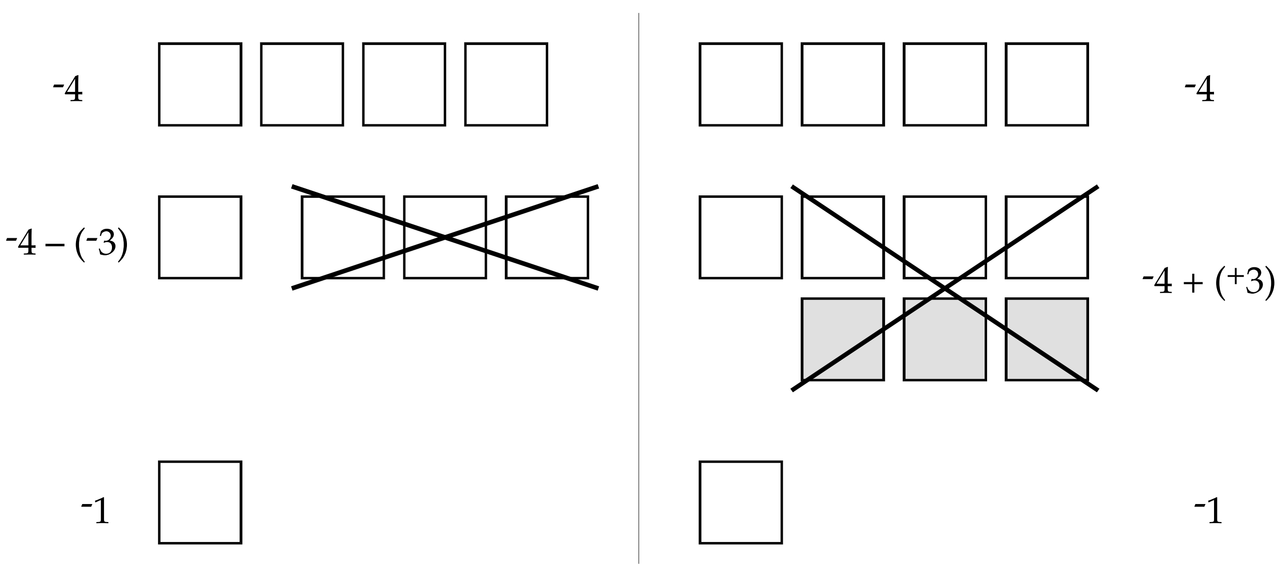
We can see that the following two examples also have the same result:

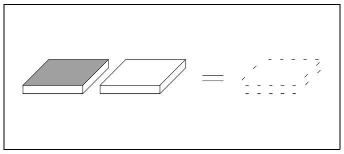
$$^{-}4 - (^{-}3) \ = \ ^{-}1$$

$$^{-}4 + (^{+}3) \ = \ ^{-}1$$

The diagram shows why this is true:

$$^{-}4 - (^{-}3) \quad = \quad ^{-}4 + (^{+}3)$$





We can see that:

- **Subtracting a positive number is the same as adding a negative number.**
- **Subtracting a negative number is the same as adding a positive number.**
- **In general, subtracting any number is the same as adding its opposite.**

$$4 - 3 = 4 + (^-3)$$

$$^-4 - (^-3) = {}^-4 + (^+3) = {}^-4 + 3$$

Here are some examples of how to use Method II with subtraction:

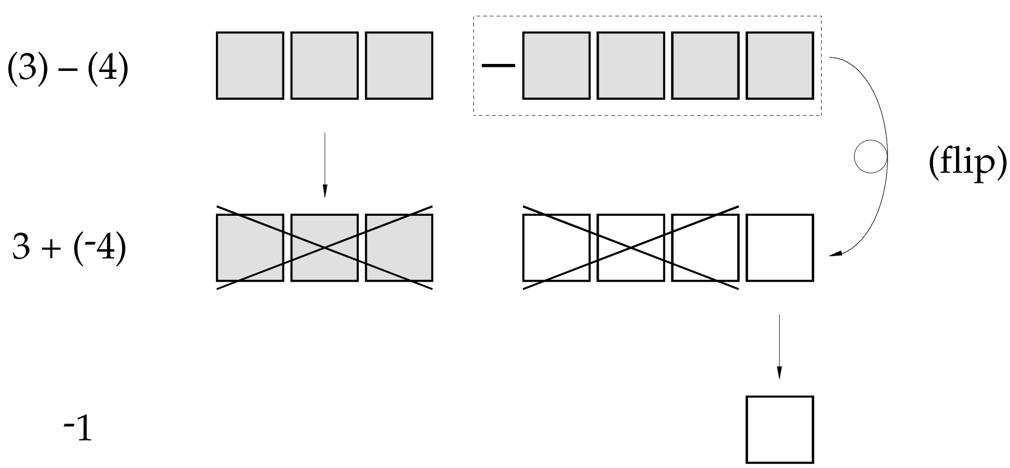
$$7 - 2 = 7 + (^-2) = 5$$

$$8 - (^-3) = 8 + (3) = 11$$

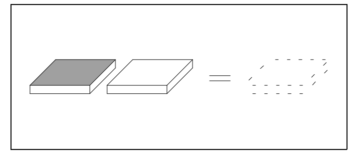
$$^-6 - 3 = {}^-6 + (^-3) = {}^-9$$

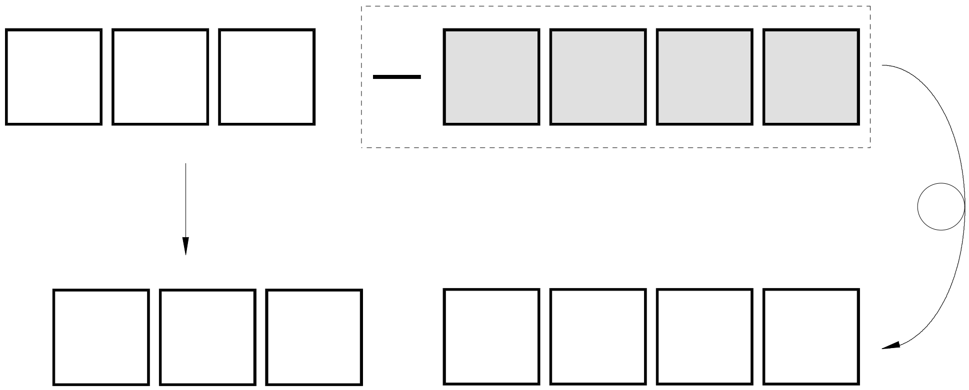
With the chips, we set up a subtraction with Method II by taking out the two groups of chips indicated. We than flip the subtracted group of chips and combine the two groups. Here are three examples:

$$3 - 4 = {}^-1$$



$^{-}3 - 4 = {}^{-}7$

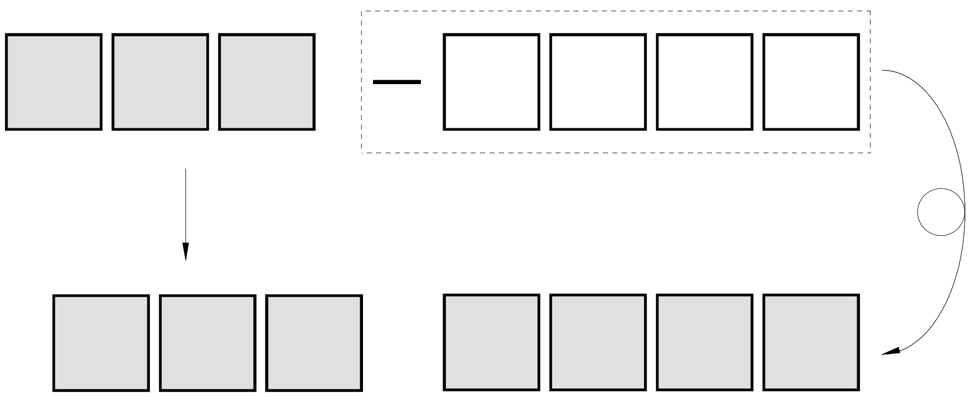




(-3) – (4)

$({}^{-}3) + ({}^{-}4) = {}^{-}7$

$3 - ({}^{-}4) = 7$



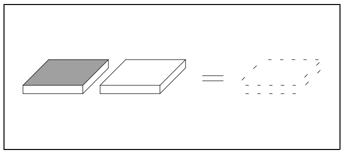
$(3) - ({}^{-}4)$

$(3) + ({}^{+}4) = 7$

Summary: Method II

To subtract (*a* and *b* stand for any numbers):

- $a - b = a + ({}^{-}b)$
- $a - ({}^{-}b) = a + ({}^{+}b)$
- **To subtract any number of chips, flip the subtracted chips and add.**



Summary: Method I and Method II

We have looked at two methods for doing subtraction. With both methods, we think of subtraction as adding. With Method I, we just look at the signs in front of each number to see what color chips to add; with Method II, we look at every subtraction as adding the opposite.

To Subtract:

Method I: Choose the color of chips by looking at the signs in front of each number, then add.

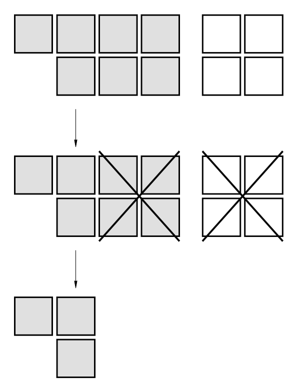
Method II: Instead of subtracting the second number, flip the chips and add the opposite.

Exercises

Use the chips to do the following subtractions:

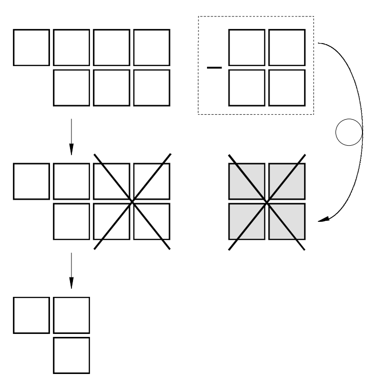
Example: 7 – 4
Solution: 3

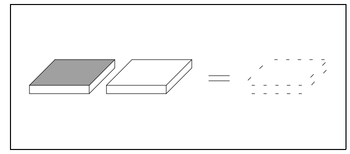
(Method I)



Example: $(^{-}7) - (^{-}4)$
Solution: $^{-}3$

(Method II)





1. $5 - (^{-}3)$

2. $^{-}5 - (^{+}3)$

3. $^{-}5 - (^{-}3)$

4. $^{-}6 - (^{-}3)$

5. $3 - 5$

6. $3 - (^{-}5)$

7. $^{-}3 - 5$

8. $^{-}3 - (^{-}5)$

9. $0 - (^{-}17)$

10. $4 - 0$

11. $6 - (^{-}0)$

12. $1 - (^{-}1)$

13. $12 - (^{-}5)$

14. $^{-}12 - (^{-}5)$

15. $^{-}7 - 9$

16. $^{-}7 - (^{-}9)$

17. $^{-}4 - 4$

18. $^{-}4 - (^{-}4)$

19. $4 - (^{-}4)$

20. $^{-}7 - 3$

21. $^{-}7 - (^{-}3)$

22. $7 - 3$

23. $7 - (^{-}3)$

24. $5 - 2$

25. $^{-}5 - (^{-}2)$

26. $2 - 5$

27. $^{-}2 - (^{-}5)$

28. $^{-}2 - 5$

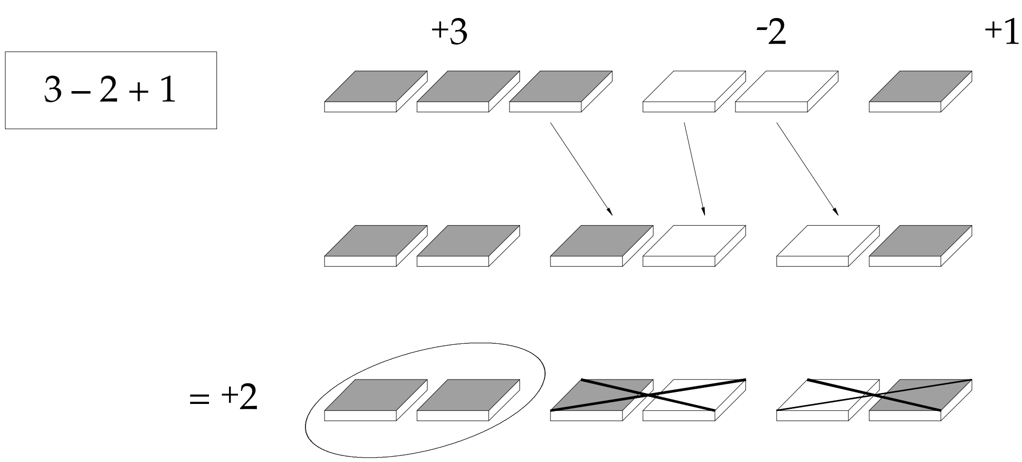
29. $2 - (^{-}5)$

30. $8 - (^{-}4)$

Section 4
Addition and Subtraction

Combining Addition and Subtraction

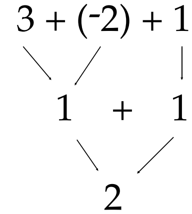
In a math sentence, if several signed numbers are written in a row with plus or minus signs in between the numbers, the sentence means that we should add the numbers by sliding the chips together and letting chips of different colors cancel out. The simplified answer is given by the sign and number of chips that are left when you're done. For example:



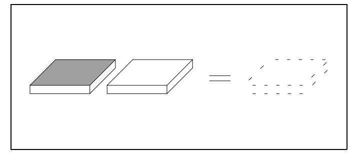
When you have three or more numbers together, we still think of them as being added. When subtraction is indicated, you may want to rewrite it as addition of the opposite kind of chips:

$$3 - 2 + 1 = 3 + (^-2) + 1$$

Then combine the chips to get the result. You can combine them in order from left to right:



Or you can rearrange the chips to add up the positives and negatives separately, and then cancel:



$$6 - 3 - 2 + 5$$

$$6 + (^{-}3) + (^{-}2) + 5$$

$$6 + 5 + (^{-}3) + (^{-}2)$$

$$11 \quad + \quad ^{-}5$$

$$6$$

Summary

When we have to add and subtract more than 2 numbers in a row, we use either method from the previous section and we consider all addition and subtraction as combining groups of chips:

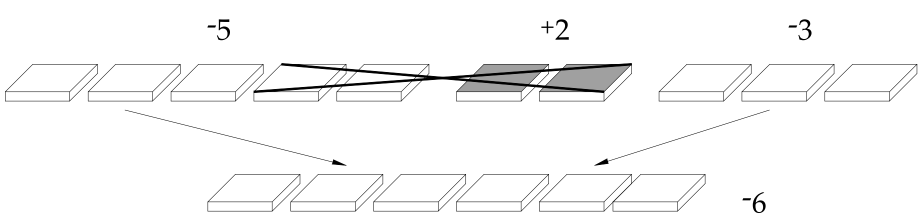
- **Combine the numbers in pairs**
- ***Or,* rearrange all of the positive numbers in one group and the negative numbers in another. Find the total negatives and total positives, then combine the totals.**

Exercises

Use the chips to find the answer and to illustrate the following problems:

Example: $^{-}5 + 2 - 3 = {}^{-}6$

Solution:

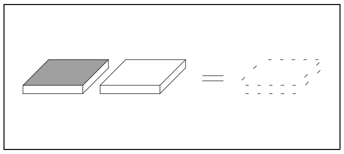


1. $^{+}1 - 4 + 3 = 0$

2. $^{-}2 + 1 - 4 =$

3. $^{+}2 + 3 - 1 - 2 =$

4. $^{+}5 - 6 - 3 - 1 =$



5. $^{+}2 - 7 + 5 - 1 =$

6. $^{+}6 + 4 + 3 + 3 =$

7. $^{-}1 + 5 - 6 + 2 =$

Use chips to show the following:

8. $^{+}(^{-}3) = {}^{-}3$

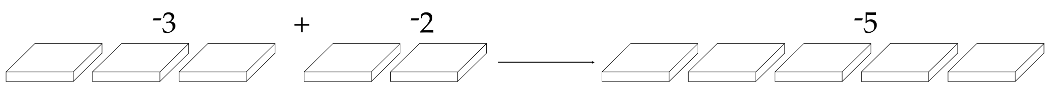
9. $^{-}(^{-}2) = {}^{+}2$

10. $^{+}(^{+}5) = {}^{+}5$

Use chips to do the following problems:

Example: $^{-}3 + (^{-}2) = {}^{-}5$

Solution:



11. $^{-}2 - {}^{-}2 = 0$

12. $^{+}2 - {}^{-}2 = {}^{+}4$

13. $^{-}1 - {}^{-}5 = {}^{+}4$

14. $^{-}(^{-}2) + 3 =$

15. $^{+}(^{-}5) - (^{-}2) =$

16. $^{-}(^{+}2) + 6 =$

17. $^{-}3 - {}^{-}7 =$

18. $^{-}3 + 5 =$

19. $3 + {}^{-}5 =$

20. $^{-}3 + {}^{-}5 =$

21. $3 - {}^{-}5 =$

22. $^{-}6 - 2 =$

23. $^{-}7 - {}^{-}3 =$

24. $^{-}7 - 3 =$

25. $7 - 3 =$

26. $7 - {}^{-}3 =$

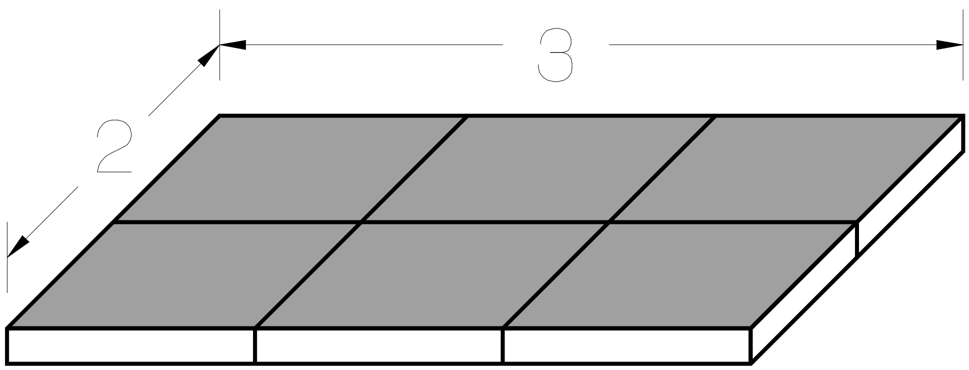
27. $^{-}8 + {}^{-}6 =$

Section **5**
Multiplication

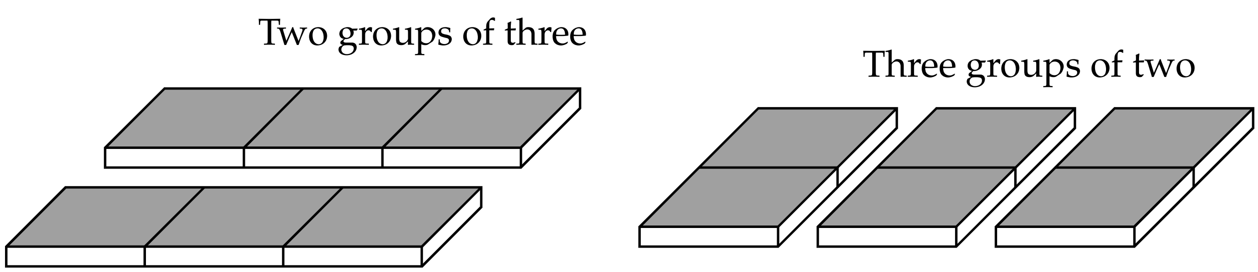
The Meaning of Multiplication

To multiply the numbers 3 and 2 using chips, make a rectangle 3 chips long and 2 chips wide, using six chips in all. We use a raised dot to indicate multiplication:

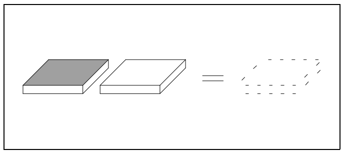
$$3 \cdot 2 = 6$$

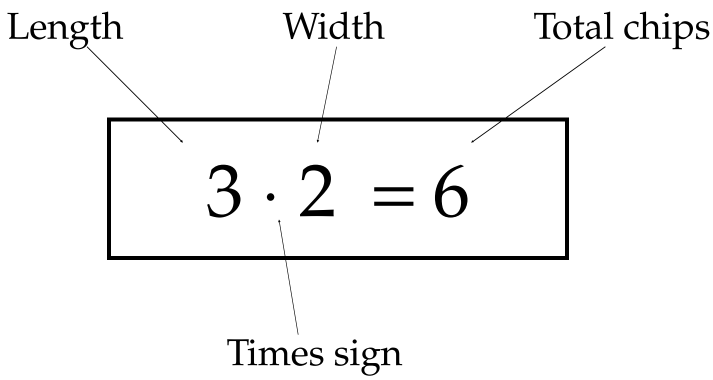


This shows either 3 groups of 2, or 2 groups of 3.



Multiplying any two numbers using chips means making a rectangle of chips with the numbers being the length and width. *Multiplying is making rectangles*. The answer to the multiplication—the product—is the total number of chips in the rectangle.



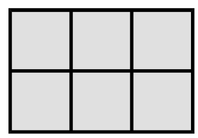


Multiplying with Signed Numbers

When multiplying signed numbers using chips we will still make a rectangle of chips, but we flip the chips once for each negative (–) sign used in the multiplication. Remember that we start with colored side up.

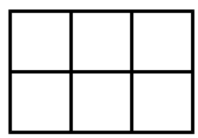
$(^+3) \cdot (^+2) = 6$

(No Flips)



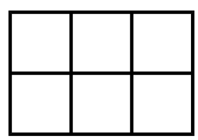
$(^+3) \cdot (^-2) = {}^-6$

(One Flip)



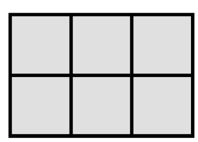
$(^-3) \cdot (^+2) = {}^-6$

(One Flip)



$(^-3) \cdot (^-2) = {}^+6$

(Two Flips)



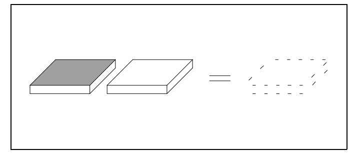
Here are some more examples:

$9 \cdot (-8) = -72$ (1 flip)

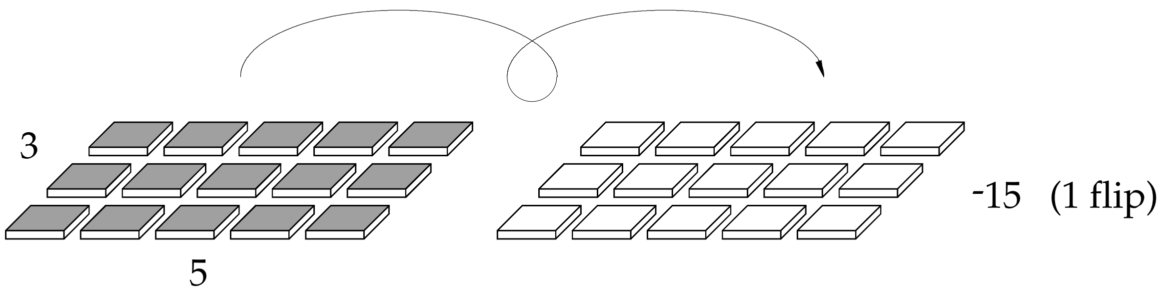
${}^-6 \cdot 3 = {}^-18$ (1 flip)

$(^-6) \cdot (^-3) = 18$ (2 flips)

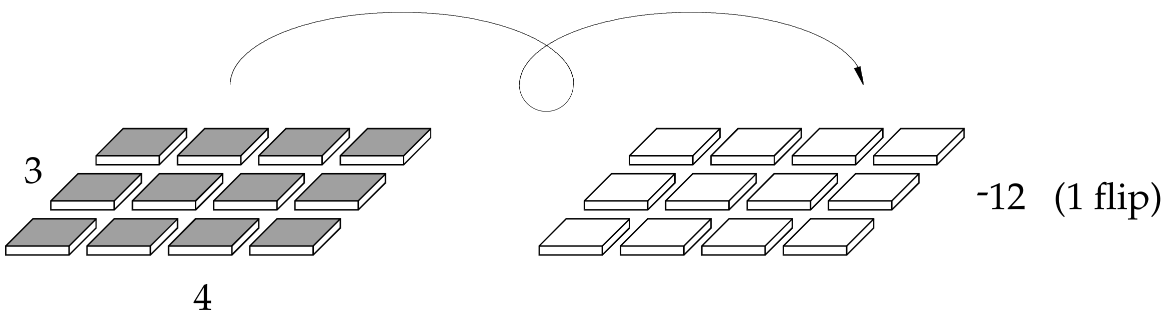
Here is how to use the chips for multiplying signed numbers:



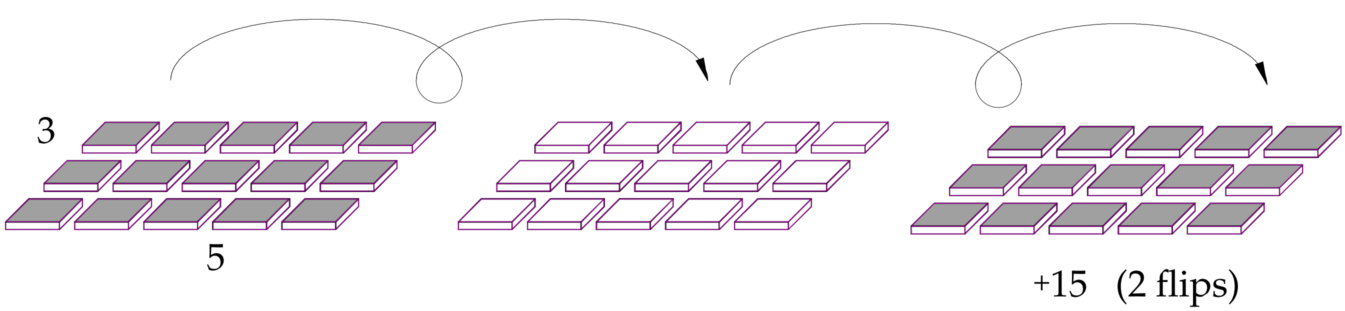
$5 \cdot (-3) = -15$

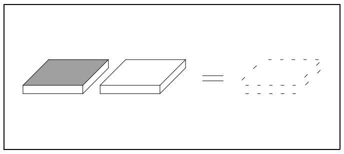


$-4 \cdot 3 = -12$



$(-3) \cdot (-5) = +15$





We can now state the procedure for multiplying:

Multiplication of Two Numbers:

Make a rectangle with one number as the length and the other as the width.

Flip all the chips once for each negative sign.

The area and the color give the result.

We can see that there is an obvious method for finding the sign of the answer in a multiplication problem:

The Sign of the Result:

If one side of the rectangle is negative and the other side is positive, the rectangle is *negative*.

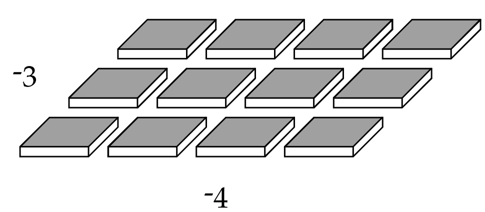
If both sides of the rectangle are positive, or both sides are negative, then the rectangle is positive.

Exercises

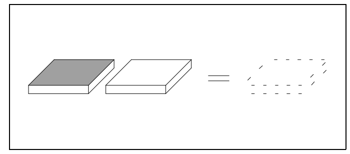
Use chips to perform the following multiplications:

Example: $(^-3)\cdot(^-4) = {}^+12$

Solution:



1. $(^{-}3)\cdot(^{+}3) = {}^{-}9$



2. $(^{-}2)\cdot(^{-}5) = {}^{+}10$

3. $(^{-}2)\cdot(^{+}5) =$

4. $(^{+}5)\cdot(^{+}3) =$

5. $(^{-}4)\cdot(^{-}3) =$

6. $(^{+}3)\cdot(^{-}1) =$

7. $(^{-}2)\cdot(^{-}2) =$

8. $(^{-}2)\cdot(^{+}2) =$

9. $4 \cdot 7 =$

10. $(^{-}4)\cdot(^{-}7) =$

11. $(^{-}4) \cdot 7 =$

12. $1 \cdot 1 =$

13. $1 \cdot (^{-}1) =$

14. $(^{-}1)\cdot(^{-}1) =$

15. $(1)\cdot(17) =$

16. $(^{-}1)\cdot(17) =$

17. $(0)\cdot(^{-}17) =$

18. $(^{-}5)\cdot(^{-}6) =$

19. ${}^{-}3\cdot(2) =$

20. $(^{-}5)\cdot(^{-}3) =$

21. ${}^{-}4\cdot3 =$

22. $2\cdot(^{-}7) =$

23. ${}^{-}2\cdot(7) =$

24. ${}^{-}2\cdot(^{-}7) =$

25. $(6)\cdot(^{-}3) =$

26. $(^{-}6)\cdot(3) =$

27. $(^{-}6)\cdot(^{-}3) =$

28. ${}^{-}1\cdot(^{-}12) =$

29. ${}^{-}3\cdot(^{-}3) =$

30. ${}^{-}5\cdot(^{+}5) =$

Section 6
Division

The Meaning of Division

Division is often described as backwards multiplication. For example, if we want to know:

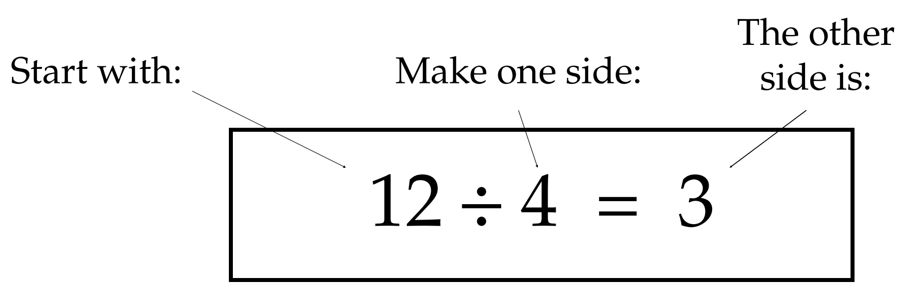
$$12 \div 4 = ?$$

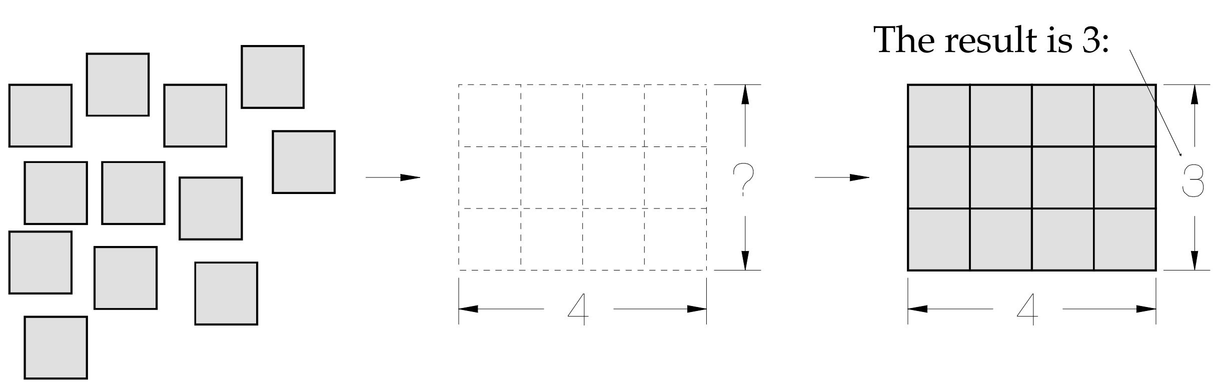
We usually think of this as:

"How many fours are in 12?"

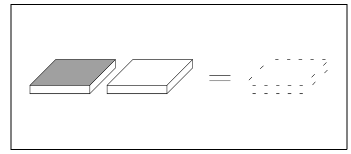
Using chips, this is also the opposite of multiplication. Since multiplication is making rectangles and counting the result, division also involves rectangles. The problem above becomes:

"Take 12 unit chips and form a rectangle with side 4.
What is the other side?"





Division with Signed Numbers



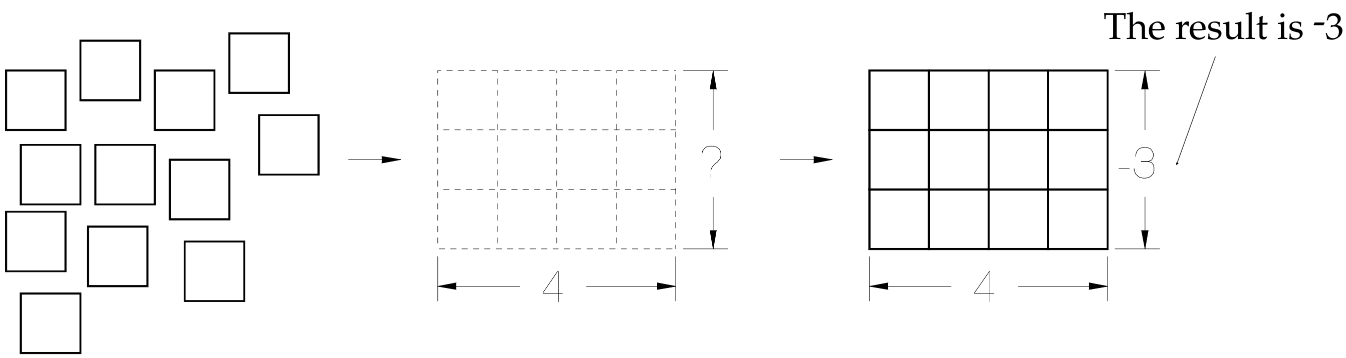
If we have a division problem with one or two negative numbers, we continue to think backwards:

$$^{-}12 \div 4 = ?$$

becomes

"What times 4 is equal to $^{-}12$?"

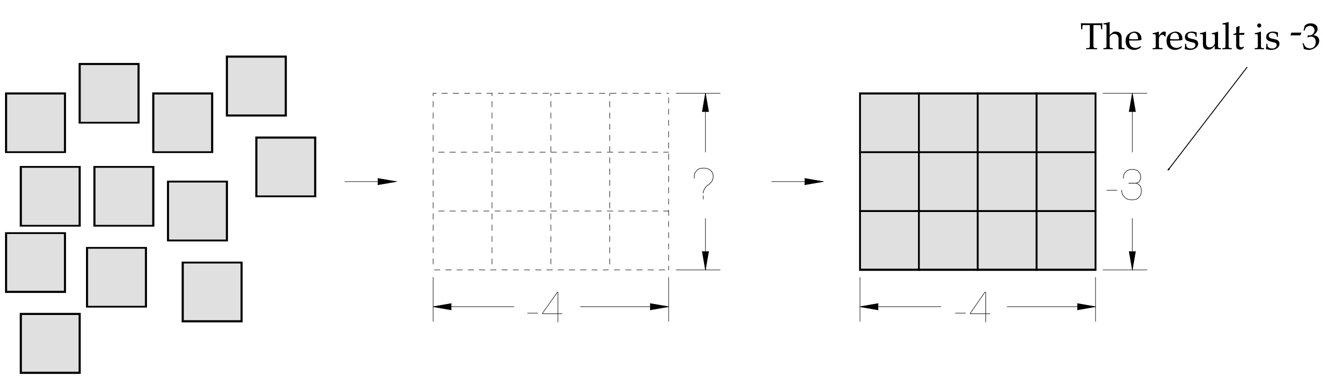
The answer is $^{-}3$ because -3 times 4 is $^{-}12$. To do this with chips, we start with $^{-}12$ unit chips and build a rectangle that is 4 on one side. The other side is 3 units. Because the answer needs to be $^{-}12$, we can see that the chips have been flipped once, so the answer—the missing side—must be negative.

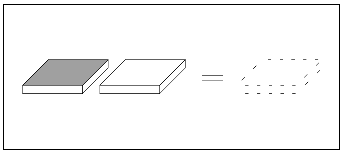


We can do other division problems in the same way. For example, what is:

$$12 \div (^{-}4)?$$

We start with 12 chips and build a rectangle with one side of $^{-}4$. The given side ($^{-}4$) is negative and acoounts for one flip. To get back to an area of $^{+}12$, we need another flip, so the other side must be negative. The answer is -3.

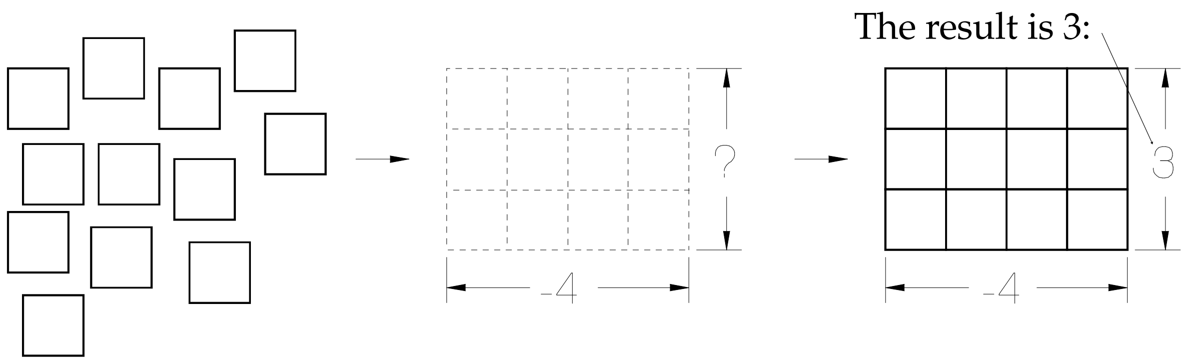




Finally, how would we illustrate:

$$^{-}12 \div (^{-}4)?$$

As we did above, we start with $^{-}12$ chips and a side of $^{-}4$ and then we can see that the other side is 3. We flip the chips once for $^{-}4$, giving the negative sign that $^{-}12$ requires, so the other side is positive 3.



Division problems in algebra are most often written as fractions; instead of writing

$$12 \div 4 = 3$$

we will commonly write

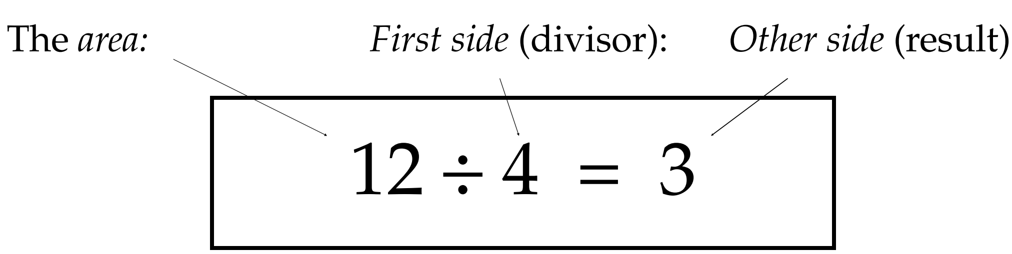
$$\frac{12}{4} = 3$$

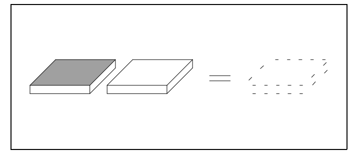
You are probably aware that we can think of fractions as division problems and we can rewrite division problems as fractions. When writing division problems as fractions, we normally will reduce all fractions and we will write "improper" fractions as mixed numbers.

For an explanation of why a division problem can be rewritten as a fraction, please see Section 3 (Compound Fractions) of the FRACTIONS *chapter.*

Summary

Division is the opposite of multiplication. Since multiplication is making rectangles, division is making rectangles in reverse:





Division:

1. **Start with unit chips (the *area*).**
2. **Build a rectangle with the divisor for the *first side*.**
3. **How long is the *other side*?**
4. **The color of the *area* and the sign of the *first side* will tell you the sign needed for the *other side* (*result*).**

Division: The Sign of the Result

1. **If the area is positive:**
   Both sides are positive,
   or both sides are negative.

2. **If the area is negative:**
   One side is negative,
   and the other side is positive.

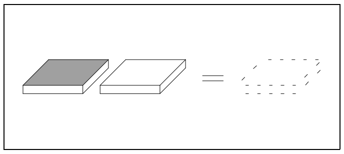
Positive	**divided by**	**Positive**	**is**	**Positive**
Positive	divided by	**Negative**	**is**	**Negative**
Negative	**divided by**	**Positive**	**is**	**Negative**
Negative	**divided by**	**Negative**	**is**	**Positive**

Exercises

Complete the following division problems using the chips:

1. $12 \div (^{-}2)$

2. $^{-}12 \div (^{+}2)$



3. $^-12 \div (^-2)$

4. $16 \div (-8)$

5. $^-16 \div (^-4)$

6. $4 \div (4)$

7. $4 \div (^-4)$

8. $^-4 \div (4)$

9. $1 \div (^-1)$

10. $^-1 \div (^-1)$

11. $0 \div 17$

12. $0 \div (^-17)$

13. $14 \div (^-7)$

14. $^-16 \div (^-2)$

15. $18 \div (^-3)$

16. $^-22 \div (^-11)$

17. $20 \div (^-5)$

18. $^-20 \div 5$

19. $^-20 \div {}^-5$

20. $^-5 \div ({}^-5)$

21. $\frac{12}{^-3}$

22. $\frac{15}{5}$

23. $\frac{^-14}{7}$

24. $\frac{^-8}{^-2}$

25. $\frac{^-20}{4}$

26. $\frac{^-20}{^-4}$

27. $\frac{^-24}{9}$

28. $\frac{^-24}{^-9}$

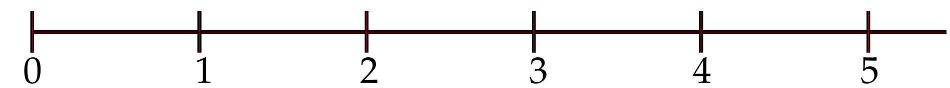
29. $\frac{9}{6}$

30. $\frac{^-12}{5}$

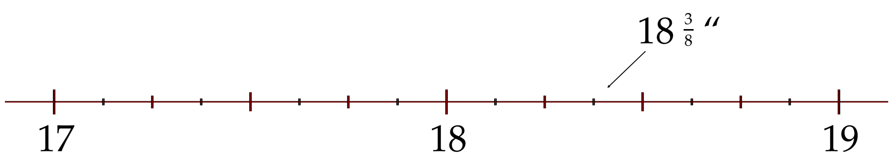
Section 7
The Number Line

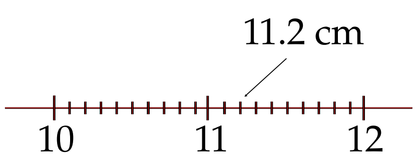
Numbers as Distance

A number line is a useful method of representing positive and negative numbers and their relationships. A number line is similar to a measuring tape; distances from the end of the tape (zero) are marked out in equal divisions along the tape. (Most measuring tapes use units of inches or centimeters.)



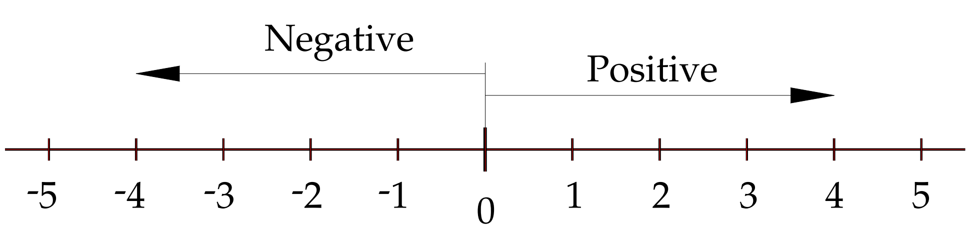
The farther you move along the tape the higher the numbers get. Between the whole numbers units are parts of units, marked off in fractions or decimals.

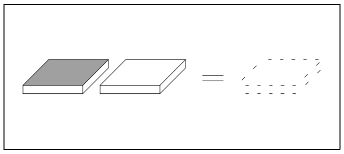




Even between the closest marks on the measuring tape, we know that *any* small fraction or decimal part of a unit could be represented if we used a magnifying glass or a micrometer. In these ways a number line is again just like a measuring tape.

A number line is different from a tape measure in that the number line marks off both *positive* and *negative* distances from zero by defining one direction as positive and the opposite direction as negative, with zero in the middle.





Generally, distances to the right of zero along the number line are called positive, and distances to the left of zero are called negative. Notice from the picture that the large, more positive numbers lie farther to the right, and the more negative numbers lie farther to the left. Since negative numbers are like being *below zero* or *in the hole*, we say that any number on the number line is greater than (more positive than) any number lying to its left.

8 is greater than (more positive than) 3.

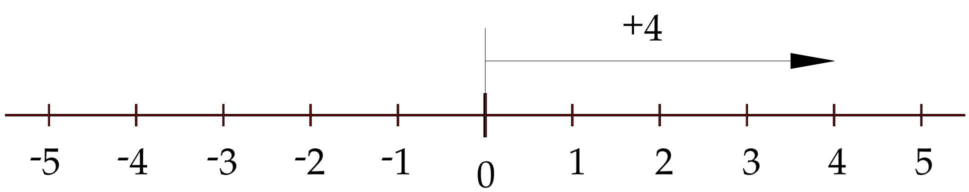
$^{-}2$ is greater than (more positive than) $^{-}5$.

A number line also differs from a measuring tape because the units on the number line don't actually represent distances like inches or centimeters. The number line is made up of what are called **pure numbers**, which don't necessarily represent any lengths or objects, but are just numerical values.

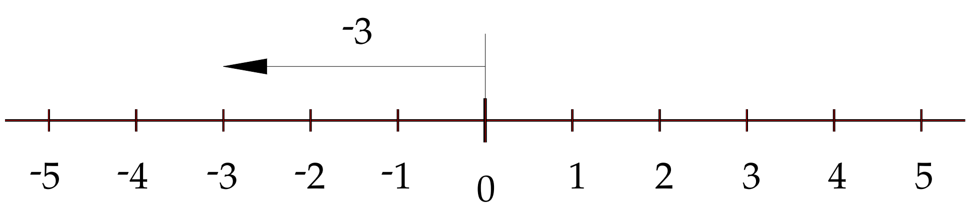
Of course numerical values might be used to represent numbers of objects, etc., but these representations are not necessary to use a number line.

Adding on a Number Line

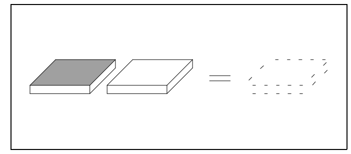
Positive numbers are represented on a number line as arrows pointing to the right and having a length showing the number of units.



Negative numbers are represented as arrows pointing to the left and also having length equaling the number of units.

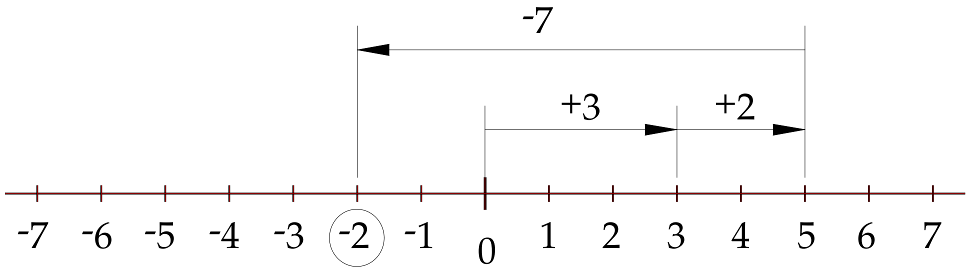


To add several numbers on the number line we represent each number as an arrow. Beginning with the tail of the first arrow at zero, we place the tail of each succeeding arrow at the tip-point of the previous arrow. The sum of the numbers is the position on the number line of the tip of the final arrow.



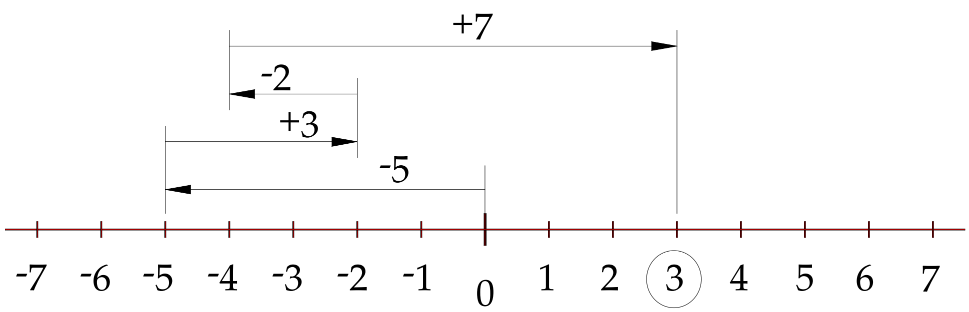
For example:

3 + 2 – 7



The sum is $^{-}2$. Another example:

$^{-}5$ + 3 – 2 + 7



The sum is $^{+}3$.

Before adding on a number line, you must simplify all double negatives to positives. The answers we get from adding on a number line will always be exactly the same as the answers we get by adding positive and negative chips; only the representation is different.

Exercises

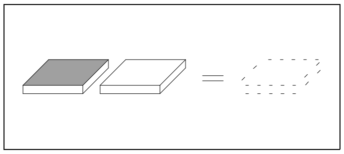
Draw number lines and arrows to complete these additions. Circle the resulting sum. (Remember, the spaces between the units on the number line must all be the same.)

1. 3 + 5 – 2

2. $^{-}2$ + 4 – 6

3. $^{-}3$ – 2 + 4

4. 2 – ($^{-}5$) – 3



Make a number line and complete the following additions by counting with your pencil point. Start with your pencil point at zero, and count steps to the right for each positive number and steps to the left for each negative number added. Get your result from the number line without drawing arrows.

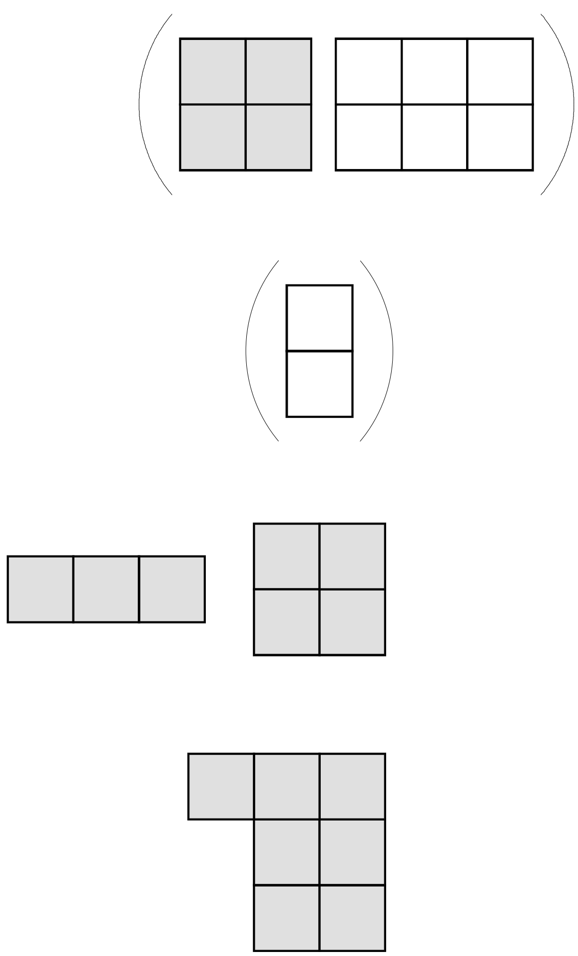
5. $2 - 5 + 3 - 1$
6. $^{-}3 - 5 + 2 + 1$
7. $7 + 1 - 5 - 3$
8. $^{-}2 + 5 - 6 + 1$
9. $4 + 3 - (^{-}2) + (^{-}5)$
10. $^{-}5 - (^{-}3) + (^{-}2) - 4$

For discussion:

11. If a tape measure is going to work, why must the separation of all the units be the same?
12. How would you multiply using a number line?

Chapter 3

Symbols and the Order of Operations



Section 1
Rules of Language

Symbols and Grammar

Algebra is a written language, and just like English or French, it has an alphabet of symbols and a set of rules. (Unlike other languages, algebra is usually written and seldom spoken). As we all know, written languages have very specific rules for things like

- **which direction to read**
- **where to start reading**
- **where to pause**
- **how to end one thought and begin a new thought**

We call these rules **grammar**. In order to write and read effectively we must all agree upon the rules of writing and reading, and upon the meaning of the symbols that we use. With this agreement, the author knows what the readers expect and the readers know what the author means to say.

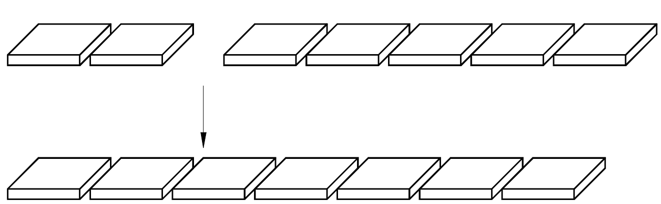
The differences between symbols can be very subtle. For example, think of the differences in meaning among these symbols in English:

, ; : .

Distinguishing Multiplication from Addition

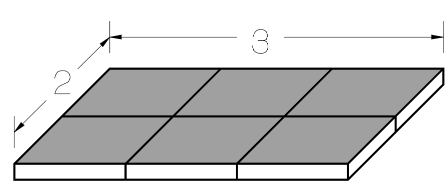
We know the difference between *adding* two numbers and *multiplying* two numbers:

Addition



${}^{-}2 + {}^{-}5 = {}^{-}7$

Multiplication



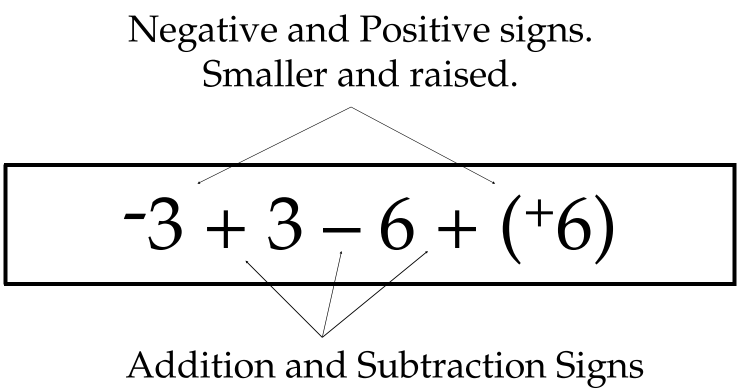
$2 \cdot 3 = 6$

In the written language of algebra, the symbols that indicate adding or multiplying can be quite confusing. There may be several different ways to write the same statement. This is especially true when using signed numbers, since they all have positive or negative signs attached to them, even when they are to be multiplied rather than added together:

$$(^{+}3)\cdot(^{-}5) + (^{-}23)$$

Positive and Negative Signs

As you may have noticed, positive and negative signs are sometimes written differently than addition and subtraction signs. While they have similar meanings, the plus and minus signs of numbers will be shown raised up and slightly smaller than addition and subtraction symbols:



Simple Addition

To indicate addition, we simply write signed numbers in a row with their signs between them, as we have already shown.

$$^{-}5 + 2 - 1$$

The signs tell which color chips to add (colored for +, white for –) and the numbers tell how many chips we have. We slide the chips together and let the different colors cancel each other out, one for one; our answer (the sum) is the number and color of chips remaining after the canceling is done.
The signs between the numbers tell us we are adding.

Notice that with signed numbers we often do not think of subtraction for negative signs. We still add, but we add white chips instead of colored chips.

{ [( )] }

Signs and Parentheses

When individual numbers are used, the positive and negative signs inside of parentheses represent the type of number—positive or negative. Exposed signs between numbers and outside of parentheses represent addition and subtraction:

Enclosed sign means positive or negative

$$^{+}5 - 3 \;=\; ^{+}5 - (^{+}3) \;=\; (^{+}5) + (^{-}3)$$

Exposed sign means add or subtract

If there are double signs on some numbers, it is the exposed signs, those not inside the parentheses, which tell us to add. You must of course carefully use the properties of double signs to know if you should add white or colored chips for each number.

Multiplication and the Dot

If a symbol is used to show multiplication, the symbol is a dot. For example, two ways to show multiplication of $^{+}5$ and $^{-}3$ are

$$(^{+}5)\cdot(^{-}3) \text{ or } ^{+}5\cdot(^{-}3)$$

The dot means multiply.

$$(^{+}5)\cdot(-3) \;=\; ^{+}5\cdot(^{-}3)$$

Dot means Multiply

Notice that even when the dot (·) is used to indicate multiplication, the plus or minus sign between the numbers is still enclosed in parentheses.

Missing Symbols: Multiplication

When two quantities are written next to each other *without* a sign between them, the meaning is multiplication:

$$({}^{+}5)({}^{-}3) = {}^{+}5({}^{-}3) = -15$$

No sign means multiply

This rule of *no sign between means multiply* works even if only one of the numbers is in parentheses.

Missing Symbols: Addition

With signed numbers we assume that a number is positive (+) unless we see a minus (–) sign. This means that when the plus (+) sign is not needed for the understanding of a number statement, it can usually be left off.

For example, in addition there must be signs (+ or –) *between* the numbers being added, but if the first number in a row is positive the plus sign can be left off that number without confusion.

If the positive number is not the first number in a row, then the sign is still necessary to show addition:

Can't leave off negative sign (–)

$$5 - 3 = {}^{-}3 + 5$$

Positive first number. Leave off + sign.

Positive number, not first. Must have + sign

{ [( )] }

When multiplying, since no sign means *multiply*, a positive number can often be written without a sign:

$$5\cdot({}^-3) = ({}^-3)(5) = {}^-15$$

The important thing to remember is that if you want numbers to be added together, then there must be an exposed sign (+ or –) *between* the numbers. If there is no exposed sign between two numbers, the expression means *multiply*. A number with no sign in front (to the immediate left) of it is understood to be positive.

Summary

- **Positive and negative signs may be shown raised and smaller than addition and subtraction signs.**
- **The dot means multiply.**
- **Enclosed signs refer to the type of number (positive or negative).**
- **Exposed plus and minus signs and signs outside of parentheses stand for addition and subtraction.**
- **No sign between numbers means multiplication.**
- **When the first number in a statement is positive, the positive sign may be omitted. Negative signs are always required.**

Exercises

Read the symbols carefully as you do these exercises:

1. $5 - 3 + 6 = 8$

2. ${}^-3({}^-4) = 12$

3. ${}^-2({}^+5) = -10$

4. $2 + 5 =$

5. ${}^-3 - 4 =$

6. ${}^-5\cdot(3) =$

7. $2 - 5 =$

8. $2({}^-5) =$

9. $7 - 5 =$

10. ${}^-7 - 5 =$

11. ${}^-7(5) =$

12. $7\cdot(5) =$

Section 2
Order of Operations

Number Statements

Number statements in algebra can be quite complex. Statements can have many numbers, some of which are multiplied while others are added together first and then multiplied. To deal with this variety, the language of algebra has a set of rules which tells us which steps to do first and which to do next.

These rules tell us in what order we do the different operations, so that we all agree on the meaning and result of the statement. In English or Spanish, an equivalent rule is that we always agree to read words from left to right, starting at the top of the page and moving down line by line. If we try to change the order of our reading, the statements don't make sense.

But in some other languages, words are read from top to bottom starting at the right edge of the page and moving to the left, column by column. In these languages, you must also follow those rules of order for the statements to make sense.

The first rule of order for algebra is that we always multiply (or divide) before we add (or subtract). When you have a choice between multiplying or adding, always multiply first and then add.

CORRECT:

$3 + 5 \cdot 6$ —— First, multiply 5 times 6

$3 + 30$ —— Then add 3

33

If you do the operations in the wrong order, you will get a different (and incorrect) result:

INCORRECT:

$3 + 5 \cdot 6$ —— Adding first

$8 \cdot 6$

48 —— A different result

{ [( )] }

You will notice that in algebra we do not always work from left to right. Wherever we find multiplications in a number statement, we do these first, and then we do the additions.

2(5) – 3(6) + 4 — Multiply first

10 – 18 + 4 — Then add

$^{-}4$ — The result

Writing each step below the one before can make the steps easier to follow. It is also helpful to line up the related numbers below each other, as illustrated above.

In summary, here is the order of operations that we have developed:

Order of Operations

1. **Multiply or divide.**
2. **Add or subtract.**
3. **Finish the operations from left to right**

Exercises

Complete the following exercises by simplifying to one number. Use the chips so you don't miss any steps. Remember, no sign between numbers means multiply.

Example:

$$^{-}3 + 4(2) - 6$$
$$^{-}3 + 8 - 6$$
$$^{-}1$$

{ [( )] }

More examples:

$$^{-}2(^{-}3) + 7(2)$$
$$^{+}6 + 14$$
$$+20$$

$$3 - 2(^{-}1) + 5(^{-}5)$$
$$3 \quad + \quad 2 \quad - 25$$
$$^{-}20$$

1. $3(^{-}4) + 5(3)$

2. $^{-}5 + 3(3) - 4(^{-}2)$

3. $5(2) - 3(^{-}2) + 7$

4. $^{-}4(^{-}2) + 6 + 3(^{-}1)$

5. $^{-}5 + 3 - 7(^{-}2) + 4$

6. $(^{-}4)(^{-}3)(^{-}2) - (2)(^{-}3)(^{-}1)$

7. $1 + 23 \cdot 2$

8. $4 - 3 \cdot 2$

9. $^{-}3 \cdot 2 + 4$

10. $1 + 2 \cdot 3 - 4 \cdot 5$

11. $12 - 4 - 3$

12. $12 \div 4 - 3$

13. $3 - 12 \div 3$

14. $12 - 4 + 3$

15. $2 \cdot 3 \cdot 4 - 5 \cdot 6$

16. $^{-}2(^{-}3) + 5(2)$

17. $^{-}2 - 3(5) + 2$

18. $^{-}3(^{-}4) - 5$

19. $^{-}3 - 4(^{-}5)$

20. $^{-}3 - 4 - 5$

21. $2(^{-}5) - 3(^{-}4)$

22. $2(^{-}5)(^{-}3) - 4$

23. $2 - 5(^{-}3)(^{-}4)$

24. $2 - 5(^{-}3) - 4$

25. $^{-}2 - 5(^{-}3) - 4$

26. $^{-}2 - 2(^{-}3)(^{-}4)$

Section 3
Parentheses ( )

Using Parentheses

Even though the first rule in our "order of operations" says that we all agree to multiply before we add, there are times when you might want to write a number statement in which the first step has to be addition, with multiplication coming later.

In the cases where a statement needs to say "add this first," the language of algebra uses special symbols called parentheses ( ). If a number statement has parentheses with some operation inside them like (3 + 5), then the parentheses ( ) are a signal which says "do this step first." For example, watch how the following number statement is simplified:

$^{-}3(2) + 5(7 - 3) - 4$	Do ( ) first
$^{-}3(2) + 5(4) - 4$	Now multiply
$^{-}6 + 20 - 4$	Now add
10	The result

First we do what is inside the ( ), then we multiply, and finally we add. So now we have three rules of order:

- **First do any operations which are inside parentheses ( ).**
- **Second, when you have a choice, multiply before you add.**
- **Finally, work the remaining operations from left to right.**

If there are both additions and multiplications inside of the parentheses, then the "multiply first" rule still holds.

{ [( )] }

For example:

3 – 2(5 + 2 · 4) — Work inside parentheses ( ) Multiply first

3 – 2(5 + 8) — Now add

3 – 2(13) — Outside ( ), multiply first

3 – 26 — Now add

-23 — The result

If the parentheses have only one number inside, there is no operation to do. It is sometimes useful to set off one number with parentheses to show multiplication or to show the effect of a positive or negative sign. Here are some examples (each is separate from the others):

$$^{-}(^{-}3)$$

$$(2)(3)$$

$$(2)(^{-}3)$$

$$5 - (6)$$

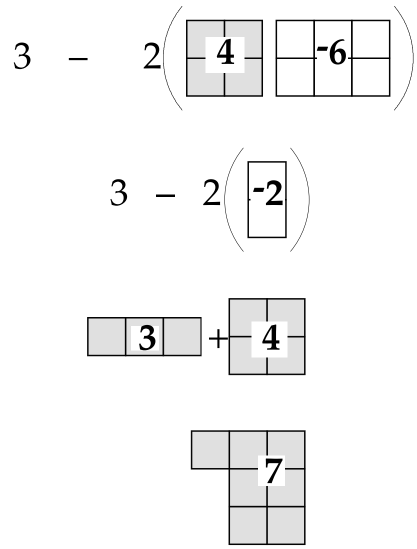
$$5 - (6)(3)$$

$$(^{-}4)(^{+}3)$$

{ [( )] }

Here is how a statement is simplified if we use the chips to do one step at a time:

$$3 - 2(4 - 3 \cdot 2)$$



There is one final order of operations rule. Sometimes, for more complex statements, it is necessary to have one set of parentheses inside of another set of parentheses. Whenever this is required, a different type of symbol is used for each pair of parentheses to avoid confusion:

First level: round **Parentheses** ( )

Second level: square **Brackets** [ ]

Third level: curly **Braces** { }

So a very complex number statement could have parentheses arranged like this:

$$\{43 + 16[9 - 6(2 + 5)] - 13\}$$

Following the rule of "do what's inside the parentheses first," it makes sense that inside the braces { } we do the bracket [ ] part first, and inside the brackets [ ] we do the parentheses ( ) first.

{ [( )] }

So our first rule becomes:

- **Do the operations in the *innermost* parentheses first, and work your way, step by step, to the outside.**

The rules for order of operations can still be written as only three rules, with both parentheses rules combined into one:

- **Do the operations in the *innermost* parentheses first. Work your way, step by step, to the outside.**
- **When you have a choice, multiply before you add.**
- **Work the remaining operations from left to right.**

Watch how these rules work together:

$3 - 2[5 + (4 - 3 \cdot 2)]$	Inner parentheses first. Multiply.
$3 - 2[5 + (4 - 6)]$	Now add
$3 - 2[5 + (^-2)]$	Now move out and add
$3 - 2[3]$	Move out and multiply
$3 - 6$	Add
$^-3$	The result!

Summary

Order of Operations

1. **Do operations in the innermost parentheses first.**
   Work your way to the outside.

2. **Inside parentheses or when there are no parentheses:**
   Multiply or divide first.
   Add or subtract next.
   Finish the operations from left to right.

{ [( )] }

Exercises

Perform the following operations. Work carefully, and *remember to do only one step at a time*, while the rest of the steps wait. Examples:

$$^-5 + 2(3\cdot 4 - 6)$$
$$^-5 + 2(12 - 6)$$
$$^-5 + 2(6)$$
$$^-5 + 12$$
$$7$$

$$3(2 - 5\cdot 2) - 2(5 + 1)$$
$$3(2 - 10) - 2(5 + 1)$$
$$3(^-8) - 2(6)$$
$$^-24 - 12$$
$$^-36$$

$$5 - 2[3 + 4(2 - 3\cdot 3)]$$
$$5 - 2[3 + 4(2 - 9)]$$
$$5 - 2[3 + 4(^-7)]$$
$$5 - 2[3 - 28]$$
$$5 - 2[^-25]$$
$$5 + 50$$
$$55$$

Simplify to give one number:

1. $^-2(4 - 2\cdot 3) + 6 - 1$

2. $(5 + 6\cdot 2) - 3\cdot 4$

3. $^-4 + 3(2 - 5\cdot 2)$

4. $2(3 - 2\cdot 4) + 3(2 + 6)$

5. $4 + 2[5 - 3(2\cdot 4 - 3)]$

6. $5 - 3 + [2 + (3 - 5\cdot 2)]$

7. $^-4\{^-3 - 2[1 + (6 - 2\cdot 3)] - 1\}$

8. $^-1 - 3\{^-1 - [2(^-4 - 3) + 2] - 1\}$

9. $(4 - 3)(2 - 4)(^-7 - 1)$

10. $(^-4 + 2)\{2 - 3[^-6 - (^-1)]\}$

Section 4
Division and Fractions

The Meaning of Fractions and Division

Most number statements in algebra do not use the sign for division (÷). Instead, division steps are usually written as fractions or ratios. For example:

$$6 \div 2 = \frac{6}{2}$$

See Section 3 of the FRACTIONS chapter for a more detailed discussion of why this is true.

As with multiplications, steps involving division or the reducing of fractions are done before steps involving addition. For example, follow through the simplification of the following number statement:

$$3 + 4 \cdot 2 - 12 \div 3 - 1$$ *First multiply and divide*

$$3 + 8 - 4 - 1$$ *Then add*

$$6$$ *The result!*

As we stated above, this number statement would usually be written with a fraction rather than with a division sign. First, simplify the fraction and do the multiplication. Next add the results and the remaining numbers:

$$3 + 4 \cdot 2 - \frac{12}{3} - 1$$ *Multiplication and fractions first*

$$3 + 8 - 4 - 1$$ *Add next*

$$6$$ *The result!*

{ [( )] }

If the top (numerator) and the bottom (denominator) of a fraction need simplification before the fraction can be reduced, then this simplification must be done first. For example:

$$\frac{-2 \cdot 9}{4 \cdot 3 - 6}$$ Multiply within fraction

$$\frac{-18}{12 - 6}$$ Now add

$$\frac{-18}{6}$$ Divide

$$-3$$

Since fractions must be simplified before they can be reduced, we can think of fractions as understood to be within their own parentheses saying "do me first." Simplifying a statement having fractions would look like this:

$$6 - \frac{2+7}{3} - 2(3-7)$$ Simplify fractions and parentheses

$$6 - \frac{9}{3} - 2(-4)$$ Divide and multiply

$$6 - 3 + 8$$ Add

$$11$$

Here is our final set of rules:

- **Do the operations in the innermost parentheses first. (Fractions have implied parentheses; they represent a division problem.) Work your way step by step to the outside.**

- **When you have a choice, multiply or divide before you add or subtract.**

- **Work the remaining operations from left to right.**

- **Do only one step at a time, leaving everything else unchanged.**

These rules are convenient agreements that we make to avoid confusion and to simplify writing number statements. This is one of the few parts of algebra that you must memorize; the rest of the properties and techniques will be easy to understand without memorization.

{ [( )] }

Review the rules as you follow these examples:

$$8\cdot(2)+\left(7-\frac{5+4}{3}\right)$$

$$8\cdot 2 \quad + \quad (7-3)$$

$$8\cdot 2 \quad + \quad (4)$$

$$16 \quad + \quad 4$$

$$20$$

$$\frac{6-2\cdot 5}{2}+3$$

$$\frac{6-10}{2}+3$$

$$\frac{-4}{2}+3$$

$$^{-}2+3$$

$$+1$$

{ [( )] }

Here is a more complicated example. Remember to work only one small step at a time, while everything else waits.

$$5 - \frac{2+4}{3 \cdot 2} - 2(2 \cdot 5 - 8)$$

$$5 - \frac{2+4}{6} - 2(10 - 8)$$

$$5 - \frac{6}{6} - 2(2)$$

$$5 - 1 - 4$$

A final example:

$$5 - 3\left[6 - 2\left(\frac{6+4}{5} - 2 \cdot 3\right)\right]$$

$$5 - 3\left[6 - 2\left(\frac{10}{5} - 2 \cdot 3\right)\right]$$

$$5 - 3[6 - 2(2 - 6)]$$

$$5 - 3[6 - 2(^-4)]$$

$$5 - 3[6 + 8]$$

$$5 - 3[14]$$

$$5 - 42$$

$$^-37$$

Exercises

{ [( )] }

Simplify:

1. $\dfrac{6+4}{2} - 3$

2. $\dfrac{5 \cdot 4}{2} + 6$

3. $6 + \dfrac{4 - 2 \cdot 7}{5}$

4. $5 - \dfrac{4 + 10}{3 \cdot 2 + 1}$

5. $3\left(2 - \dfrac{7+5}{4}\right)$

6. $4 - 3\left(\dfrac{9 \cdot 2}{3} - 8\right)$

7. $\dfrac{7 \cdot 2 + 1}{3 + 2} - (2 \cdot 3 + 1)$

8. $18 - 2\left(\dfrac{3 + 2 \cdot 6}{5} - 4\right)$

9. $3 - \left[18 - 2\left(\dfrac{3 + 2 \cdot 6}{5} - 4\right)\right]$

10. $2\left\{3 - \left[18 - 2\left(\dfrac{3 + 2 \cdot 6}{5} - 4\right)\right]\right\}$

Reviewexercises:

1. $^{-}2(3 + 5) + 1$
2. $2(3) + 5(3 - 8)$
3. $^{-}3 - 2(5 - 1)$
4. $5 - 3(2 - 9)$
5. $7 + 2[3 - (4 - 6)]$
6. $^{-}3 + 5(^{-}4 - 2(8 - 3)]$
7. $2 + [7 - 3(2 - 6) + 2]$

{ [( )] }

8. $(2 + 3)[4 - 5(3 - 1)]$

9. $(3 - 5)[^{-}3(2 - 7) + 1]$

10. $(7 + 2)[(3 - 4) - 6]$

11. $\dfrac{5 \cdot 2 - 1}{3 + 6} - (5 \cdot 3 - 1)$

12. $2\left(\dfrac{3 + 2 \cdot 6}{5} - 4 \cdot 2\right) - (0 \cdot 17)$

13. $2 + \left[{}^{-}3 - 2\left(\dfrac{2 + 2 \cdot 6}{7} + 3\right)\right]$

14. $\dfrac{3 + 2 - 4}{2 \cdot 2 + {}^{-}3} + \dfrac{3 + 2}{1 - {}^{-}4}$

15. $1\left\{1 - \left[1 - 1\left(\dfrac{1 + 1 \cdot 1}{1} - 1\right)\right]\right\}$

Section 5
Absolute Value

The Size Of A Number

The *size* of a number (independent of its sign) has a special name in the language of algebra. The size of a number is called its **absolute value.**

We indicate the absolute value of a number by putting a straight vertical line on each side of the number. Thinking of the chips, the absolute value of a group of chips is just the number of chips, independent of their color.

$$|5| = 5 \qquad |{}^{-}5| = 5$$

$$|{}^{-}3| = 3 \qquad |3| = 3$$

The absolute value of a number is *always* positive. Notice that the bars $|\ \ |$ around a number indicate an operation to be done to that number. The bars mean that we should *make the number positive and rewrite it without the bars*.

Operations Inside The Absolute Value Sign

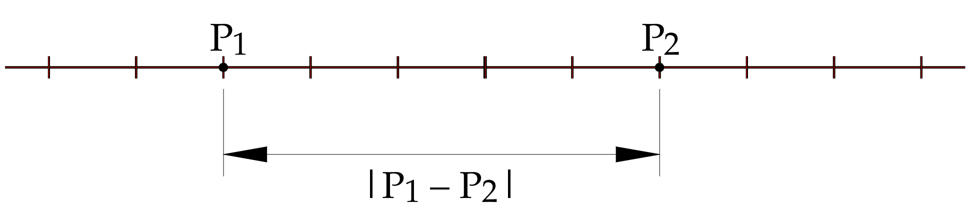
If the absolute value bars are around an expression, and the expression has several numbers and operations, perform the operations first, *before* taking the absolute value.

$$|3 - 7| = |{}^{-}4| = 4$$

$$|2{\cdot}3 - 7| = |6 - 7| = |{}^{-}1| = 1$$

When we say that the absolute value is always positive, this *doesn't* mean that we should turn all the signs inside that absolute value into plus signs. Do the operations as they are indicated, and then take the absolute value of the simplified number at the very end.

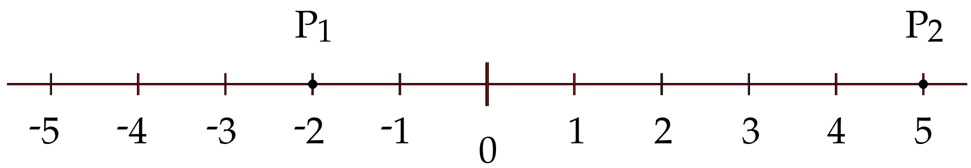
Absolute value is often used to describe the separation between two points on the number line. This separation is obviously the difference between the values of the points, but we generally want the separation to be expressed as a *positive* number. This can be indicated using the absolute value of the difference in the values of the points.



{ [( )] }

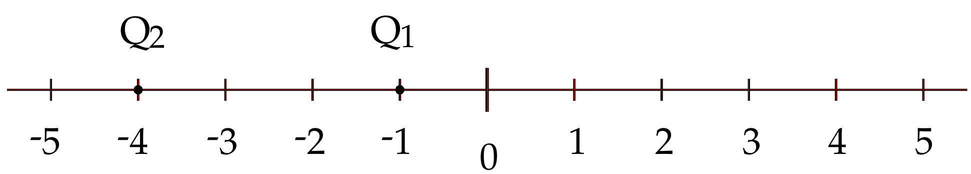
The separation between points P_1 and P_2 is $|P_1 - P_2|$. Written in this way, it does not matter which of the points has the larger value; the separation between them will always be expressed as a positive number.

Here are two other examples:



The separation between P_1 and P_2 is

$$|P_1 - P_2| = |{}^-2 - 5| = |{}^-7| = 7$$



The separation between Q_1 and Q_2 is

$$|Q_1 - Q_2| = |{}^-1 - {}^-4| = |3| = 3$$

Altitude and temperature are other quantities where a difference is often discussed as a positive number. For example, consider the statements:

> "From the mountain top to the valley floor was 8,000 vertical feet."

> "The variation in the temperature during the experiment was 52°."

In mathematical language these statements would be expressed using absolute values.

Operations Outside The Absolute Value

If an absolute value is indicated as part of a larger expression, first simplify inside the absolute value; then take the absolute value and put the result inside parentheses ( ) within the larger expression. Finally, continue to simplify the remaining expression.

{ [( )] }

For example, consider the expressions below:

$$
\begin{aligned}
5 - 2|6 - 10| &= 5 - 2|{}^{-}4| \\
&= 5 - 2(4) \\
&= 5 - 8 \\
&= {}^{-}3
\end{aligned}
$$

$$
\begin{aligned}
3 \cdot 5 - |2 - 8| &= 15 - |{}^{-}6| \\
&= 15 - (6) \\
&= 9
\end{aligned}
$$

Notice that a negative sign inside an absolute value is not directly affected by the negative sign outside the absolute value. In simplifying the absolute value, the negative sign on the outside is not used until the whole expression inside is simplified.

Absolute Value Signs Differ From Parentheses

The concept of the absolute value of a number is not difficult to understand, but the notation for absolute value can sometimes be confusing because it resembles parentheses or brackets.

Absolute value signs are different from parentheses and brackets in that absolute value signs indicate an operation to be done to the number inside. We *never multiply across* an absolute value sign; we must simplify and remove the absolute value, putting its result inside parentheses, before continuing with simplifying the rest of the expression.

For example:

$$\mathbf{3 + {}^{-}2|{}^{-}6 + 4|}$$

CORRECT	NOT CORRECT
$3 + {}^{-}2|{}^{-}6 + 4|$	$3 + {}^{-}2|{}^{-}6 + 4|$
$3 + {}^{-}2|{}^{-}2|$	$3 + {}^{-}2|{}^{-}2|$
$3 + {}^{-}2 \cdot (2)$	$3 + |({}^{-}2) \cdot ({}^{-}2)|$
$3 + ({}^{-}4)$	$3 + (4)$
-1	7

Simplify absolute value first

Wrong: Do not multiply through absolute value signs!

{ [( )] }

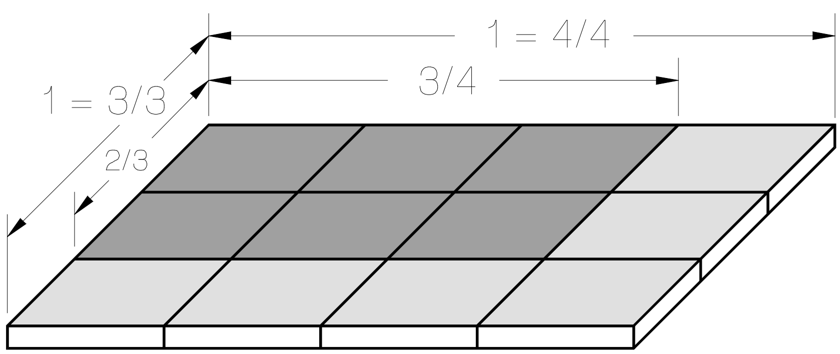
Exercises

Simplify to a single number:

1. $|{}^{-}7|$
2. $|12|$
3. $|8-3|$
4. $|2-11|$
5. $|6-9|$
6. $|10-3|$
7. $|5-3\cdot 8|$
8. $|2\cdot 4-6|$
9. ${}^{-}|7+2|$
10. ${}^{-}|{}^{-}3-6|$
11. $3|15-3|$
12. ${}^{-}2|3-11|$
13. ${}^{-}5|{}^{-}3-{}^{-}5|$
14. $3|2-14|$
15. $8-|7-5|$
16. $3+|5-9|$
17. ${}^{-}5+2|7-11|$
18. ${}^{-}3|5-9|+7$

Chapter 4

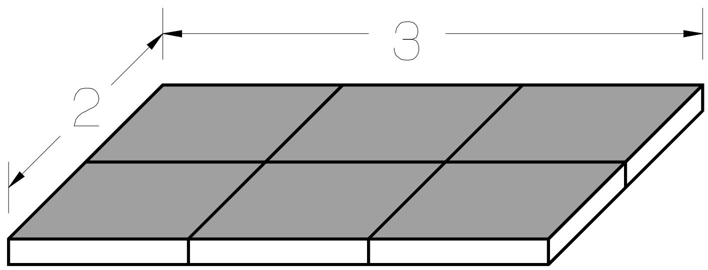
Multiplication and Division of Fractions



Section 1
Multiplication of Fractions

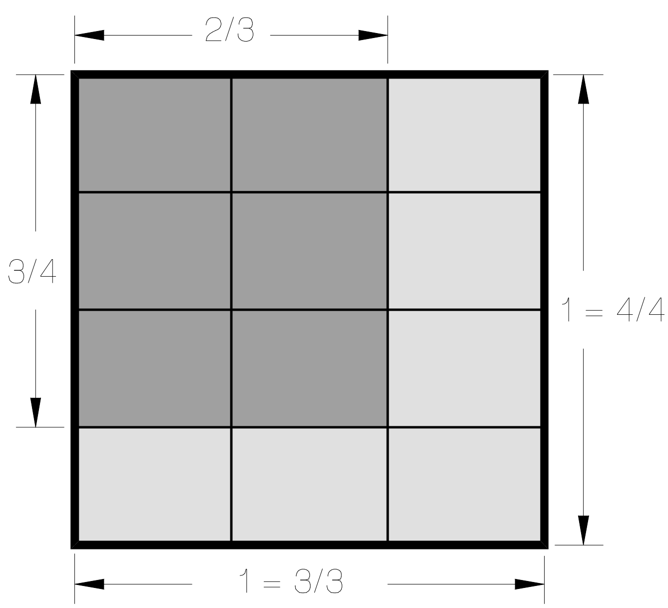
The Meaning of Multiplication

Multiplication has the same meaning with fractions as it does with positive and negative numbers. When we multiply 2 times 3, we make a rectangle that is two units wide and three units long:

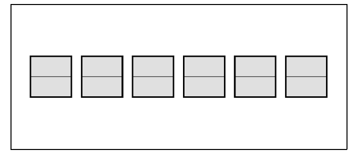


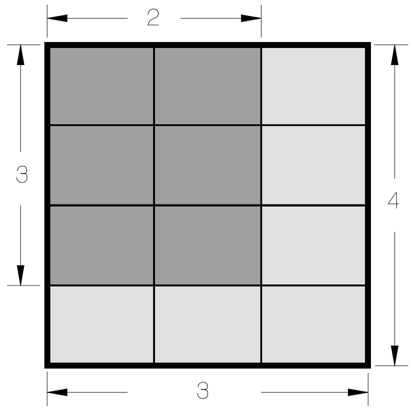
Multiplying Fractions

To multiply two fractions, we also make a rectangle. We start with a unit chip (a small square) and we cut it into smaller pieces. For example, $\frac{2}{3} \cdot \frac{3}{4}$ means:

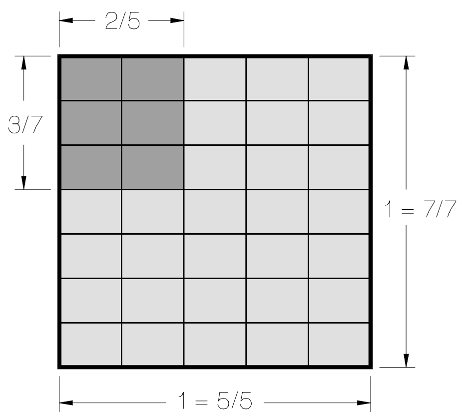


The result is a rectangle smaller than a single unit which is ⅔ units on one side and ¾ on the other side. To make this smaller rectangle, we must cut a unit square (1 by 1) into 4 parts (fourths) along one side and 3 parts (thirds) on the other side. Because there are now 12 equal pieces, and our rectangle has six (3 · 2) of them, we say that the result is 6 out of 12 or $\frac{6}{12}$ of a unit:

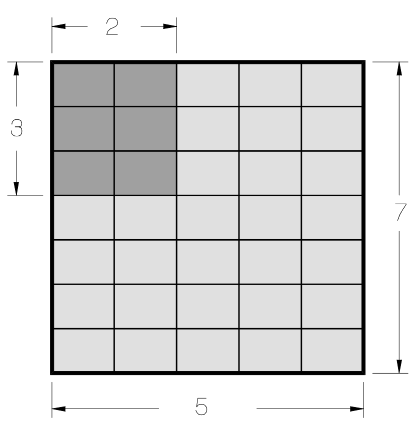




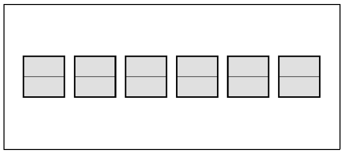
To find $\frac{3}{7} \cdot \frac{2}{5}$ we make another rectangle:



This results in a rectangle that is 6 out of 35 or $\frac{6}{35}$. We have 35 total sections because we ruled the two sides into 7 and 5 pieces and 7·5 is 35. The pattern should now be clear.

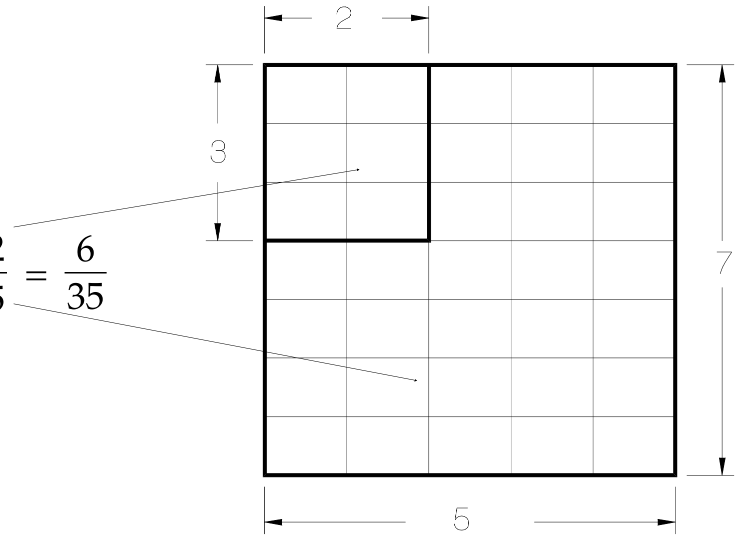


Since we chose a rectangle **three**-sevenths by **two**-fifths, we obtained a 3 by 2 rectangle with 3·2 or 6 pieces out of 35 total.



Multiplying the denominators (bottom numbers) gives us the total number of pieces. Multiplying the numerators (top numbers) gives us the number of pieces in the result. The result is the fraction of the unit square that the multiplication represents.

$$\frac{3}{7} \cdot \frac{2}{5} = \frac{3 \cdot 2}{7 \cdot 5} = \frac{6}{35}$$

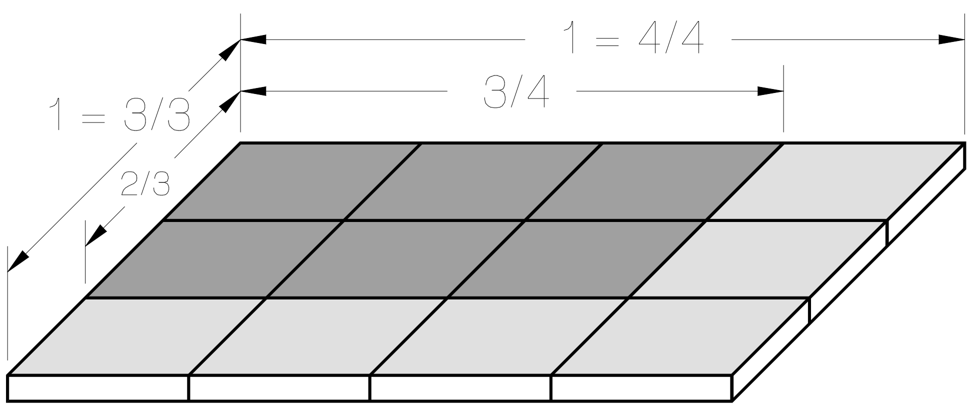


The ratio or fraction of 6 out of 35 ($^6\!/_{35}$) is the final result.

Using Chips

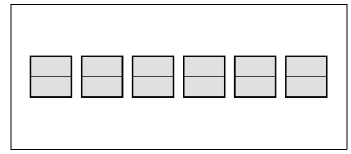
To use the chips for this process, we will have to change our system of representing fractions. Instead of a using a square for one whole unit, we will use a rectangle that is made up of small chips. If we choose the correct number of chips for each side, we will not need to draw lines or cut up the chips.

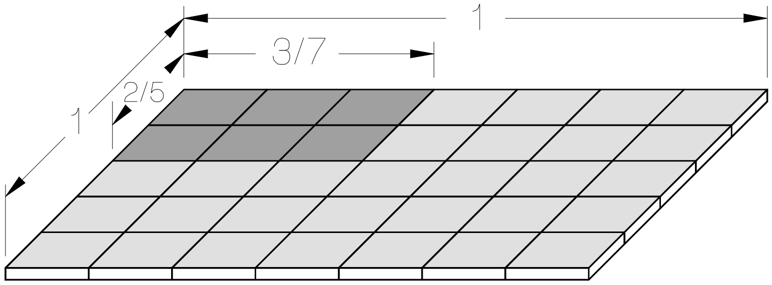
For the first example above of $^2\!/_3 \cdot\ ^3\!/_4$ we will use a 3 by 4 rectangle to represent one whole unit. This will allow us to measure thirds in one direction (where there are 3 chips) and fourths in the other direction:



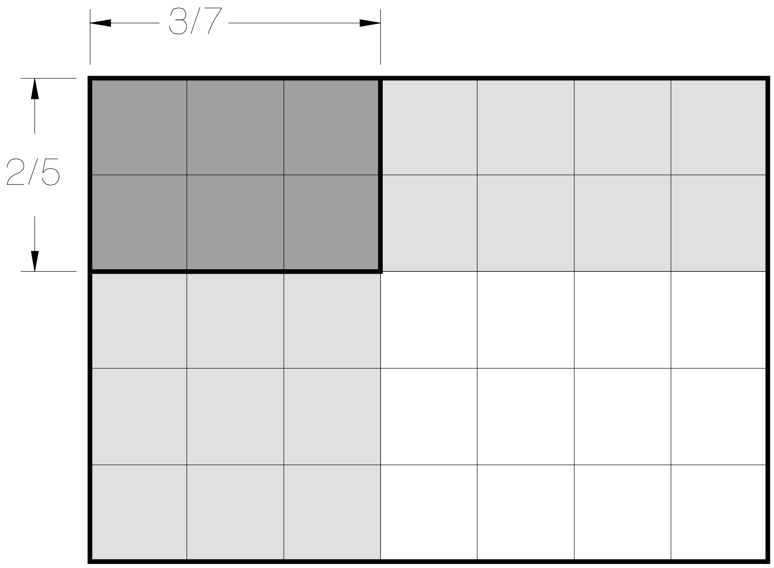
The large rectangle represents one whole. Again, we see that $^2\!/_3 \cdot {}^3\!/_4 = {}^6\!/_{12}$.

To multiply $\frac{3}{7} \cdot \frac{2}{5}$ with the chips, we lay out a 7 by 5 rectangle and then measure three-sevenths and two-fifths:





You may also have realized that $\frac{3}{7}$ times $\frac{2}{5}$ can mean three-sevenths *of* two-fifths.



To summarize this property with symbols:

Multiplying fractions

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}$$

Exercises

Multiply the two fractions. Use chips or pictures to illustrate each problem.

1. $\frac{1}{2} \cdot \frac{1}{4}$ (Use a 2 by 4 rectangle)

2. $\frac{1}{3} \cdot \frac{2}{3}$ (Use a 3 by 3 rectangle)

3. $\frac{2}{3} \cdot \frac{3}{5}$

4. $\frac{3}{4} \cdot \frac{4}{5}$

5. $\frac{4}{5} \cdot \frac{4}{5}$

6. $\frac{5}{8} \cdot \frac{2}{3}$

7. $\frac{5}{9} \cdot \frac{3}{4}$

8. $\frac{1}{3} \cdot \frac{3}{4}$

9. $\frac{5}{8} \cdot \frac{3}{4}$

10. $\frac{5}{7} \cdot \frac{4}{5}$

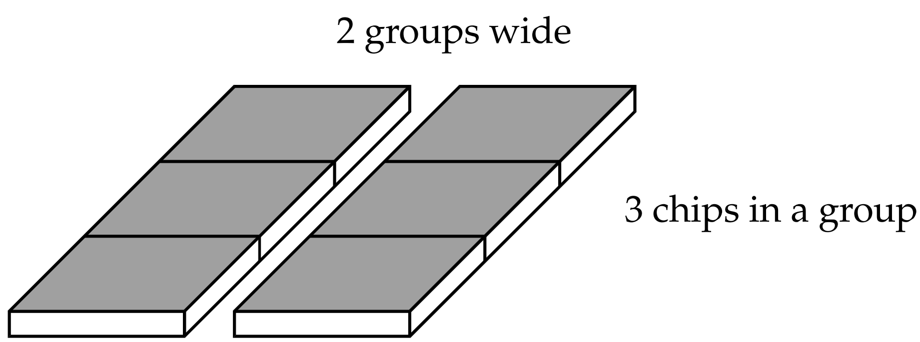
Section 2
Division of Fractions

The Meaning of Division

With whole numbers, 6 ÷ 3 has meant

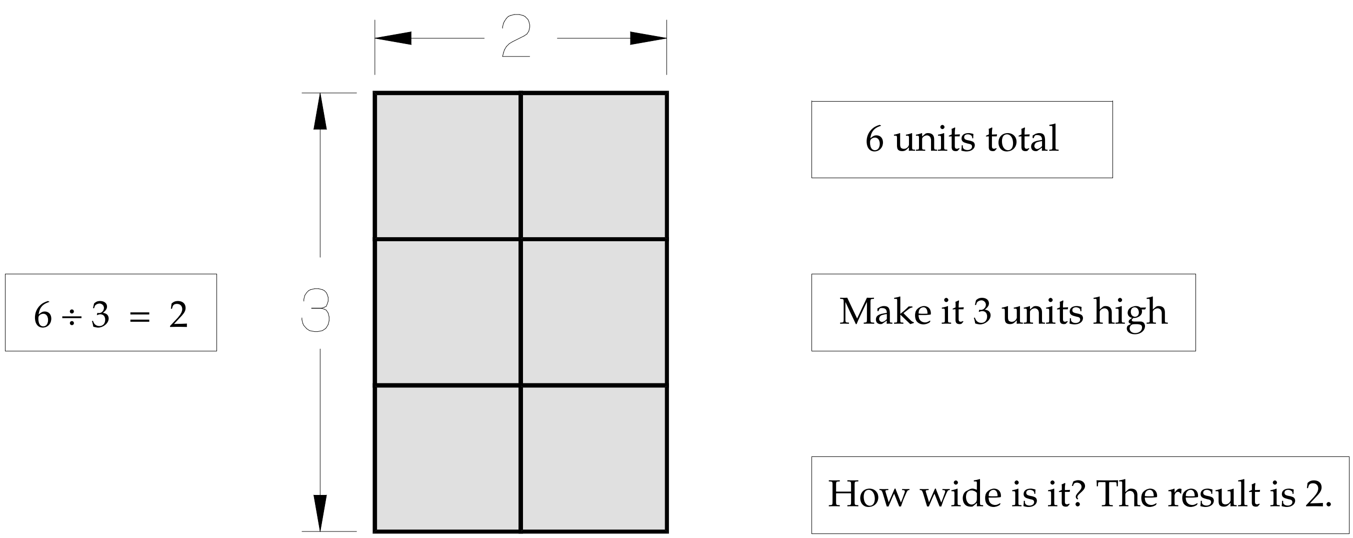
"How many threes in six?"

This means that we get out six chips and make groups of three; the result is 2 because there are 2 groups.

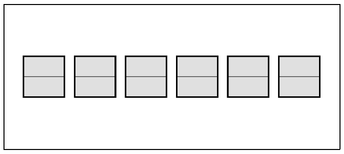


Another way to state the problem is to ask:

"If we put 6 into a rectangle that is 3 units high, how wide will it be?"



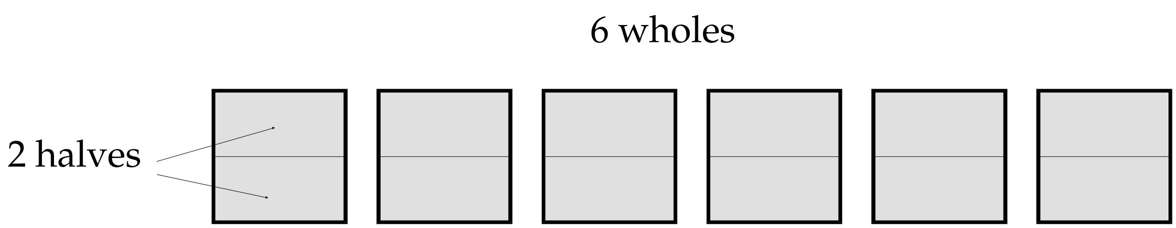
Dividing Fractions



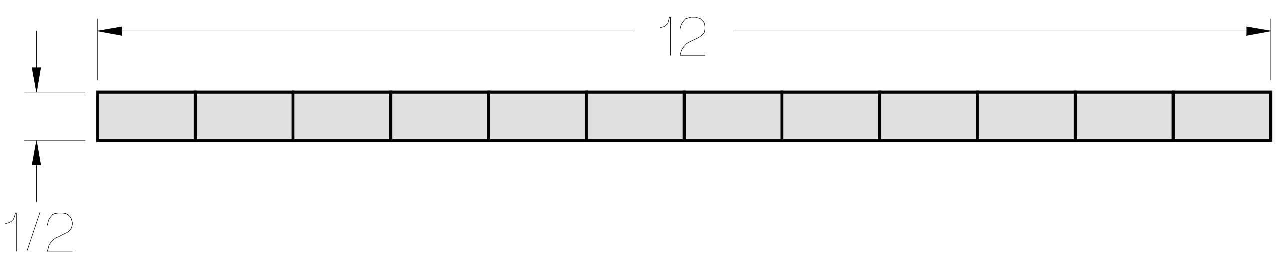
Division problems with fractions have the same meaning as the familiar examples above:

$6 \div \frac{1}{2}$ means "How many halves in six?"

We take six wholes and count the number of halves. There are 12 halves, so the result is 12:



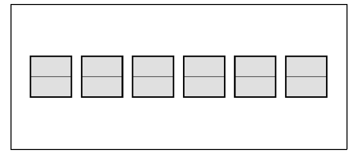
Again, we can look at the problem as a question of using 6 units to construct a rectangle that is ½ unit high. The result is the width, or 12:



1. Take 6 units.
2. Make a rectangle that is ½ unit high.
3. How wide is it? This is the result.

We could count each half, but it would be faster to notice that each whole has 2 halves and that there are 6 groups of 2 or $6 \cdot 2 = 12$ in all. This suggests that

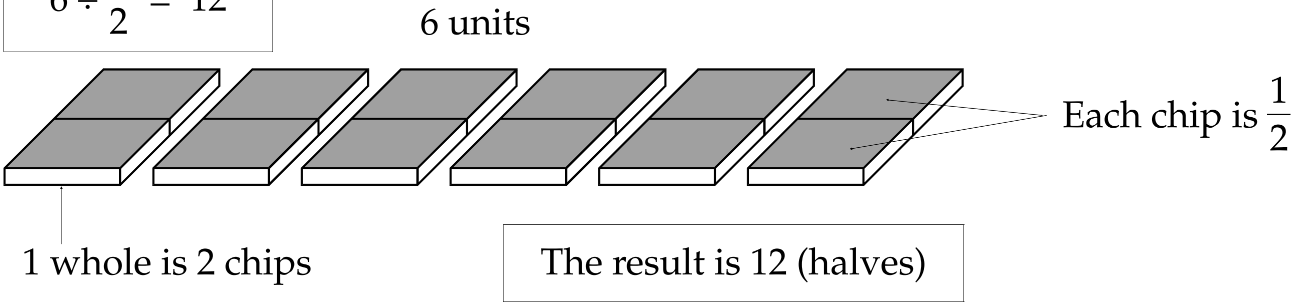
$$6 \div \frac{1}{2} = 6 \cdot 2 = 12$$



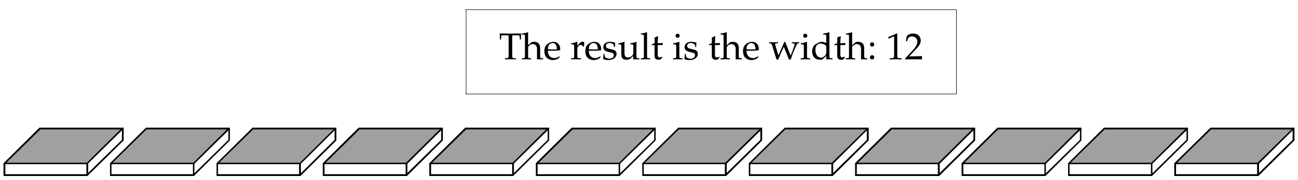
Using Chips

To use the chips, we do the same thing that we did for multiplication—we change the size of a whole so that we do not have to cut up the chips. In this case, because we need 2 halves in each whole, we use 2 *chips* for each whole:

$$6 \div \frac{1}{2} = 12$$

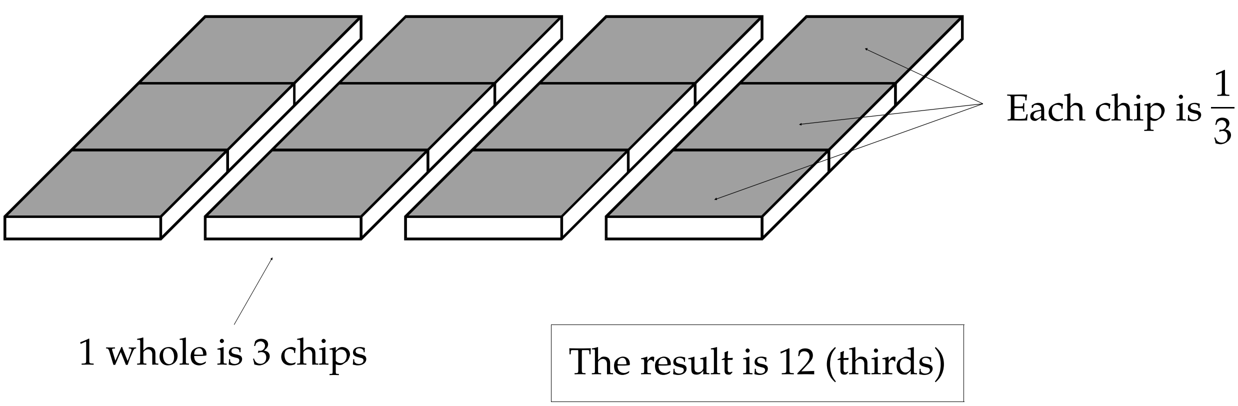


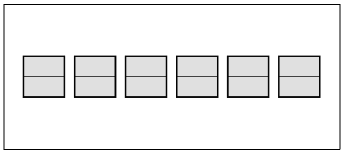
If we arrange the chips in a rectangle that is ½ (1 chip) high, the result is a width of 12:



Here is a second example using the chips: How many thirds in 4?

$$4 \div \frac{1}{3} = 12$$



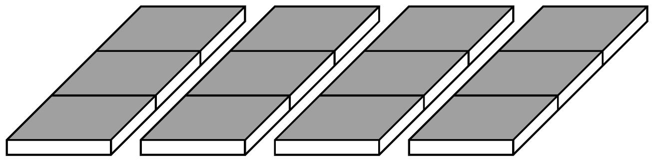


Division: Method I

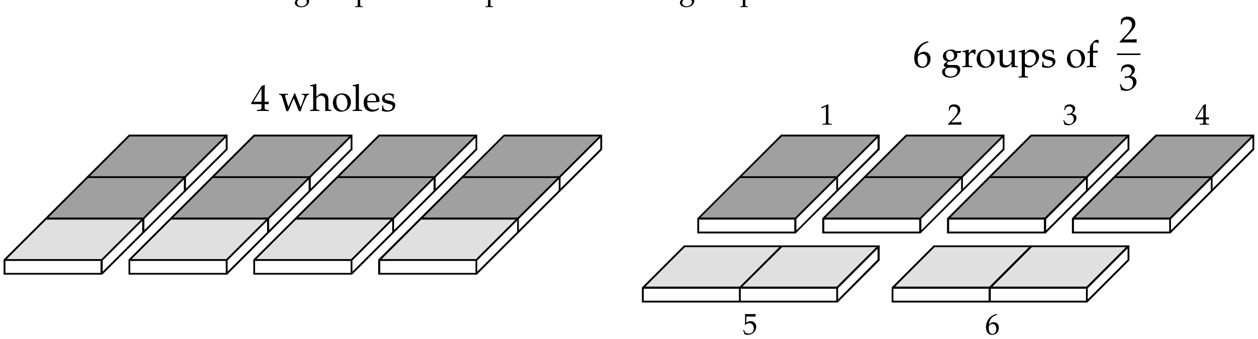
There are two different ways of visualizing the division of more complicated fractions. Consider:

$$4 \div \frac{2}{3}$$

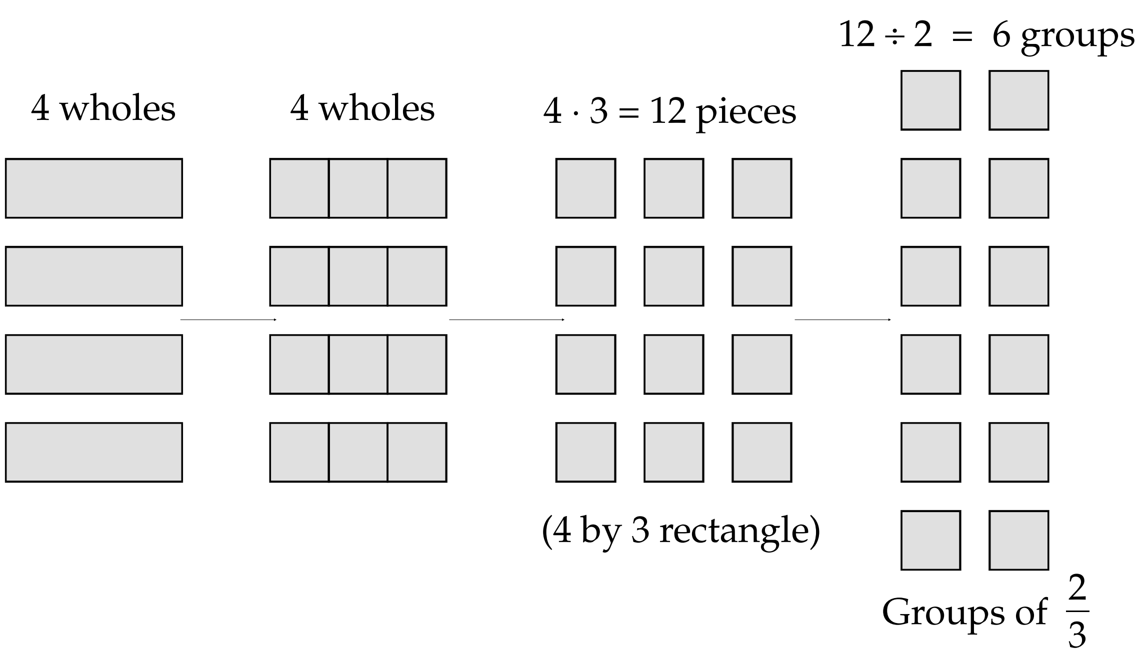
This means we should count how many groups of $\frac{2}{3}$ are in 4, so we set up 4 wholes, each made of 3 chips:



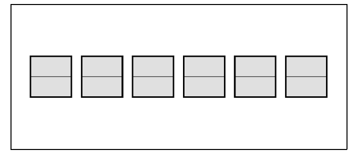
There are really 12 chips, with each chip representing $\frac{1}{3}$, so groups of $\frac{2}{3}$ are groups of 2 chips. There are 6 groups in all:



When making groups of two chips, be sure to use all the chips—separate pieces from different wholes are joined to be one group. Here is a summary of the process:



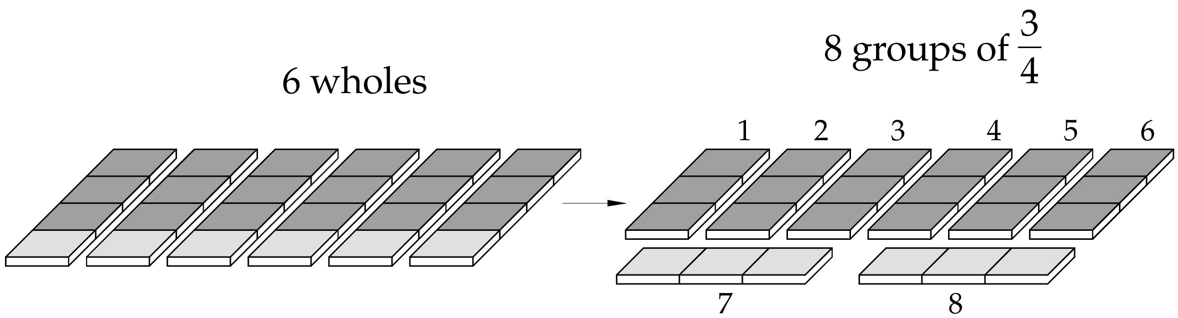
We multiply the first number (4) by the denominator of the fraction (3) to find out how many pieces we have (12). Then we divide by the numerator (2) to find out how many groups we have (6).



Consider another example:

$$6 \div \frac{3}{4}$$

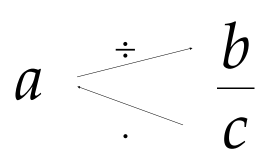
We make each whole from 4 chips, where each chip is ¼. This gives us a total of 6·4 or 24 chips. Since we want groups of ¾ (3 chips), we divide 24 by 3 to get 8:



Since we actually multiply by the *denominator* of the fraction and then divide by the *numerator* we write the technique this way:

Method I: Dividing by a fraction

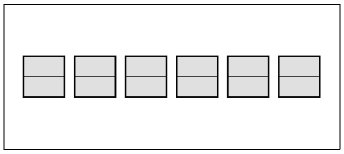
$$a \div \frac{b}{c} = (a \cdot c) \div b = \frac{ac}{b}$$



Division: Method II

There is a second way to visualize the division of fractions. In our first examples, we looked at problems like

$$6 \div \frac{1}{2}$$

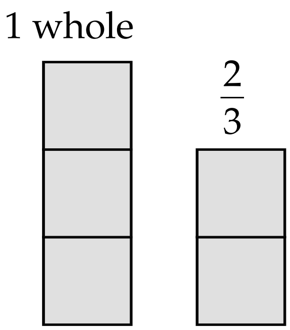


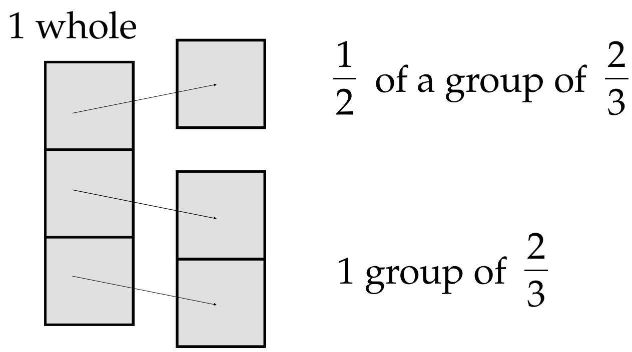
We noticed that there were 2 halves in each whole and then multiplied 6 times 2.

For the problem

$$4 \div \frac{2}{3}$$

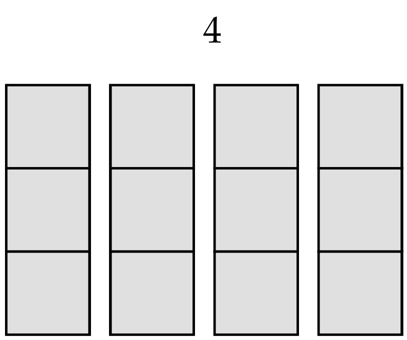
we could find out how many groups of $\frac{2}{3}$ are in each whole, and then multiply that times 4:

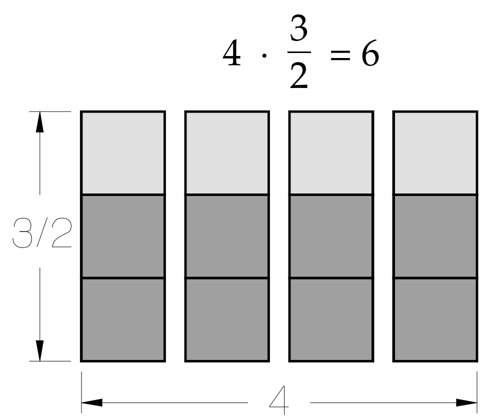




We can see that there are 1 ½ or $\frac{3}{2}$ groups in each whole. *Notice that we are looking at 1 ½ groups of $\frac{2}{3}$, not at simply 1 ½ wholes.*

Now we can count 4 groups of 1 ½, or 4 times 1 ½:





It is not an accident that $\frac{3}{2}$ looks like $\frac{2}{3}$ "flipped over." When we ask

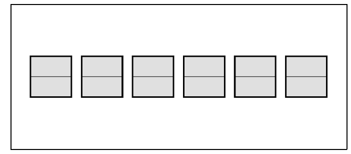
"How many groups of $\frac{2}{3}$ are in 1 whole?"

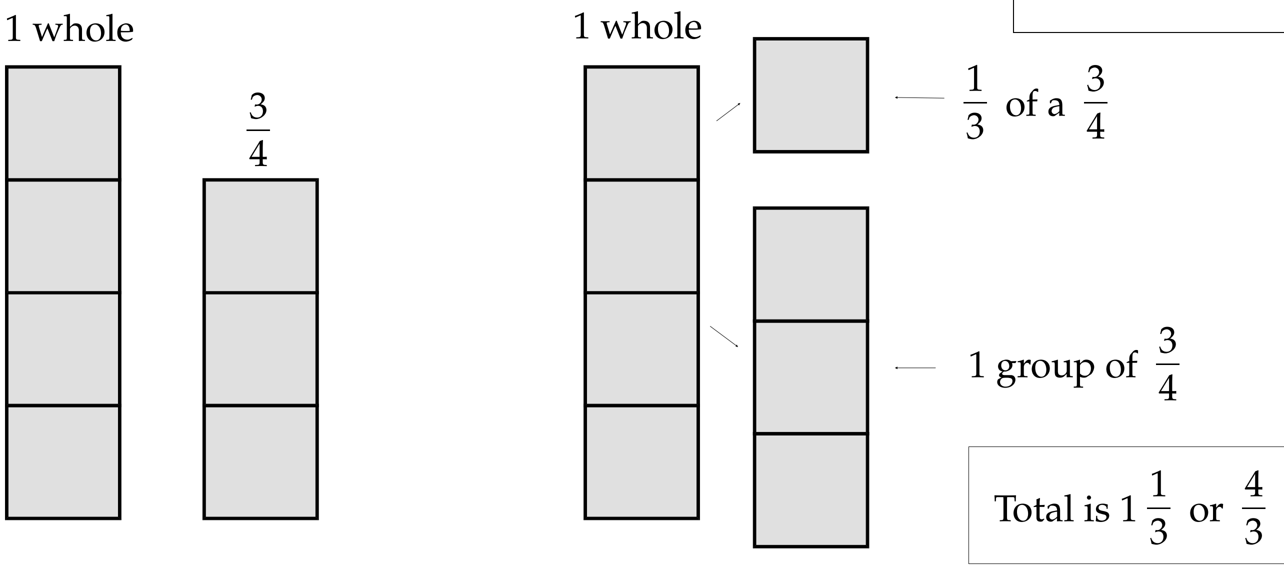
We are asking

"What times $\frac{2}{3}$ is 1?"

The answer is $\frac{3}{2}$ because $\frac{3}{2} \cdot \frac{2}{3} = \frac{6}{6} = 1$.

Here is an illustration of finding how many groups of $\frac{3}{4}$ can be made out of one whole. The result is the **reciprocal** of $\frac{3}{4}$ or $\frac{4}{3}$. There are $1\frac{1}{3}$ or $\frac{4}{3}$ groups:

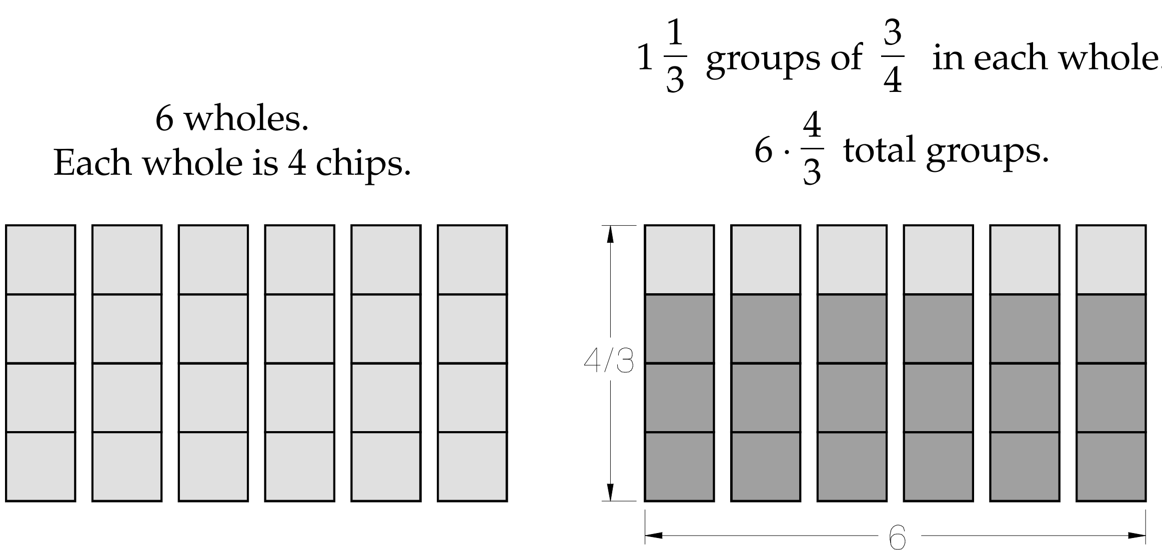




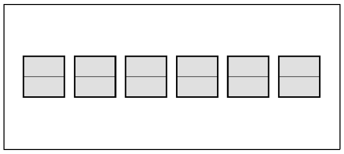
Now we can do the same problem that we did in the previous topic:

$$6 \div \frac{3}{4}$$

There are $\frac{4}{3}$ groups of $\frac{3}{4}$ in every whole, and we have 6 wholes, so we have a total of 6 times $\frac{4}{3}$ or 8 groups as a result:



$$6 \div \frac{3}{4} = 6 \cdot \frac{4}{3} = \frac{6}{1} \cdot \frac{4}{3} = \frac{6 \cdot 4}{1 \cdot 3} = \frac{24}{3} = 8$$



Summary

To divide by a fraction, we have two methods. With Method I, we multiply by the denominator and divide by the numerator of the divisor. With Method II, we multiply by the reciprocal of the divisor. Both methods accomplish the same thing, but they are slightly different ways of visualizing the same process.

Method I

$$a \div \frac{b}{c} = (a \cdot c) \div b = \frac{ac}{b}$$

Method II

$$a \div \frac{b}{c} = a \cdot \frac{c}{b} = \frac{a}{1} \cdot \frac{c}{b} = \frac{a \cdot c}{1 \cdot b} = \frac{ac}{b}$$

Note: We can use the chips to illustrate more complex situations such as the division of two fractions or two mixed numbers. See the APPENDIX for more information.

Exercises

Use your chips to illustrate the solution to these problems. Try both methods.

1. $2 \div \frac{1}{2}$

2. $2 \div \frac{2}{3}$

3. $5 \div \frac{5}{6}$

4. $6 \div \frac{2}{3}$

5. $3 \div \frac{2}{3}$

6. $4 \div \frac{2}{5}$

7. $1 \div \frac{2}{3}$

Section 3
Compound Fractions

Division and the Meaning of Fractions

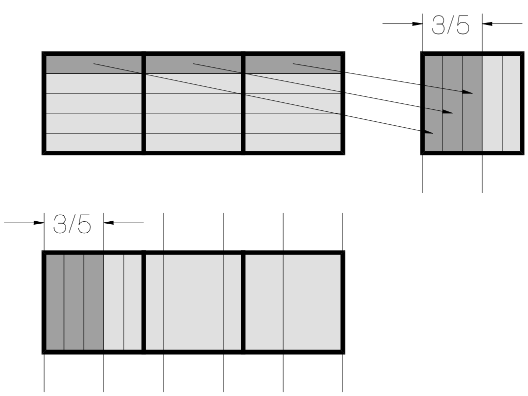
A fraction can be thought of as a division problem. For example:

$$\frac{3}{5} \text{ means } 3 \div 5$$

It is not necessary to take this for granted. If we look at the meaning of the division $3 \div 5$, we will see that the fraction $\frac{3}{5}$ represents an equivalent amount. We have thought of division in several different ways:

- **1. Divide 3 into 5 equal pieces. How large is each piece?**
- **2. How many 5's in 3?**
- **3. Arrange 3 units in a rectangle with a width of 5. How high is the rectangle?**

The first case:

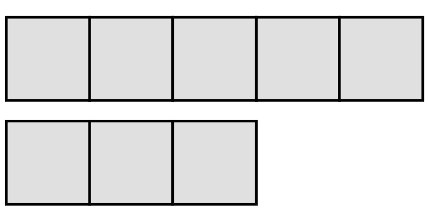


Start with 3 units.
Divide into 5 equal pieces.
Each strip is $\frac{3}{5}$.

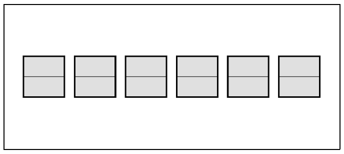
OR

Divide 3 into five equal sections.
Each section is $\frac{3}{5}$.

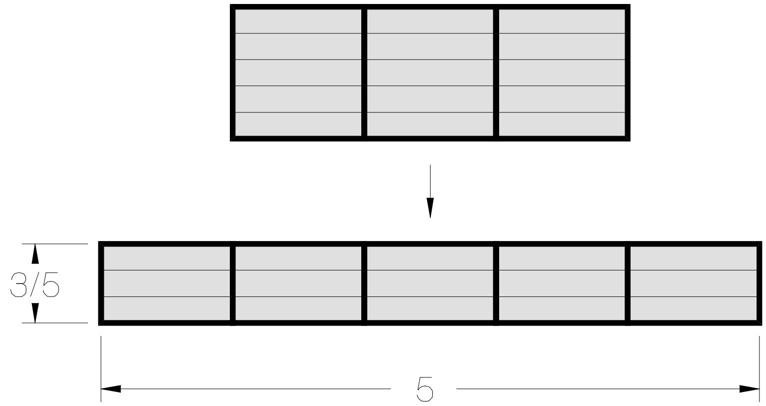
The second case:



How many 5's can we make from 3?
3 is $\frac{3}{5}$ of a 5. The result is $\frac{3}{5}$.



The third case:



Take 3 units. Divide into fifths. Build a rectangle of width 5. How high is it? The result is ³⁄₅.

To summarize:

Fractions and Division

$$\frac{a}{b} = a \div b$$

Compound Fractions

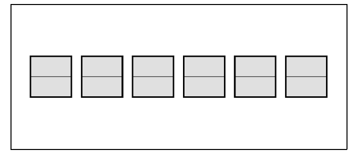
We can use this idea of division to simplify more complex fractions that contain fractions. A fraction containing other fractions is called a **compound fraction**. *To simplify compound fractions, think of the larger fraction as a division problem. Then multiply the first number by the reciprocal of the second.* For example:

$$\frac{\frac{1}{2}}{\frac{3}{4}} = \frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \cdot \frac{4}{3} = \frac{1 \cdot 4}{2 \cdot 3} = \frac{4}{6} = \frac{2}{3}$$

Other problems can all be done in a similar manner:

$$\frac{\frac{5}{8}}{\frac{4}{3}} = \frac{5}{8} \div \frac{4}{3} = \frac{5}{8} \cdot \frac{3}{4} = \frac{5 \cdot 3}{8 \cdot 4} = \frac{15}{32}$$

If one number is not a fraction, you may want to write it as fraction before continuing:



$$\frac{3}{\frac{1}{3}} = \frac{\frac{\mathbf{3}}{\mathbf{1}}}{\frac{1}{3}} = \frac{3}{1} \div \frac{1}{3} = \frac{3}{1} \cdot \frac{3}{1} = \frac{3 \cdot 3}{1 \cdot 1} = \frac{9}{1} = 9$$

With algebra symbols, we can summarize the process like this:

Compound Fractions

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Exercises

Illustrate these examples with pictures to demonstrate why the fraction and the division problem are equivalent.

1. $\frac{2}{3} = 2 \div 3$

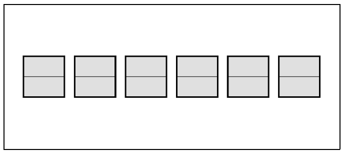
2. $\frac{5}{4} = 5 \div 4$

3. $\frac{1}{2} = 1 \div 2$

Simplify these fractions by rewriting the fraction as a division problem. Complete the division to find the answer.

4. $\dfrac{\frac{7}{16}}{\frac{3}{8}}$

5. $\dfrac{\frac{3}{8}}{\frac{7}{16}}$



6. $\dfrac{8}{\frac{3}{7}}$

7. $\dfrac{6}{\frac{3}{4}}$

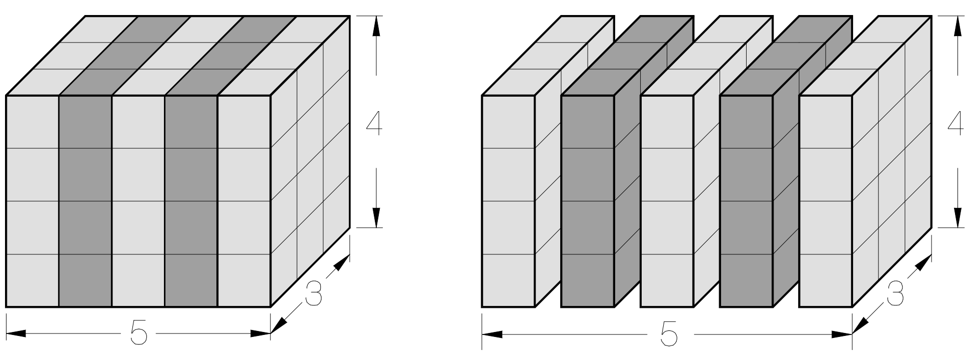
8. $\dfrac{\frac{2}{3}}{\frac{3}{4}}$

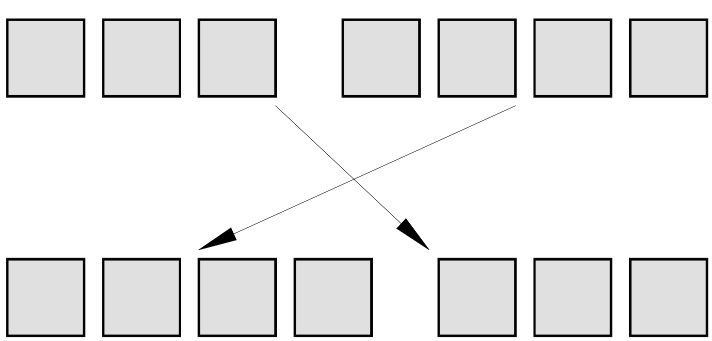
9. $\dfrac{\frac{3}{4}}{\frac{3}{4}}$

10. $\dfrac{\frac{3}{8}}{\frac{3}{4}}$

Chapter 5

Properties





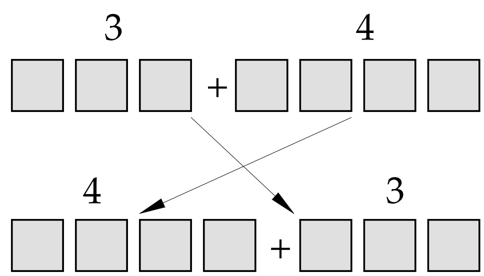
Section 1
Properties of Addition and Multiplication

Commutative Property of Addition

We know that

$$3 + 4 = 4 + 3$$

because



In symbols:

Commutative Property of Addition

$$\mathbf{3 + 4 = 4 + 3}$$
or
$$\boldsymbol{a + b = b + a}$$
for any numbers $\boldsymbol{a}$ and $\boldsymbol{b}$

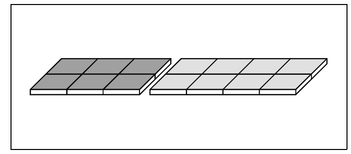
When we add two numbers, the order does not matter because we get the same total of chips in either case.

Associative Property of Addition

What is the meaning of **3 + 4 + 5** ?

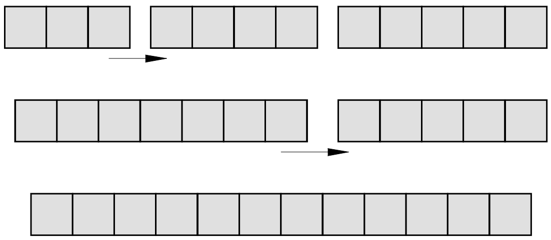
Because we think of adding only two numbers at one time, do we mean

$$(3 + 4) + 5 \quad \text{or} \quad 3 + (4 + 5)\ ?$$

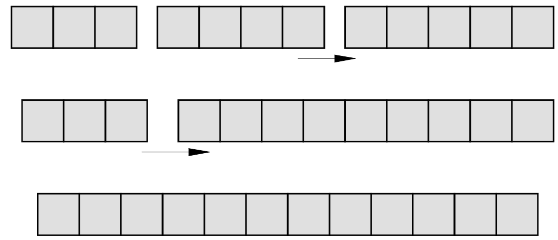


Of course, we can see that it doesn't matter because we will get the same answer in either case:

$(3 + 4) + 5$



$3 + (4 + 5)$



In summary:

Associative Property of Addition

$$(3 + 4) + 5 = 3 + (4 + 5)$$

or

$$(a + b) + c = a + (b + c)$$

for any three numbers a, b, c

Commutative Property of Multiplication

We all know that

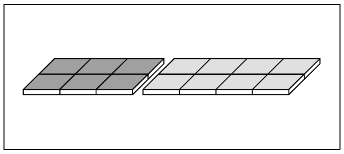
$$3 \cdot 4 = 4 \cdot 3$$

Why is this true? We can check it by doing the addition

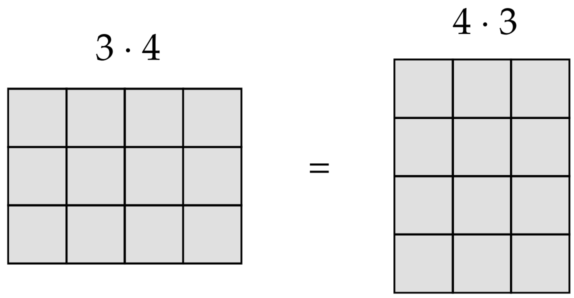
$$3 \cdot 4 = 4 + 4 + 4 = 12$$
$$4 \cdot 3 = 3 + 3 + 3 + 3 = 12$$

but this only confirms that it is true; we still don't know why.

Since multiplication is making rectangles, we can build two rectangles–one will be three by four and the other will be four by three.



Then we have:



With symbols, we have:

Commutative Property of Multiplication

$3 \cdot 4 = 4 \cdot 3$
or
$\boldsymbol{a \cdot b = b \cdot a}$
for
any two numbers $\boldsymbol{a}$ and $\boldsymbol{b}$

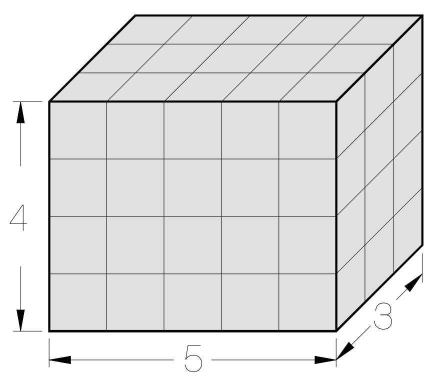
When we multiply two numbers, the order doesn't matter because we are making a rectangle that is the same size in either case.

Many properties are this easy to understand; many properties describe ideas that you already know.

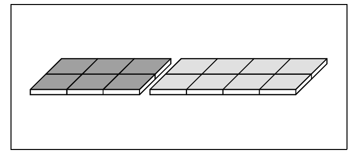
Associative Property of Multiplication

What is the meaning of $\mathbf{3 \cdot 4 \cdot 5}$?

Since we have looked at multiplication of two numbers as making a rectangle, we will look at the third number as making a three-dimensional box:

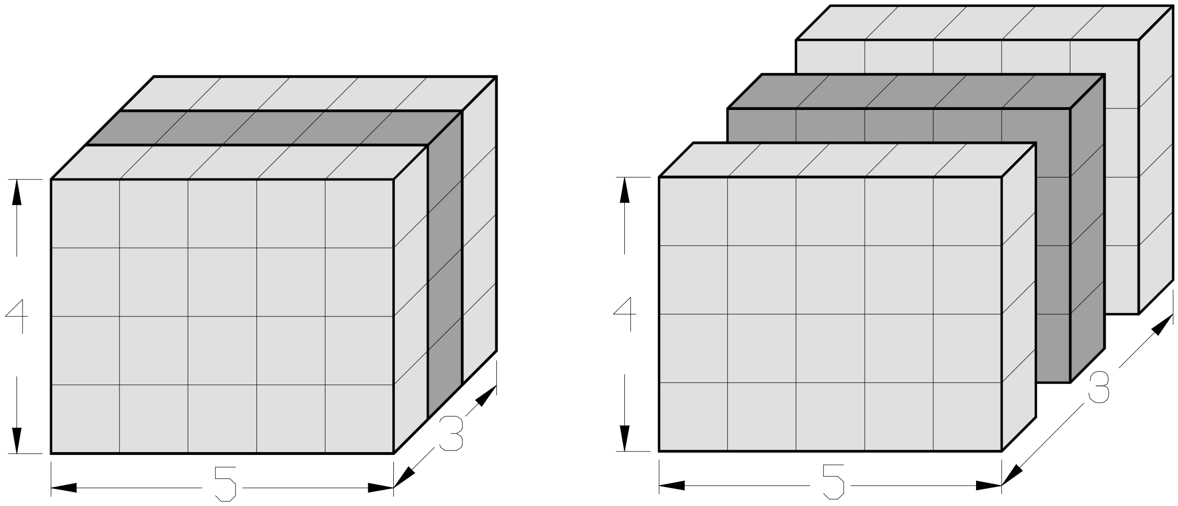


Let's look at



$$3 \cdot (4 \cdot 5)$$

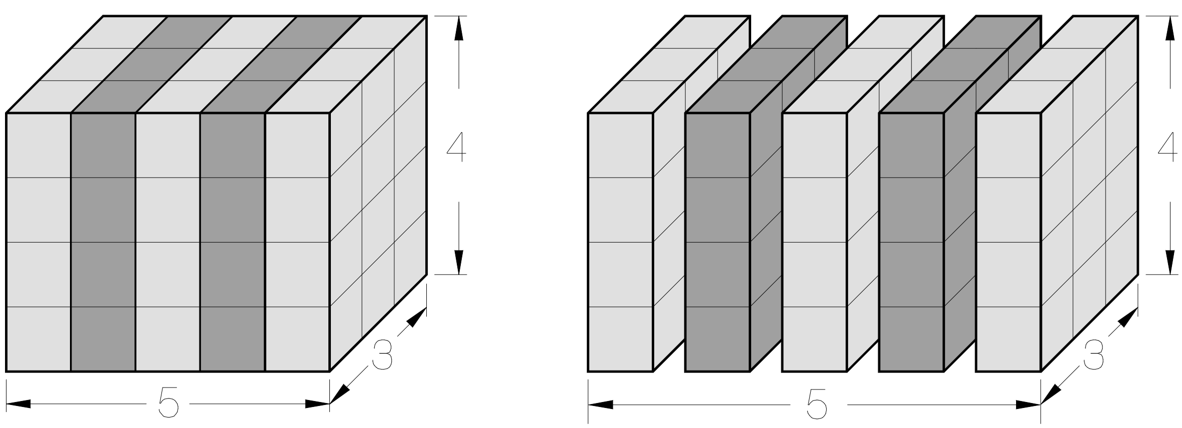
as three (3) groups or slices that are four by five $(4 \cdot 5)$:

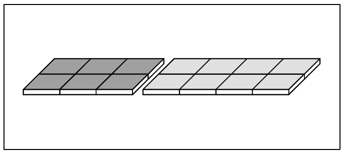


What is

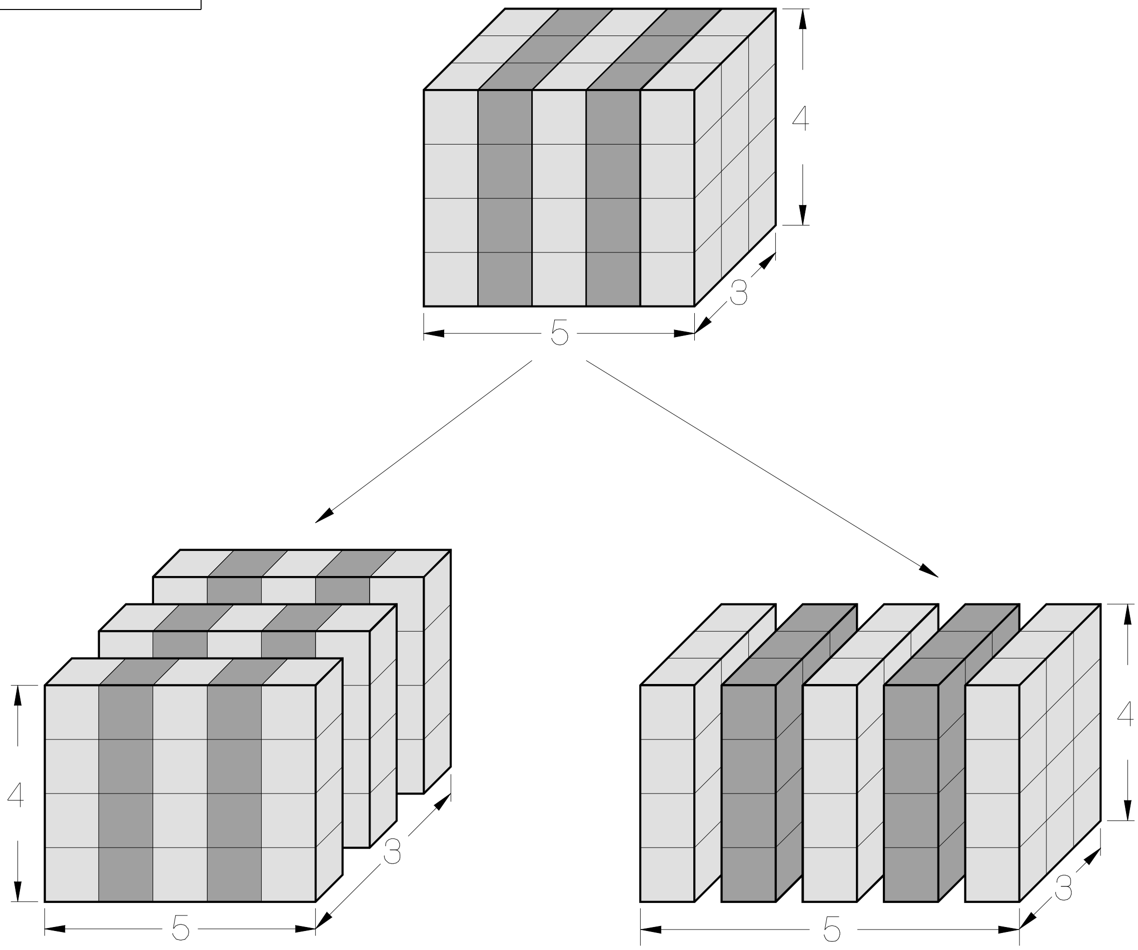
$$(3 \cdot 4) \cdot 5\ ?$$

It is five (5) slices, each three by four $(3 \cdot 4)$. We count the small cubes (60):





These two amounts are obviously equal, because we are simply counting the same number of blocks in a different order.



3 groups of (4·5)

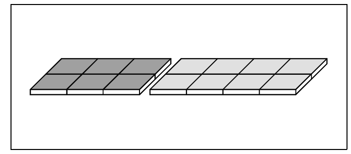
5 groups of (3·4)

Associative Property of Multiplication

$\mathbf{3 \cdot (4 \cdot 5) = (3 \cdot 4) \cdot 5}$
or
$\boldsymbol{a \cdot (b \cdot c) = (a \cdot b) \cdot c}$
for any numbers $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$

All of these commutative and associative properties are examples of the same idea: *if you count a group of chips or blocks in two different ways, you get the same answer.*

The words used for these properties—associative and distributive—have a meaning in mathematics that is similar to their meaning in everyday language:



Associative Properties describe the grouping or *association* of numbers.

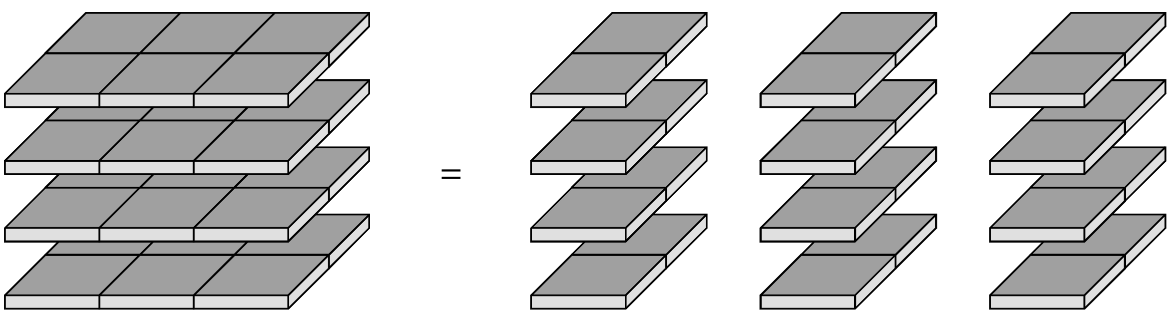
Commutative Properties describe the order or *commuting* of numbers.

Exercises

Use your chips to show why each statement is true, then identify the property (or properties) that you have used. To show the statement is true, arrange chips to represent each side of the equal sign and show why the two pictures are equal.

Example: $4 \cdot (2 \cdot 3) = (4 \cdot 2) \cdot 3$

Solution: $4 \cdot 6 = 8 \cdot 3 = 24$



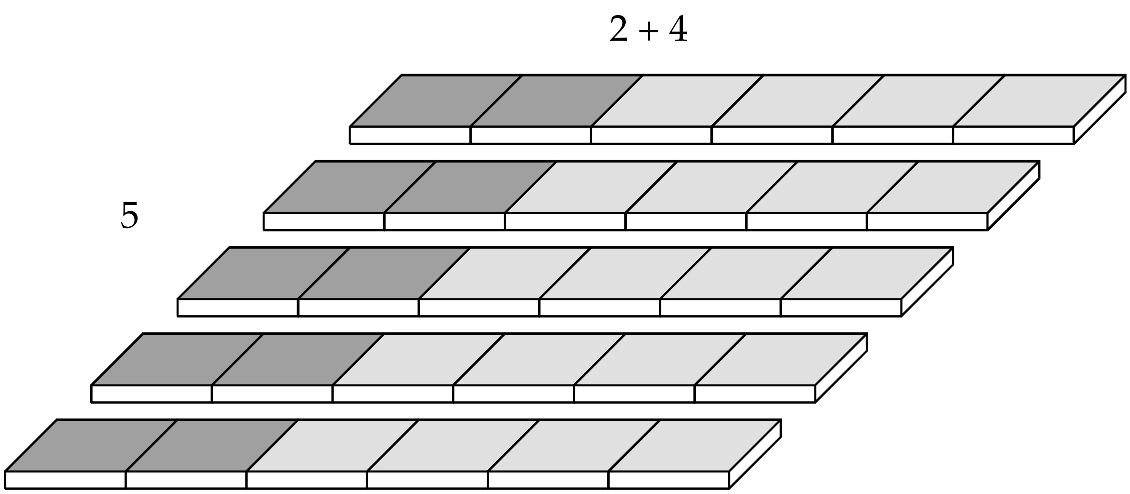
1. $7 \cdot 4 = 4 \cdot 7$
2. $1 + 2 = 2 + 1$
3. $5 + (6 + 7) = (5 + 6) + 7$
4. $1 \cdot (2 \cdot 3) = (1 \cdot 2) \cdot 3$
5. $^-2 + (^-3 + 4) = (^-2 + {}^-3) + 4$
6. $4 + (^-3 + {}^-2) = (^-2 + {}^-3) + 4$
7. $2 \cdot (3 \cdot 4) = 2 \cdot (4 \cdot 3)$

Section 2
The Distributive Property

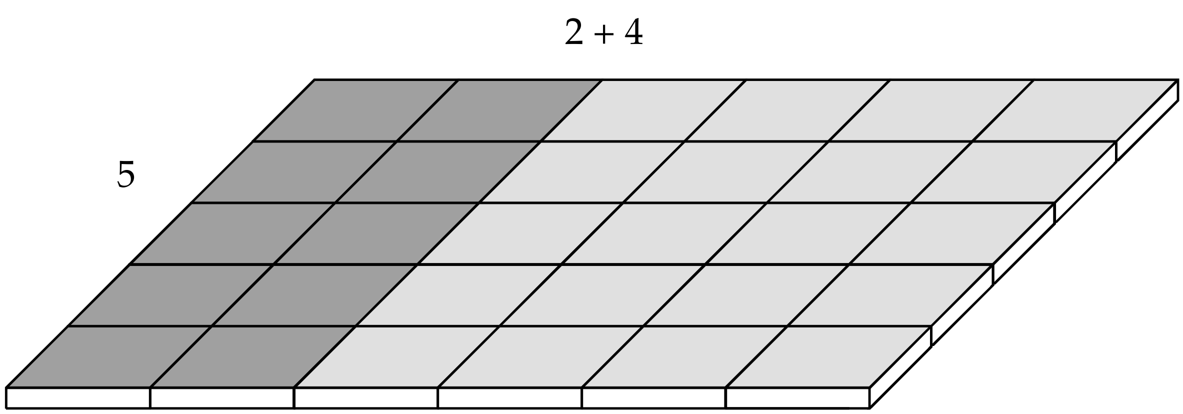
Multiplying a Number Times a Sum

When we multiply a number times a sum (the addition of two other numbers) we discover the **distributive property**. It is a very easy concept; in fact, you may already know it.

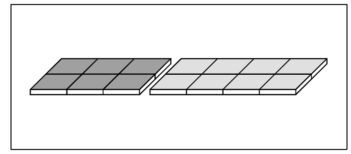
5(2 + 4)   means: 5 groups of (2 plus 4):

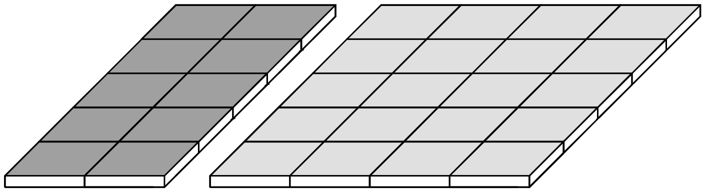


This is a large rectangle composed of two smaller ones:



If we count the rectangles separtely, the total number of chips will be the same as if we count them together:





$5 \cdot 2$ $\qquad$ $5 \cdot 4$

Again, you can add first, then multiply:

$$5(2 + 4) = 5 \cdot 6 = 30$$

or you can multiply first, then add:

$$5(2 + 4) = (5{\cdot}2) + (5{\cdot}4) = 10 + 20 = 30$$

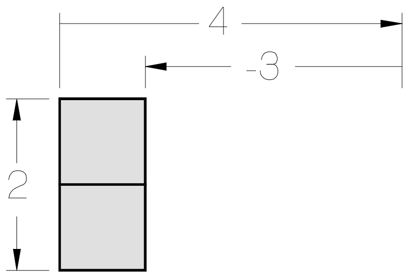
If you multiply first, you must multiply the 5 times both the 2 *and* the 4.

Multiplying Times a Difference

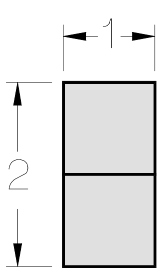
Is there a property that will tell us about

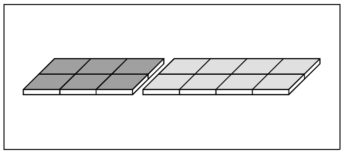
$$2 \cdot (4 - 3) \text{ ?}$$

First, let's think of it as a rectangle that is two (2) on one side and four minus three (4 – 3) on the other:

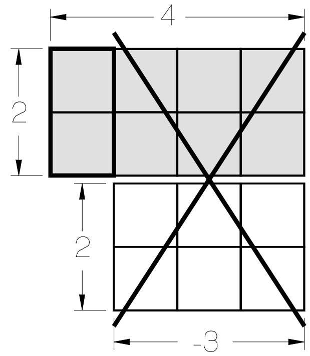


This is really a two by one rectangle equal to two (2):





To try another way, we can think of it as a two by four $(2 \cdot 4)$ rectangle and a two by negative three $(2 \cdot {}^{-}3)$ rectangle:



This is also equal to two (2).

To summarize, we can subtract first and then multiply, or we can multiply first and then subtract. The answer is the same.

Distributive Property

$$3(4 + 5) = (3 \cdot 4) + (3 \cdot 5)$$

or

$$a(b + c) = (a \cdot b) + (a \cdot c)$$
$$a(b - c) = (a \cdot b) - (a \cdot c)$$

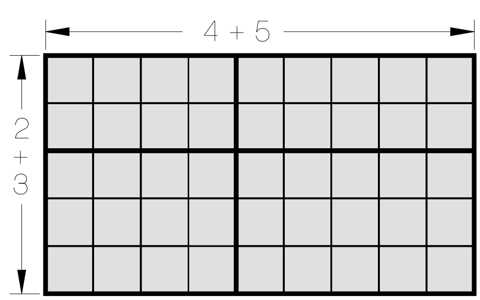
for any three numbers a, **b**, and **c**

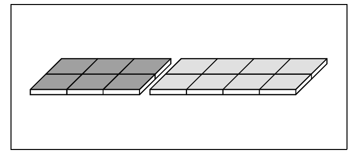
Multiplying Two Sums

The next case we will look at is the idea of multiplying two sums. For example, consider:

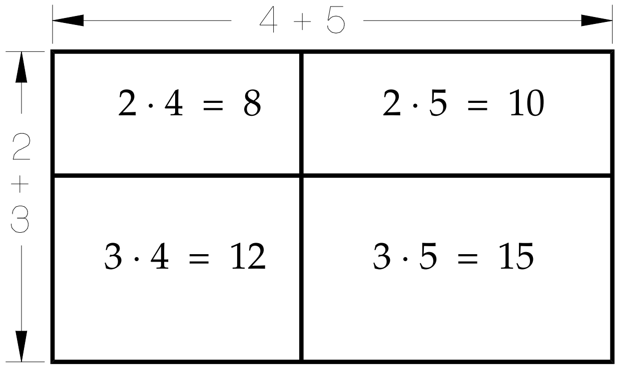
$$(2 + 3) \cdot (4 + 5)$$

Since multiplication is making rectangles, we need to make a rectangle that is **(2 + 3)** on one side and **(4 + 5)** on the other side.





This large rectangle is made up of *four* smaller rectangles, and the result is the sum of these four products:

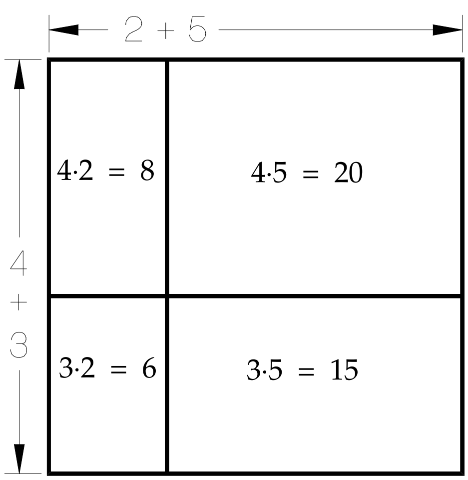


$$\mathbf{(2 + 3)(4 + 5) = (2 \cdot 4) + (2 \cdot 5) + (3 \cdot 4) + (3 \cdot 5)}$$

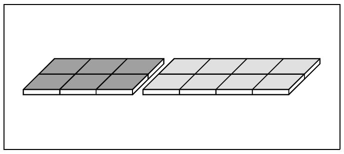
The four rectangles come from the four possible products of each length and each width. In symbols, here is how we do it:

$$\begin{aligned}(2 + 3)(4 + 5) &= (2 \cdot 4) + (2 \cdot 5) + (3 \cdot 4) + (3 \cdot 5)\\ &= 8 + 10 + 12 + 15\\ &= 45\end{aligned}$$

Again, it is important to work on the idea of these problems rather than attempting to memorize the pattern. Here is another example:



$$\begin{aligned}(4 + 3)(2 + 5) &= 4{\cdot}2 + 4{\cdot}5 + 3{\cdot}2 + 3{\cdot}5\\ &= 8 + 20 + 6 + 15\\ &= 49\end{aligned}$$



A Familiar Example

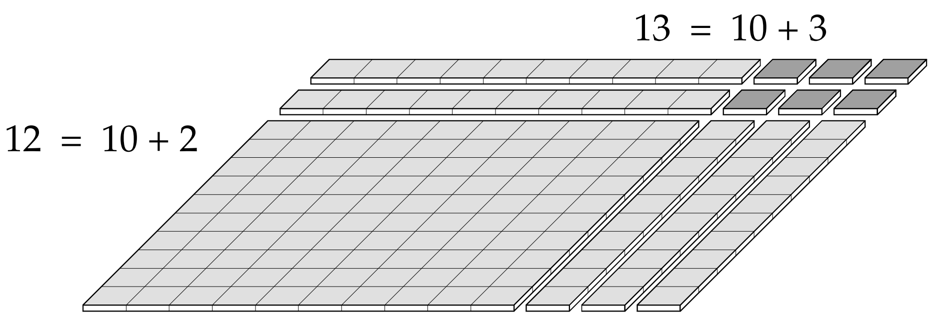
When we multiply two-digit numbers, we have a familiar process that is actually an example of the use of the distributive property. Here is the way it usually works:

$$
\begin{array}{r}
13 \\
\times\ 12 \\
\hline
26 \\
13\ \ \\
\hline
156
\end{array}
$$

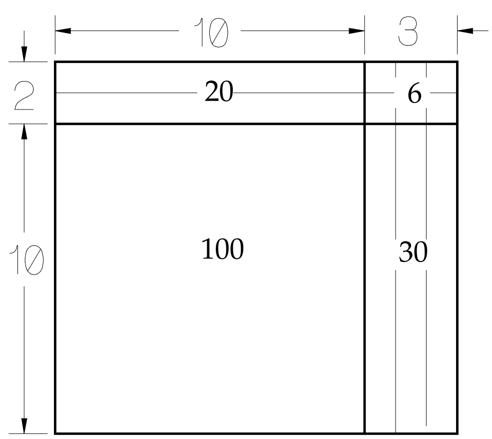
With pictures, think of the problem as

$$13 \cdot 12 = (10 + 3) \cdot (10 + 2)$$

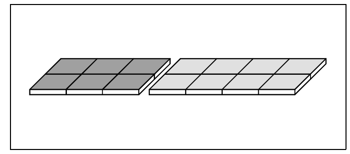
This is a rectangle 13 wide and 12 deep



which is made up of four smaller rectangles:



With symbols, here is what we are doing:



$$\begin{array}{r} 13 \\ \times\ 12 \\ \hline 6 \\ 20 \\ 30 \\ 100 \\ \hline 156 \end{array}$$

6	$2 \cdot 3$	=	6
20	$2 \cdot 10$	=	20
30	$10 \cdot 3$	=	30
100	$10 \cdot 10$	=	100

Distributive Property

(2 + 3)(4 + 5) = (2·4) + (2·5) + (3·4) + (3·5)
or
(a + b)(c + d) = (ac) + (ad) + (bc) + (bd)

for any four numbers **a**, **b**, **c**, and **d**

Division and the Distributive Property

Without using pencil and paper, calculate half of $2.50.

Did you need to do long division, or did you think of it as

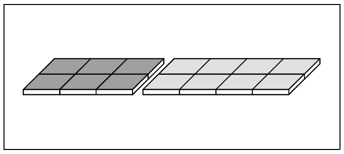
Half of $2.00 is $1.00. Half of .50 is .25
The answer is $1.00 plus .25 or $1.25

This is another example of the distributive property. The new idea is that the property works when you are taking *prt* of something (dividing by two or multiplying by one-half) as well as when you are multiplying by whole numbers.

Next, think about making a recipe smaller. If your recipe calls for four and two-thirds cups of milk, and you want to make half, how do you do it?

Half of $4\frac{2}{3}$ is half of 4 plus half of $\frac{2}{3}$ is

2 plus $\frac{1}{3} = 2\frac{1}{3}$



The procedure that you naturally use is to take half of the parts and then add these two halves together. This is much easier than:

$$4+\frac{2}{3} = \frac{12}{3}+\frac{2}{3} = \frac{14}{3}$$

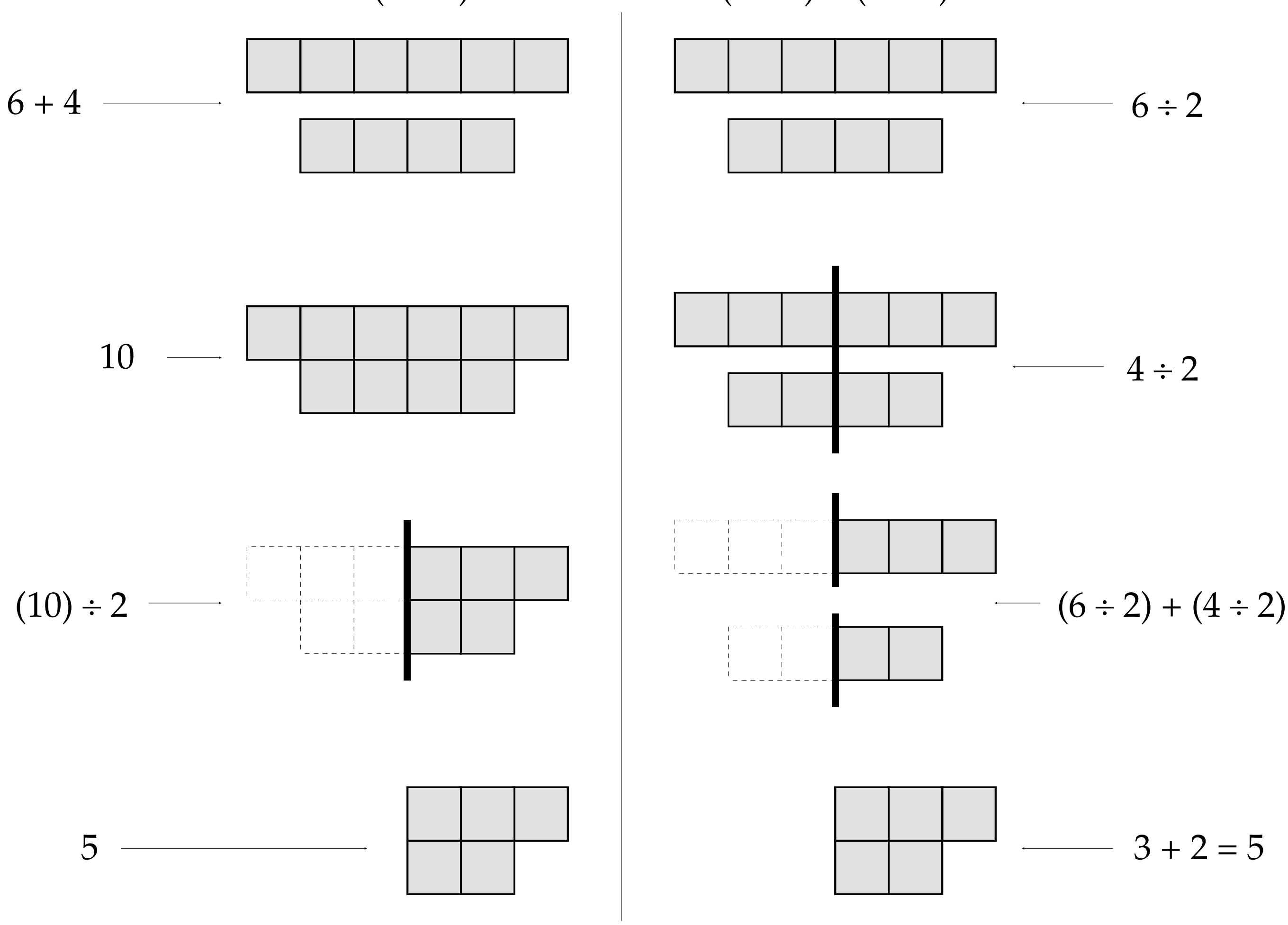
$$\frac{1}{2}\cdot\frac{14}{3} = \frac{14}{6} = \frac{7}{3} = 2\frac{1}{3}$$

OR

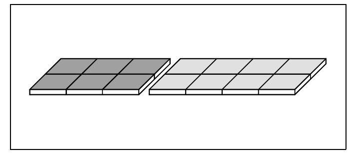
$$\frac{1}{2}\left(4+\frac{2}{3}\right) = \frac{1}{2}\left(\frac{12}{3}+\frac{2}{3}\right) = \frac{1}{2}\left(\frac{14}{3}\right) = \frac{14}{6} = \frac{7}{3} = 2\frac{1}{3}$$

The next example illustrates this property with pictures. The two sets of pictures demonstrate that the two sides of the equation are equal.

$$(6+4)\div 2 \quad = \quad (6\div 2)+(4\div 2)$$



With symbols alone, here is how it looks:



Distributive Property of Division over Addition

$$(6 + 4) \div 2 = (6 \div 2) + (4 \div 2)$$
or
$$(a + b) \div c = (a \div c) + (b \div c)$$

for any numbers **a**, **b**, and **c**

Exercises

Use pictures or chips to showeach problem and calculate the answers. *Multiply or divide first, using the distributive property.* Check by adding or subtracting the quantity in parenthese first.

1. $3(5 + 6)$

2. $2(1 + 4)$

3. $(4 - 1) \cdot 3$

4. $4(5 - 2)$

5. $(1 + 2)(3 + 4)$

6. $(3 + 2)(1 + 1)$

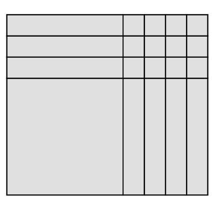
7. $(8 + 10) \div 2$

8. $(6 + 12) \div 3$

For the following problems, use the large square in your kit to represent 100 (10 by 10), the long bar to represent 10 (1 by 10), and the small chips to represent 1 (1 by 1).

Example: $14 \cdot 13 = (10 + 4)(10 + 3)$

Solution: $100 + 30 + 40 + 12 = 182$



$$\begin{array}{r} 100 \\ 30 \\ 40 \\ +\ 12 \\ \hline 182 \end{array}$$

9. $13 \cdot 15$

10. $14 \cdot 16$

11. $21 \cdot 16$

Section 3
Identities and Inverses

Operations

Addition, subtraction, multiplication, and division are called **operations**. These are **binary** operations because *two* numbers are required to find the result.

The previous sections of this chapter have dealt with some basic ideas about operations:

- **The *order* of the numbers (Commutative Properties)**
- **The *grouping* of three numbers into pairs (Associative Properties)**
- ***Combinations* of operations (Distributive Properties)**

This section will discuss certain special numbers that are called **identities** and **inverses**.

Identity Elements

When we add zero to a number, the number is unchanged. This i certainly not a great mystery, because adding zero means taking a group of chips and doing nothing to it; since the original number has an *identical* value, zero is called the **identity element for addition**.

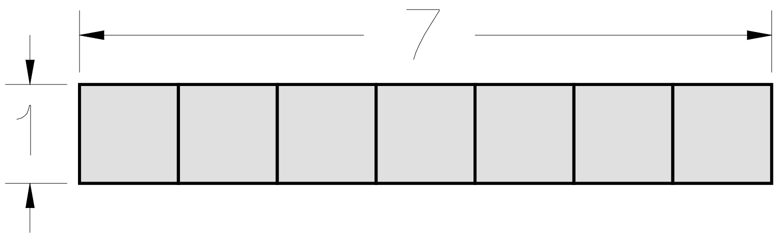
The identity element for an operation is the number that has no effect for that operation.

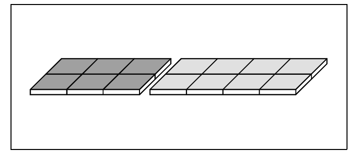
For subtraction, zero also has no effect; taking away zero chips leaves the identical number with which we started.

Multiplication is somewhat different. The identity element is the number which, when multiplying any other number, leaves it unchanged:

$$7 \cdot 1 = 7$$
$$8 \cdot 1 = 8$$
$$1 \cdot 7 = 7$$
$$1 \cdot (^-3) = {}^-3$$

*The identity element for multiplication is **one***. With chips, this means that to multiply **7 · 1** we make a rectangle that is seven chips long and one chip wide. The result is obviously seven:



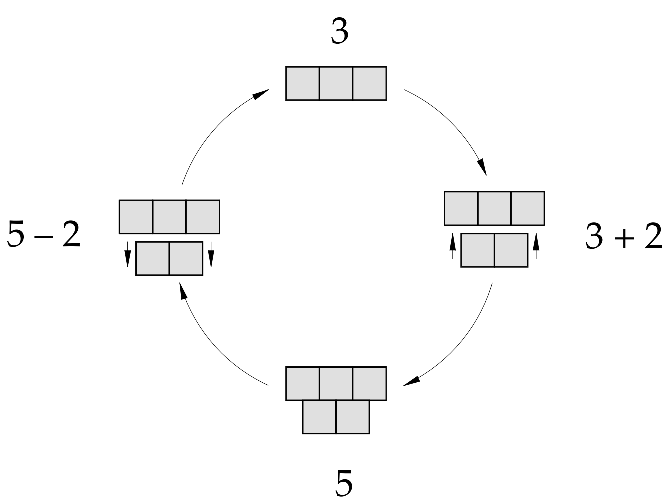


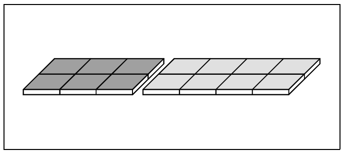
The identity element for division is also one. How many ones are there in seven? The answer is seven. How many ones in negative five? Negative five. Here is a summary of identity elements:

Operation	**Identity**	**Example**
Addition	0	$32 + 0 = 32$
Subtraction	0	$53.6 - 0 = 53.6$
Multiplication	1	$23 \cdot 1 = 23$
Division	1	$53 \div 1 = 53$

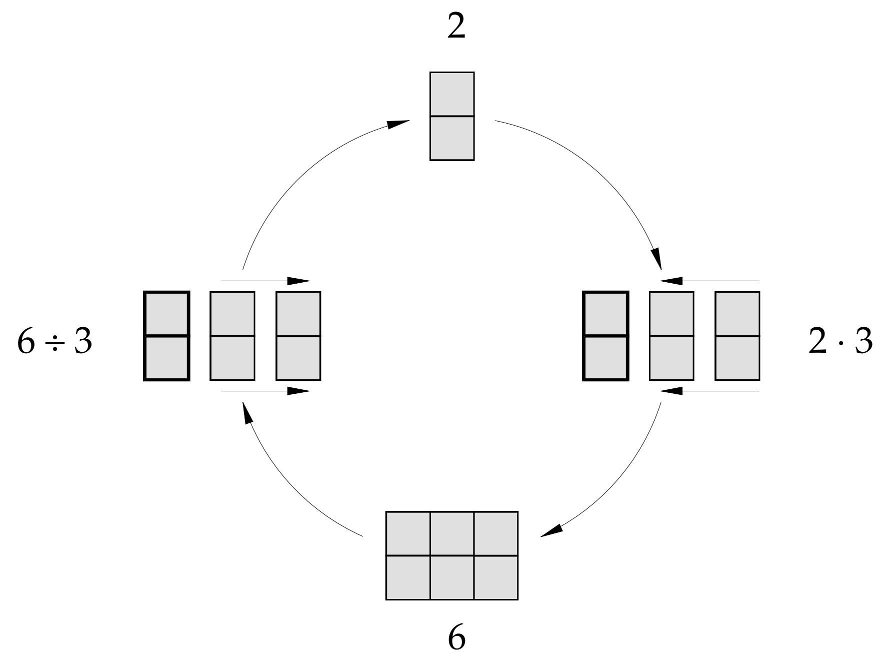
Inverse Operations

Addition and subtraction are called **inverse operations** because they represent opposite actions with numbers and chips. Addition is putting chips together; subtraction is taking chips away:





Multiplication and division have a similar relationship. Multiplication is building a number of rows; division starts with the finished rectangle and counts the number of rows:



Can you explain why the operations in each pair (Multiplication/Division and Addition/Subtraction) have the same identity element?

Inverses of Numbers

The effect of adding 4 to a number can be cancelled out by adding $^{-}4$:

$$5 + 4 + {}^{-}4 = 5$$

This occurs because

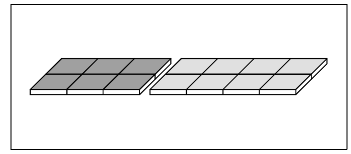
$$5 + 4 + {}^{-}4 = 5 + (4 + {}^{-}4) = 5 + 0 = 5$$

As we can see, 4 and $^{-}4$ cancel out the effects of each other because

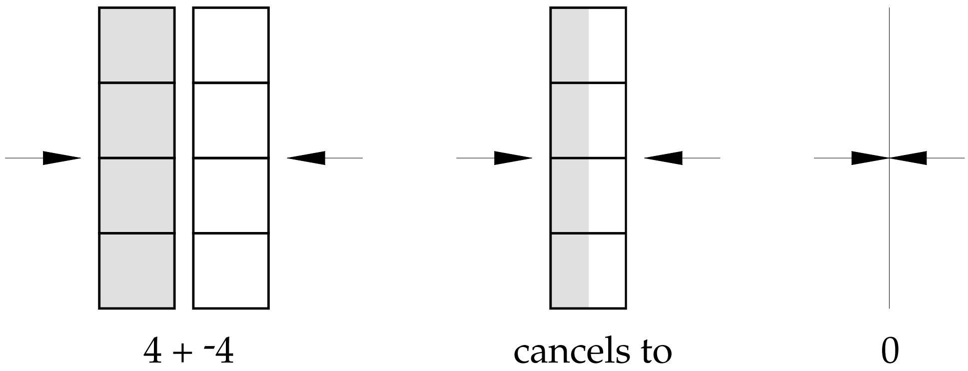
$$4 + {}^{-}4 = 0$$

Since zero is the identity element, there is no effect on the total. Numbers like these that have opposite effects are called **inverses**. With addition, opposites of positives are negatives and opposites of negatives are positives. The opposite of zero is zero.

Notice that inverses exist with respect to a particular operation only; when we multiply $4 \cdot {}^{-}4$ one does *not* cancel out the effect of the other as they did when we added.



Here is an example of **additive inverses**:



For multiplication, we can find a similar property of some familiar numbers:

$$23 \cdot 2 \cdot \frac{1}{2} = 23$$

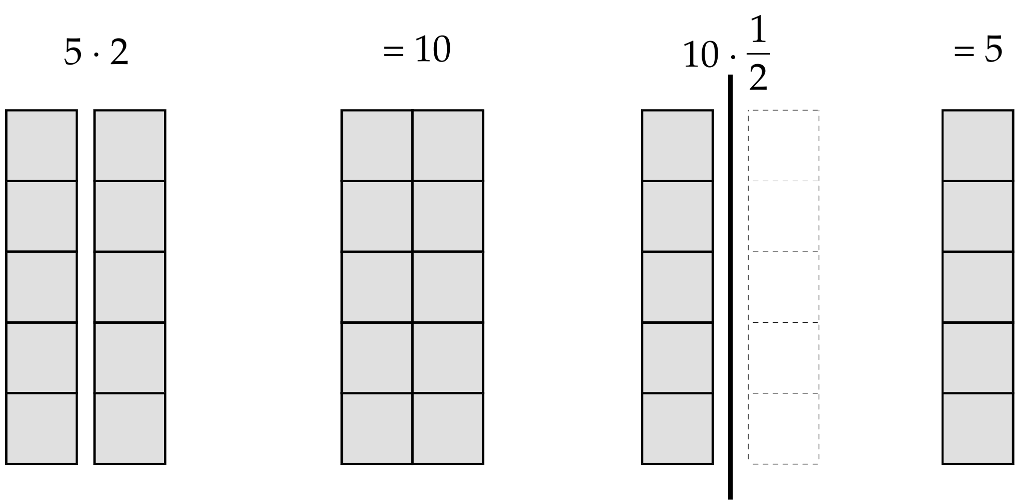
$$8 \cdot \frac{2}{3} \cdot \frac{3}{2} = 8$$

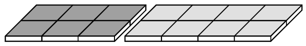
This occurs because

$$23 \cdot 2 \cdot \frac{1}{2} = 23 \cdot \left(2 \cdot \frac{1}{2}\right) = 23 \cdot 1 = 23$$

Pairs of numbers such as 2 and ½ are called **multiplicative inverses** or **reciprocals**. Each cancels out the effect of the other because their product is one—the identity element for multiplication.

Here is an example of multiplicative inverses:





Exercises

Perform the operations using chips and with symbols.

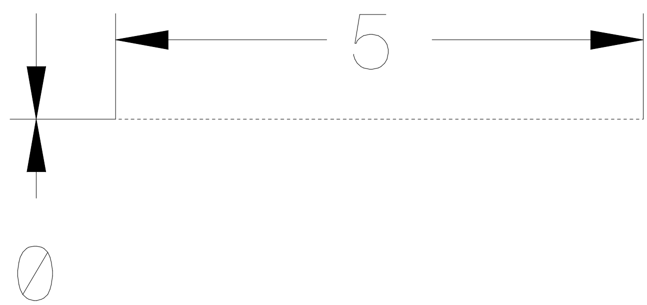
1. $3 + 5 + {}^-5$
2. $3 + 21 + {}^-3$
3. $14 \cdot 2 \cdot \dfrac{1}{2}$
4. $8 \cdot \dfrac{1}{3} \cdot 3$
5. $(3 + {}^-3) + \left(\dfrac{1}{5} \cdot 5\right)$
6. $7 \cdot ({}^- 7)$
7. $6 + {}^-({}^-6)$
8. ${}^-6 + {}^-({}^-6)$
9. $12{,}345.8 + {}^-12{,}345.8$
10. $17 \cdot 45 \cdot \dfrac{1}{45}$

Section 4
Properties of Zero

Multiplication by Zero

Zero times any number is zero. This is an obvious fact and does not need to be memorized or practiced; it is clearly true.

Five times zero is zero because five *groups* of zero are zero. Zero times five is zero because if you have no five dollar bills, you have no money at all. With chips, it looks like this:



Here, you can see a rectangle with sides of zero and five representing $\mathbf{0 \cdot 5}$. How many chips do you see?

Dividing with Zero

Consider two interesting questions:

- **What is the meaning of** $\mathbf{0 \div 5}$ **or** $\frac{0}{5}$ **?**
- **What is the meaning of** $\mathbf{5 \div 0}$ **or** $\frac{5}{0}$ **?**

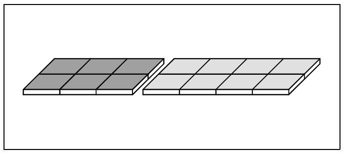
Dividing zero by five or any other number (except zero) is quite straightforward. We are asking

"How many fives in zero?"

or

"What is a fifth of zero?"

In each case, the answer is clearly zero.



Dividing *by* zero is much more troublesome. Here we are asking:

"How many zeroes in five?"

or

"Divide 5 into 0 equal parts. How much is in each part?"

It is clear that there is no sensible answer to either question. First, you can add as many zeroes as you wish, but you will never reach five.

0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+
0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+
0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+
0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+
0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+
0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+

Secondly, the idea of dividing something into zero parts (a zeroeth?) does not make much sense. We conclude that **division by zero is not defined**. If you are tempted to guess that the answer should be zero, consider these examples:

$$12 \div 3 = 4 \text{ because } 3 \cdot 4 = 12$$
$$16 \div 8 = 2 \text{ because } 8 \cdot 2 = 16$$

If 5 divided by zero were equal to zero, then

$$5 \div 0 = 0 \text{ would mean that } 0 \cdot 0 = 5?$$
$$\text{so } \quad 5 \div 0 \quad \text{is undefined}$$

This illustrates the difficulty—if we ever define division by zero, we will have to create new concepts for our old ideas about multiplying, dividing, and even numbers themselves.

Exercises

Find the answers or determine if the problem is not defined:

1. $(5 \cdot 0) \div 6$

2. $0 \div 0$

3. $(0 \div 3) \div 3$

4. $(3 \div 3) \div 0$

5. $(3 \div 0) \cdot 0$

6. $14 \div (^{-}23 - {}^{-}23)$

Section 5: (Optional) Properties or Rules?

Introduction: An Example

This section is for people who have had previous difficulties with the traditional system of memorizing rules. The purpose is to help you think about the best way to learn mathematics.

The Distributive Law of Multiplication over Addition is traditionally explained as a pattern in this way:

Distributive Law

$$\mathbf{a(b + c) = ab + ac}$$

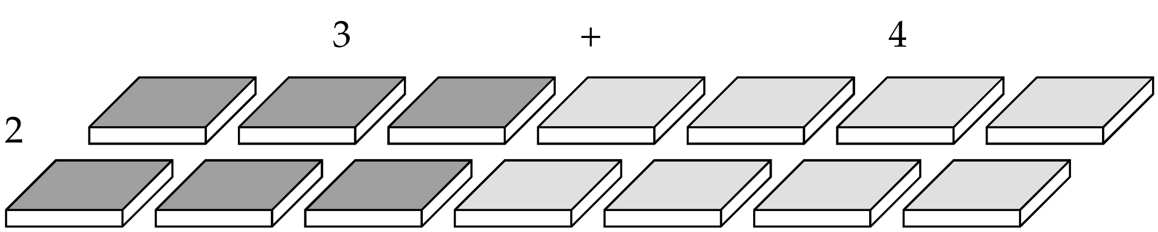
The instructions are: To multiply one number times a sum of two numbers, you multiply the first number times each of the other numbers and then add the products.

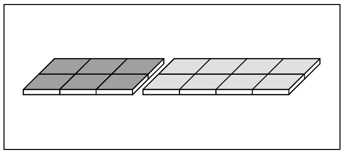
This type of explanation is often very confusing. We don't know *when* to use this property and we don't know *why*. Many students continue to have difficulty with this property even at the college level.

Here is how we have done it in this chapter:

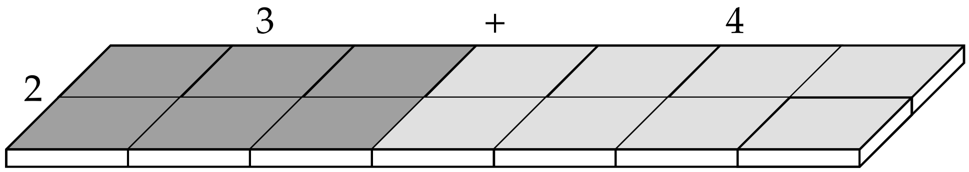
Consider (2)(3 + 4)

Multiplying **(2)** times **(3 + 4)** means making a rectangle that has dimensions of **(2)** and **(3 + 4)**:



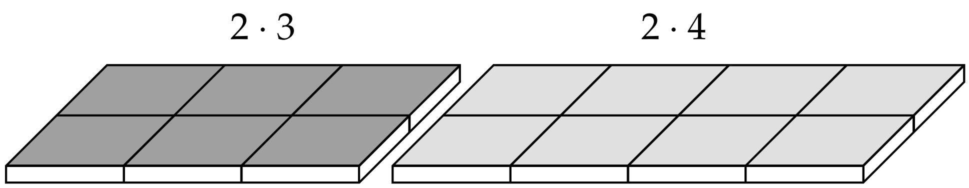


This is a large rectangle made up of two smaller rectangles; one is **(2)** by **(3)** and the other is **(2)** by **(4)**:

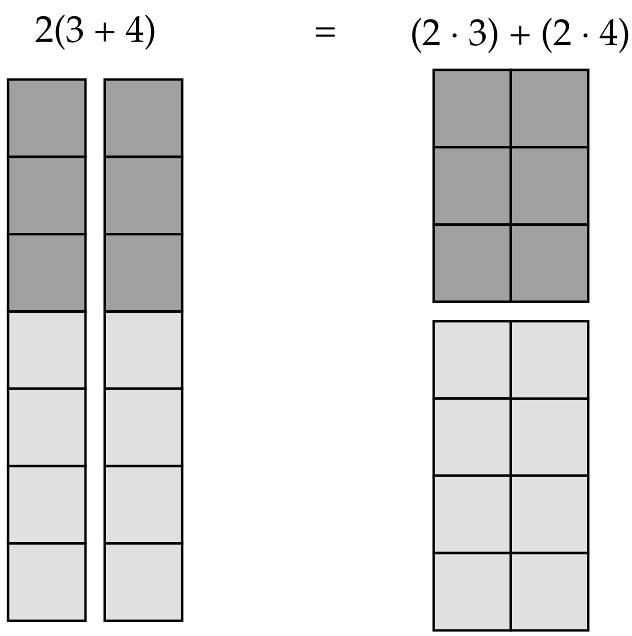


If we separate the two rectangles, we have:

$$(2 \cdot 3) + (2 \cdot 4)$$

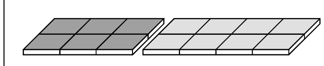


Here is the summary of the property:



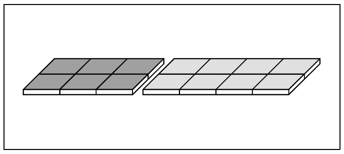
This has been an example of explaining a property with objects instead of symbols. Can you see the difference?

Rules Versus Properties



In this book, we will usually talk about **properties** instead of **rules or laws**. *A property is something real that you can discover and understand*. Properties are easy to learn and and easy to use precisely because they are understood, not memorized. On the other hand, rules are unexplained instructions that must be memorized through lengthy practice. Here are some differences between these two different ways of learning algebra:

	Rules or Laws	**Properties**
Invented by:	Someone else	You
Learned by:	Memorization	Discovery
Practiced by:	Repetition	Investigation
Believed because of:	Authority of others	Your own knowledge and experience
Enjoyable?	Not usually	Yes
Time it takes to learn	Varies, but usually a lot	Varies, but usually less
Length of time retained	A month, an hour, or until the next test.	A long time!



We will always concentrate on why things work , not on memorizing what to do. If you try this method, you will find that algebra will be more interesting and much easier to learn.

Here are some of the advantages of learning *why* instead of memorizing *what to do*:

- **Learning each topic will take less time. When you know *why*, you don't have to do as much practice.**

- **Learning will be more fun. The boredom of repetition will be replaced by the excitement of real investigation and problem solving.**

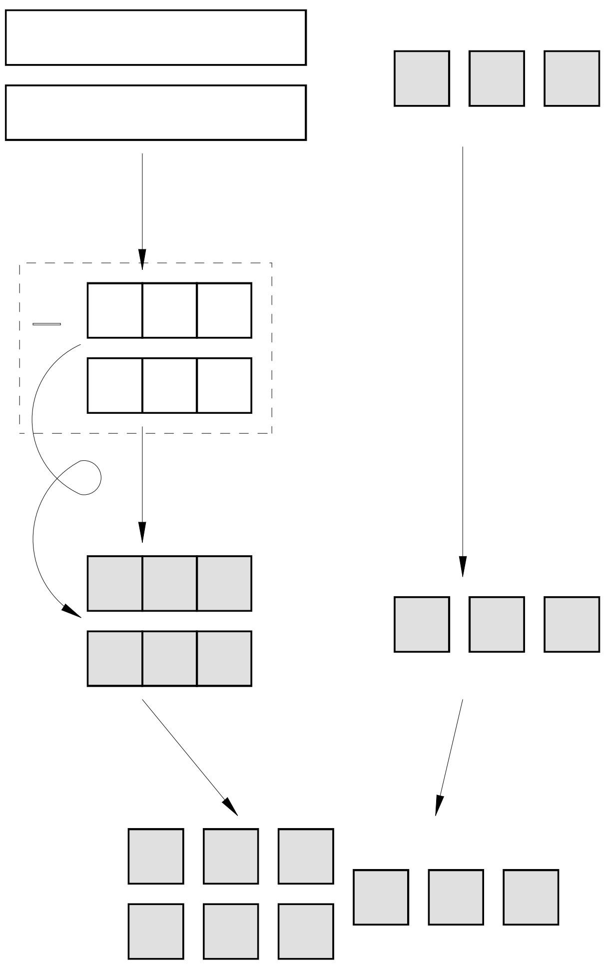
- **You will retain the material for a long time. Memorized, meaningless rules are often forgotten in a few days. Material that is truly understood will never really be lost.**

- **Mathematics will seem less complex. Many properties and techniques are really very similar; many properties are already known to you.**

- **You will be in control. You will know when you understand the material and when you need more work.**

Chapter 6

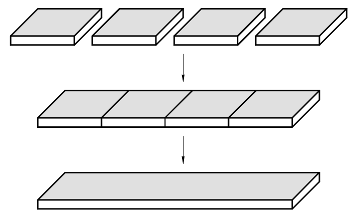
Expressions



Section 1
Simple Expressions

The Meaning of Unknowns

A group of unit chips can be represented by an **unknown** or **variable** like x. We join an unspecified number of chips together to form a bar called x:



Unknowns can be positive, negative, or zero. We use unknowns to represent quantities that will be known at a later time. Because unknowns are actually numbers, we treat them in the same manner as any other numbers.

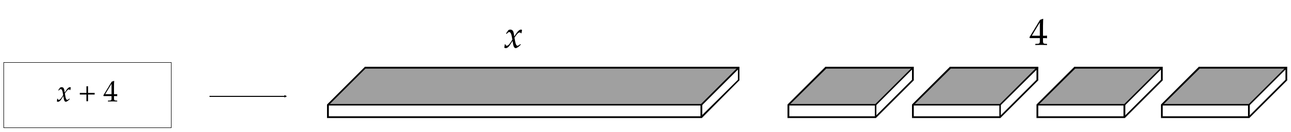
Expressions

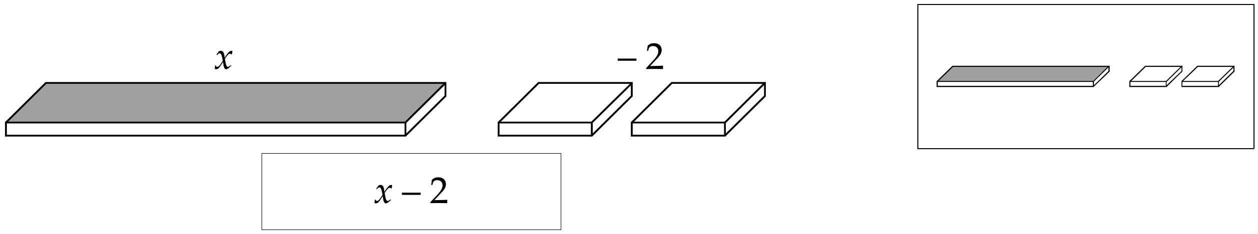
An **expression** is any quantity that stands for a number. Expressions may be as simple as one number or unknown, or they be lengthy statements including many numbers, unknowns, and operations:

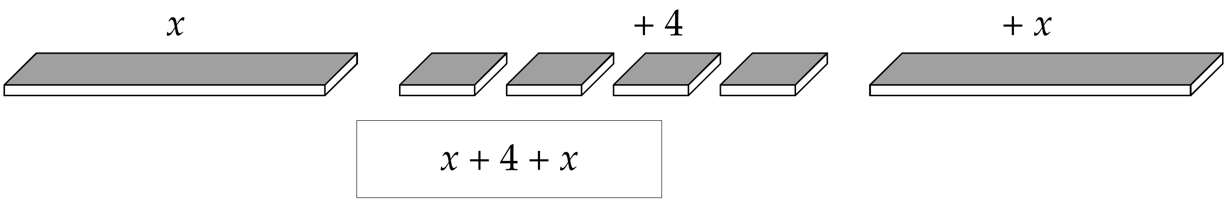
Examples of Expressions
$3x$
$3x + 1$
$^{-}17$
$3x + 2 + 6x - 2$

Simple Expressions

It is easy to visualize expressions that include only one or two symbols:

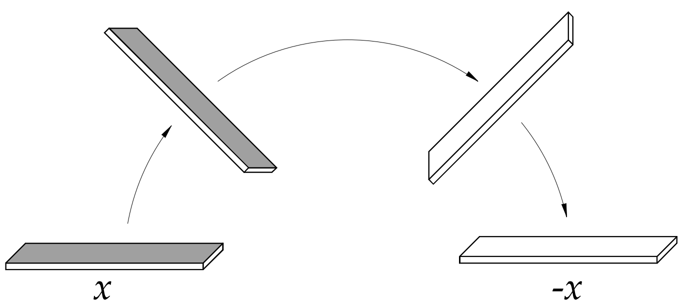






The Opposite of *x*

Just as we can find opposites of numbers by flipping the chips, the opposite of *x* can be shown as the *x*-bar flipped over:

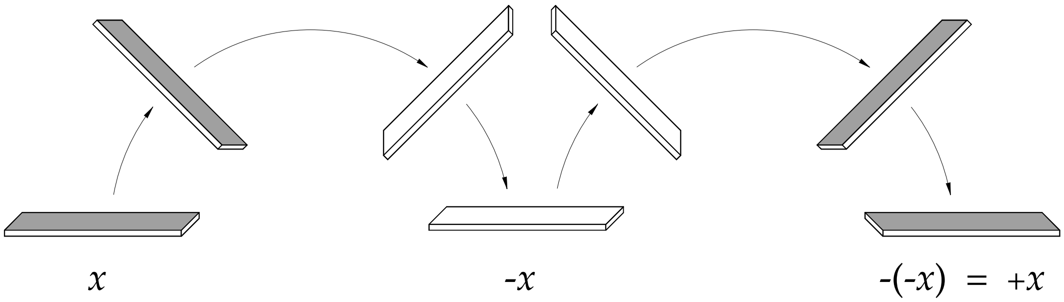


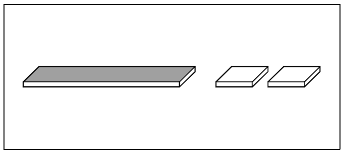
This -*x* may be called **the opposite of *x*, the additive inverse of *x***, or **negative *x***. The last term—negative *x*—should be used with care. Because *x* may stand for either a positive or negative number, negative *x* stands for the opposite of *x*; it is not necessarily a negative number.

The opposite of the opposite is still the original amount:

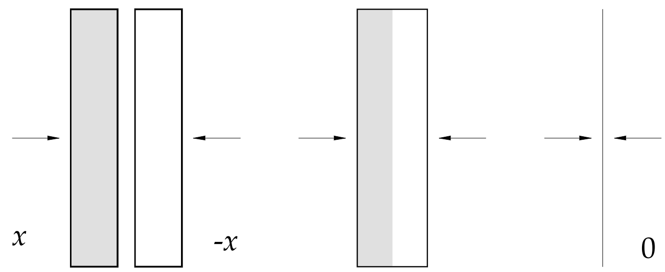
$$^{-}(^{-}5) = {}^{+}5$$

$$-(-x) = +x$$





Finally, x and $-x$ are additive inverses. When added, they “cancel” to zero in the same way that $^+3$ and $^-3$ cancel:



Evaluating Expressions

An expression that includes unknowns has no exact value because we do not know the value of the unknown. If we choose a value for the unknown, we can then **evaluate the expression** to determine its value.

To evaluate an expression, simply substitute the value of the unknown into the expression and then carry out the indicated operations:

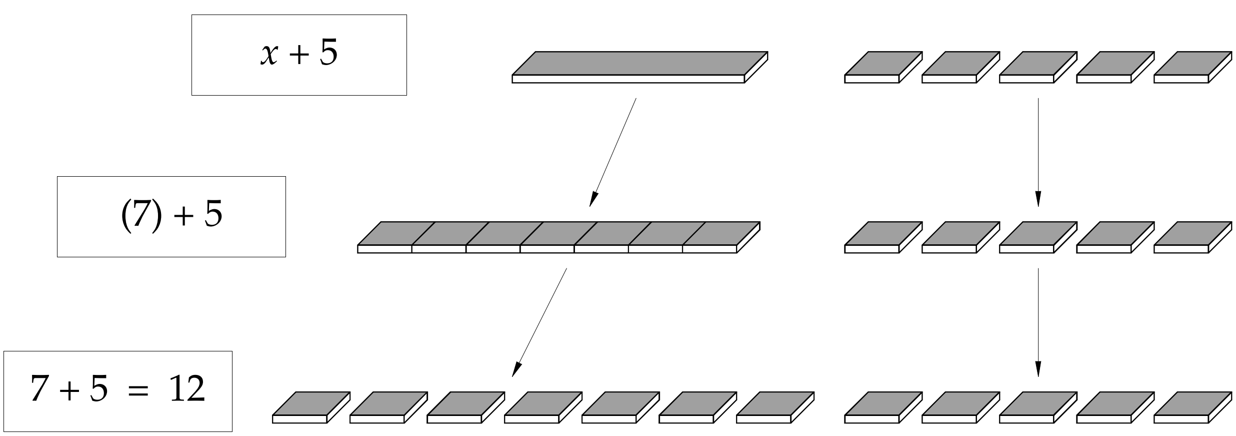
If $x = 7$, to evaluate $x + 5$:

$$x + 5$$
$$(7) + 5$$
$$12$$

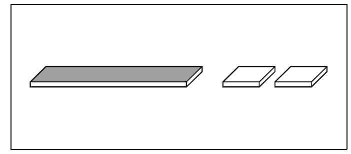
If $x = {}^-5$, to evaluate $4 - x$:

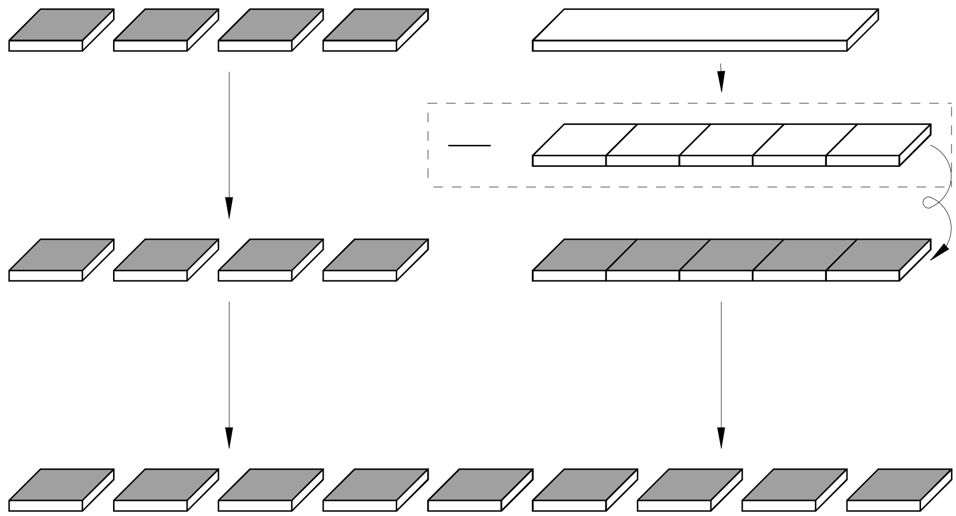
$$4 - x$$
$$4 - ({}^-5)$$
$$4 + 5$$
$$9$$

With the chips, we simply substitute the indicated number of units for the x-bar and then complete the count of unit chips. For $x + 5$, where $x = 7$:



For the second example, we start with a diagram of $4 - x$, then we figure out the value of $-x$ when x is $^{-}5$, and then we complete the count of unit chips:





The x-bar can stand for ***any*** *number—positive, negative, or zero.*

Exercises

Draw pictures of the following expressions. Evaluate each expression three different times—when x is 3, 0, and $^{-}2$:

1. $x + 6$

2. $2 + -x$

3. $x + 5 + x + x$

4. $^{-}5 + x + 5 + -x$

5. $3 + x + 5 + (-x) - x - 1$

6. $x + x + x - 5$

7. $x + x + x + x + x - 3$

8. $4 + x$

9. $x - x$

10. 5

11. $x + 3 + (^{-}3)$

12. $x - 3 - x - 2 - x - 1$

13. $3 - x$

14. $-x + 3$

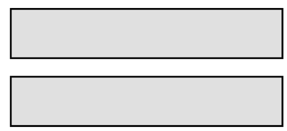
15. $x + (^{-}3)$

Section 2
Multiples of *x*

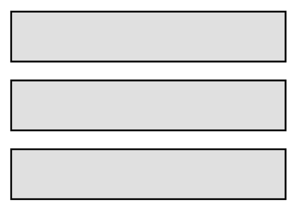
More than One *x*

If x is a certain quantity, the idea of several x's is a natural extension of our idea of one unknown:

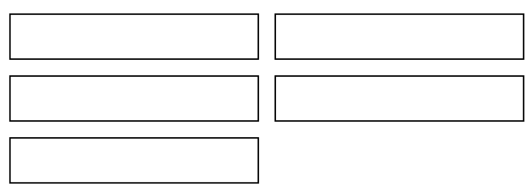
$2x$



$3x$



$^{-}5x$



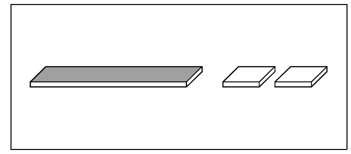
An expression like $\mathbf{2x}$ also can be thought of as a multiplication problem:

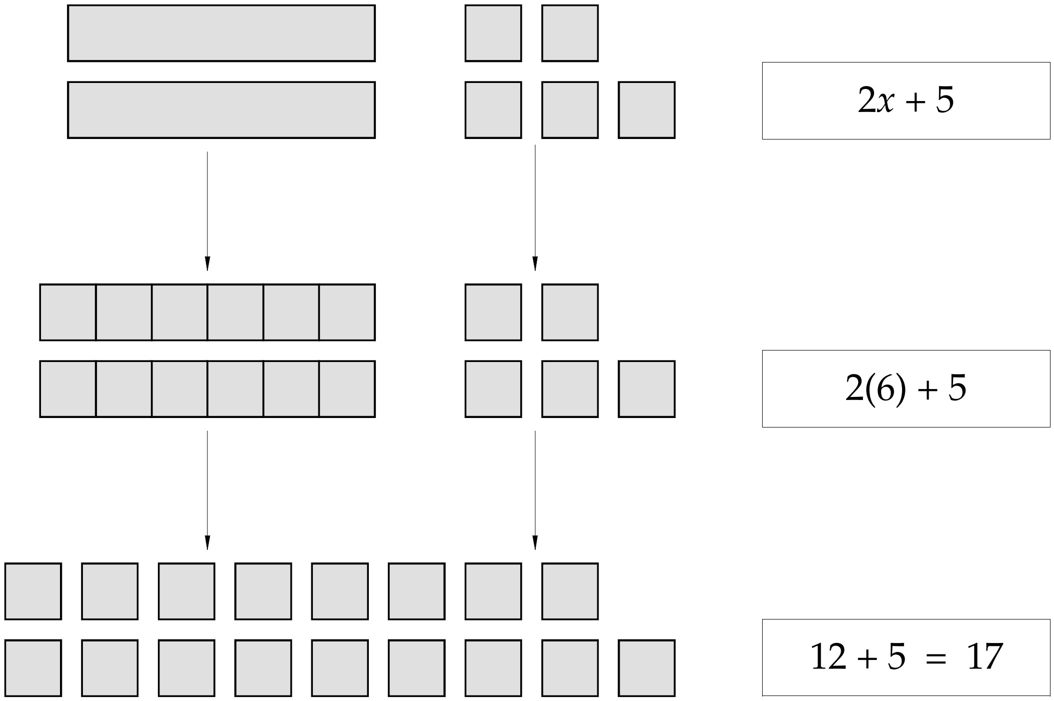
$$2x = 2 \cdot x = x + x$$

As we can see, we can call this expression "two x" or " two times x" and the meaning is still the same. Expressions such as $^{-}5x$ will be shown as 5 negative x-bars. In later chapters, we will see that the idea of $(^{-}5)\cdot(x)$ is also appropriate.

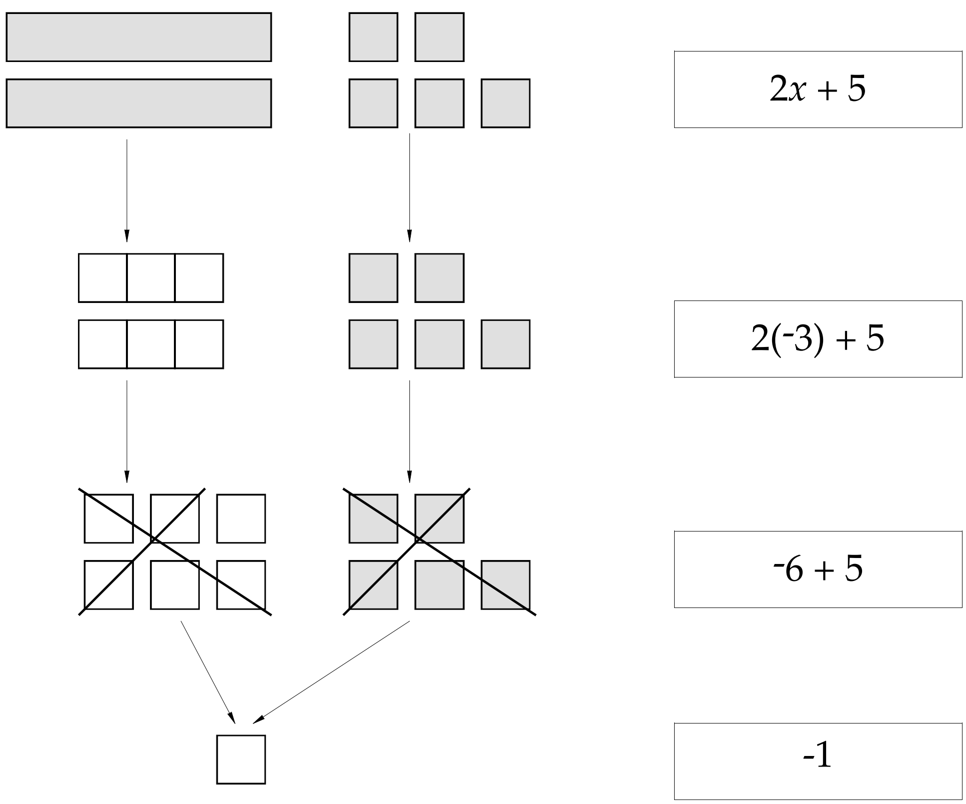
Evaluating Expressions with Multiple *x*'s

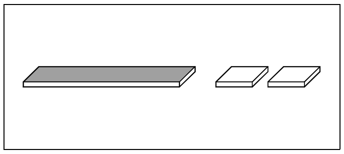
To evaluate an expression like those shown above, we still substitute a certain quantity for x, but we must be careful to carry out the indicated multiplications where a number is multiplied times x. For example, evaluate $2x + 5$, where $x = 6$:



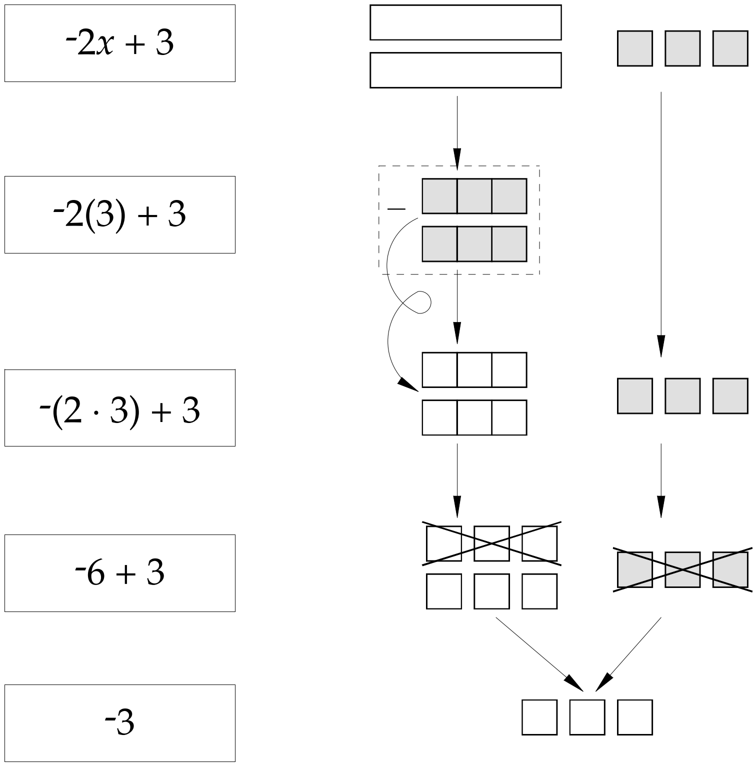


Repeating this example for a different value, where $x = {}^{-}3$:

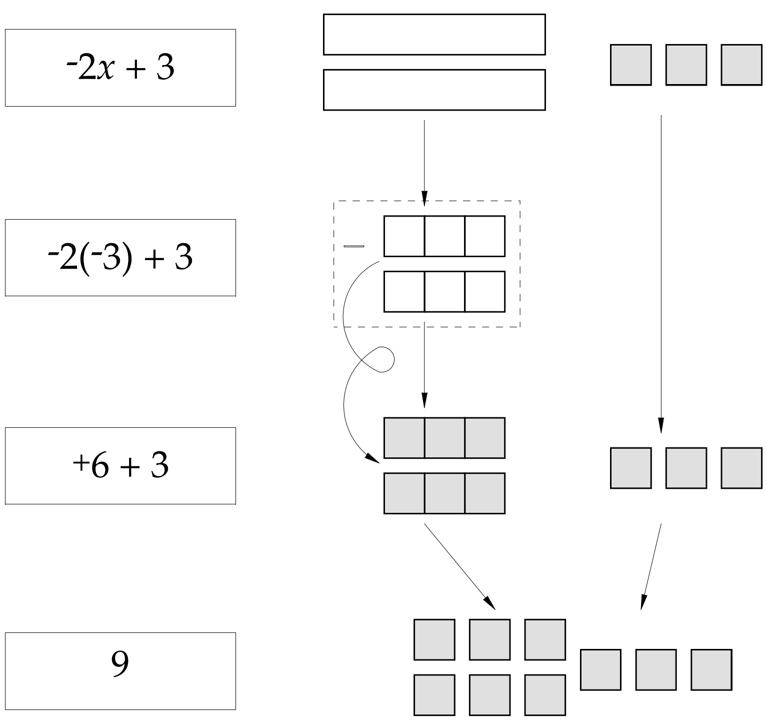




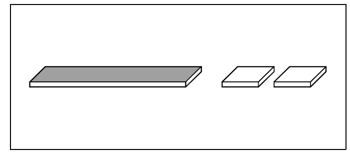
If the expression contains negative x's, we must first substitute the correct value for x; then we flip over the substituted chips to show the opposite of x. For example, to evaluate $^-2x + 3$, where x is 3:



Here is a second evaluation of $^-2x + 3$, where this time x is $^-3$:



Exercises



Use chips to represent the following expressions. Evaluate each expression four different times for $x = 1$, $x = 5$, $x = 0$, and $x = {}^{-}1$.

1. $3x - 15$

2. ${}^{-}3x - 12$

3. $2x + 5 + {}^{-}3x$

4. ${}^{-}3x + 2 + x + ({}^{-}3)$

5. $225x + 1 + {}^{-}225x$ (Do you need the chips?)

6. $5 - x$

7. $2 - 3x$

8. $0 - x + 16$

9. $3x - 2x + 6 - 2x$

10. $2 - ({}^{-}3x)$

11. $2x - 3 + x - 5$

12. ${}^{-}4 - x + 2 + 3x$

13. $5x - 2 + x$

14. $3 + 2x - 1 - x$

15. ${}^{-}({}^{-}2) + 3x + 5$

16. $x - 5 - 3x + 1$

17. ${}^{-}({}^{-}4x) + 3 + x$

18. ${}^{-}3 - 2x - ({}^{-}5)$

19. $x + x + 3x - 5$

20. $2 - 3x - 4x + 1$

Section 3
Combining Similar Terms

Combining Terms

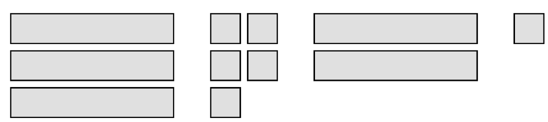
Each group of similar chips in an expression is called a **term**:

Expressions	Terms
$3x + 5 + 2x$	$3x$, 5, $2x$
$17 - 2x$	17, $^{-}2x$

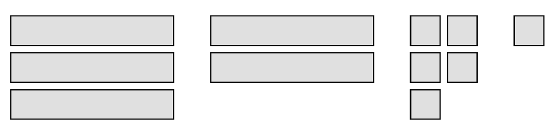
Notice that the **2** in **2*x*** is not a term.

Before we evaluate or use an expression, it is usually best to combine similar terms:

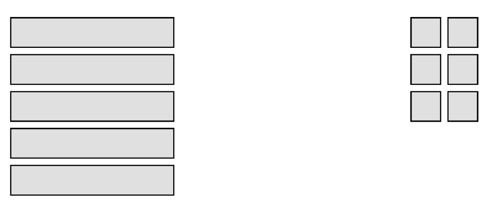
$3x + 5 + 2x + 1$



$3x + 2x + 5 + 1$

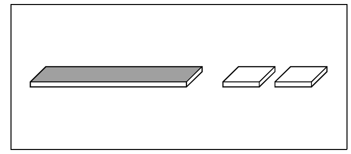


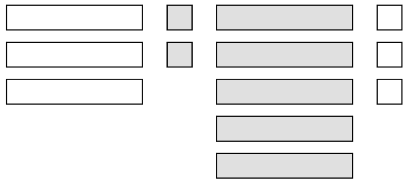
$5x + 6$



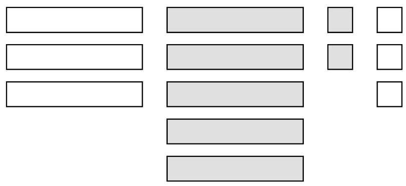
The process of combining similar terms will save time when evaluating expressions and will also be helpful in techniques presented in future chapters.

Positive x-bars and negative x-bars are combined in the same way as positive and negative chips:

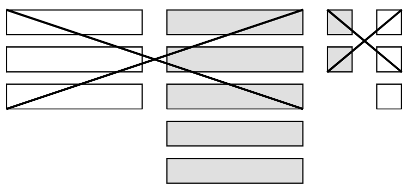


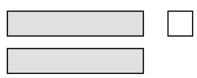


$^{-}3x + 2 + 5x - 3$



$-3x + 5x + 2 - 3$



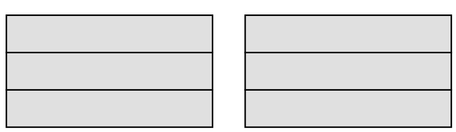


$2x - 1$

Multiplying Numbers and *x*'s

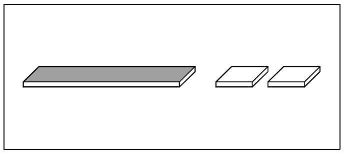
What is the meaning of:

$$2 \cdot 3x \ ?$$



Using symbols, we can write:

$$\begin{aligned} 2 \cdot 3x &= 2 \cdot (3 \cdot x) \\ &= 2 \cdot 3 \cdot x \\ &= (2 \cdot 3) \cdot x \\ &= 6 \cdot x \\ &= 6x \end{aligned}$$



If this seems too formal, think of the answer as 2 groups of $3x$. This is $3x + 3x$ or $6x$. Here are other examples:

$$
\begin{aligned}
6x \cdot 5 &= (6 \cdot x) \cdot 5 \\
&= (x \cdot 6) \cdot 5 \\
&= x \cdot 6 \cdot 5 \\
&= x \cdot 30 \\
&= 30 \cdot x \\
&= 30x
\end{aligned}
$$

$$
\begin{aligned}
6x \cdot ({}^-5) &= (6 \cdot x) \cdot ({}^-5) \\
&= (x \cdot 6) \cdot ({}^-5) \\
&= x \cdot 6 \cdot ({}^-5) \\
&= x \cdot [6 \cdot ({}^-5)] \\
&= x \cdot ({}^-30) \\
&= {}^-30 \cdot x \\
&= {}^-30x
\end{aligned}
$$

Common Errors

Most of the errors made by students can be avoided by paying attention to what the symbols *mean*. For example, consider the following errors made while combining similar terms:

$3x - x = 3$ (*Not* true)

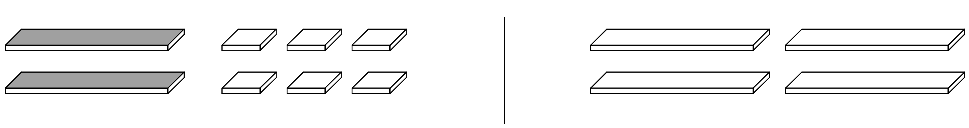
$3x + 6 = 9$ (*Not* true)

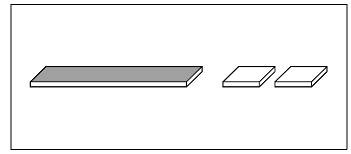
$3x + 6 = 9x$ (*Not* true)

$2x - 6 = {}^-4x$ (*Not* true)

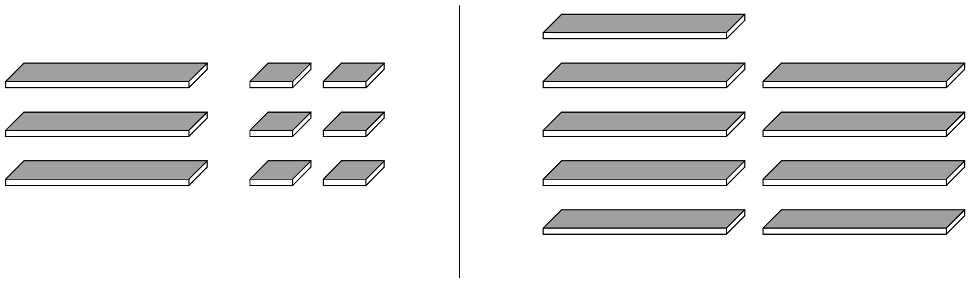
These errors usually occur when students are attempting to manipulate symbols by memorizing rules. When the chips are used, this type of mistake is much more obvious:

$2x - 6$ *is not* ${}^-4x$





$3x + 6$ *is not* $9x$



Exercises

Identify the terms in these expressions:

1. $3x - x + 5$

2. 0

3. $4x + 1 + 1$

4. $1 - 2 + x$

Simplify these expressions by combining similar terms:

5. $3x + (2x)(5) + 1$

6. $4x + 3 + (3)(2x) + 2$

7. $4x + 1 + 6x + 3 + x$

8. $^{-}4x + 3x + 1$

9. $2 + {}^{-}3x + 5x + {}^{-}6$

Evaluate these expressions *before* combining similar terms and *after* combining similar terms. Are the results the same? Do each problem with $x = 4$ and with $x = {}^{-}1$.

10. $3x + 2x + 1$

11. $4x + 3 + 2x + 2$

12. $4x + 1 + 6x + 3 + x$

13. $^{-}4x + 3x + 1$

14. $2 + {}^{-}3x + 5x + {}^{-}6$

Section 4
Expressions and Parentheses

Using Parentheses

Parentheses have the same meaning with unknowns as they do with exact numbers. You will remember that with exact numbers, parentheses indicated an operation or group of operations that were to be done first, before any other operations were done outside of the parentheses.

With unknowns, parentheses still indicate a group of terms that are together, but we cannot always complete the operations indicated because we do not know the value of the unknown term. Three examples are:

$$3(x + 5)$$

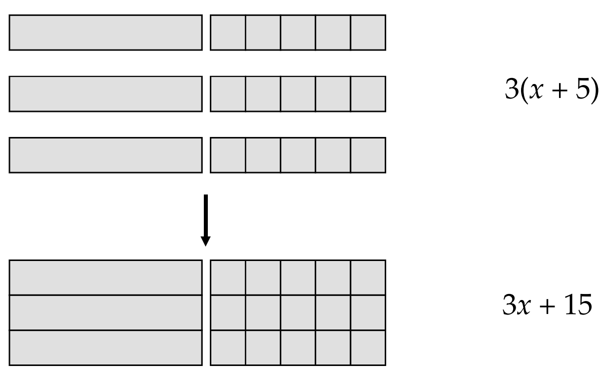
$$5 + 2(x - 3)$$

$$6 + 2(3x + 4) + x$$

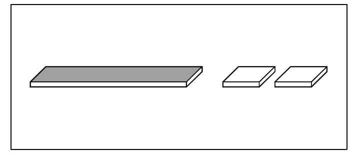
As we can see, the operations inside the parentheses cannot be immediately completed. *The symbols inside the parentheses still represent a group*. We can use the distributive property to finish the multiplication; then we combine similar terms. Here are the same examples worked out:

Example 1:

$$3(x + 5) = 3(x) + 3(5)$$
$$= 3x + 15$$



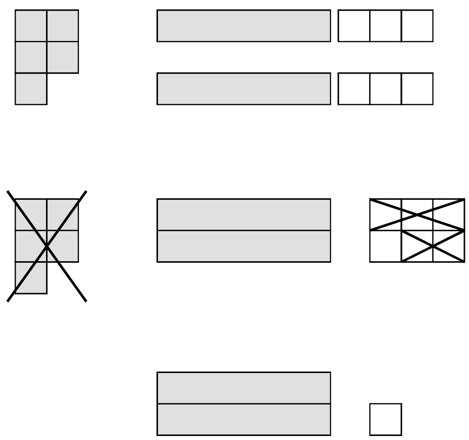
The distributive property shown in this diagram states that three groups of $(x + 5)$ is the same as $3x$ and 15. Three times the whole quantity is the same as three of each term.



Example 2: $5 + 2(x - 3) = 5 + 2(x + {}^{-}3)$

$= 5 + 2x + {}^{-}6$

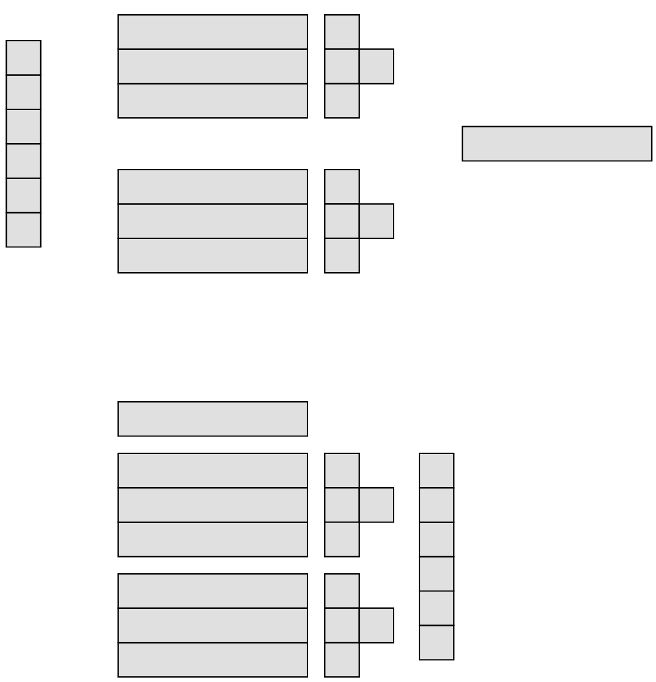
$= 2x - 1$

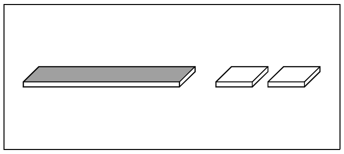


Example 3: $6 + 2(3x + 4) + x = 6 + 2(3x + 4) + x$

$= 6 + 6x + 8 + x$

$= 7x + 14$





Negative Signs and Multiplication

Consider

$$3 - 6(x + 3)$$

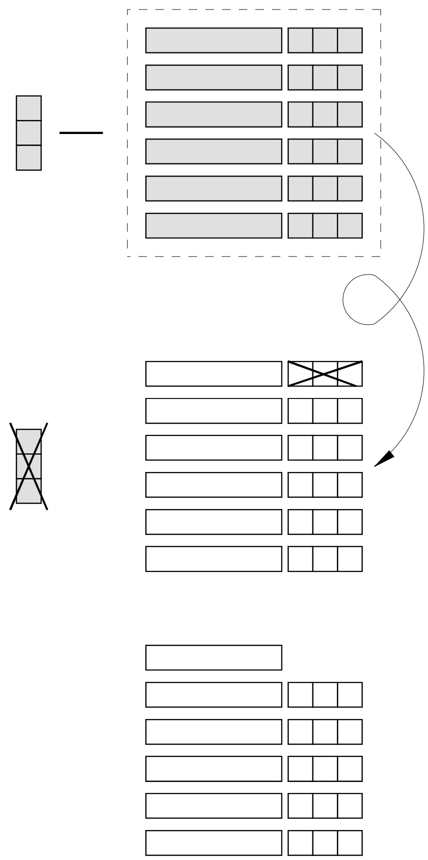
The best way to work with this expression is to rewrite the subtraction as an addition:

$$\begin{aligned} 3 - 6(x + 3) &= 3 + {}^-6(x + 3) \\ &= 3 + {}^-6x + {}^-18 \\ &= {}^-6x + {}^-15 \end{aligned}$$

$3 - 6(x + 3)$

$3 + {}^-6x + {}^-18$

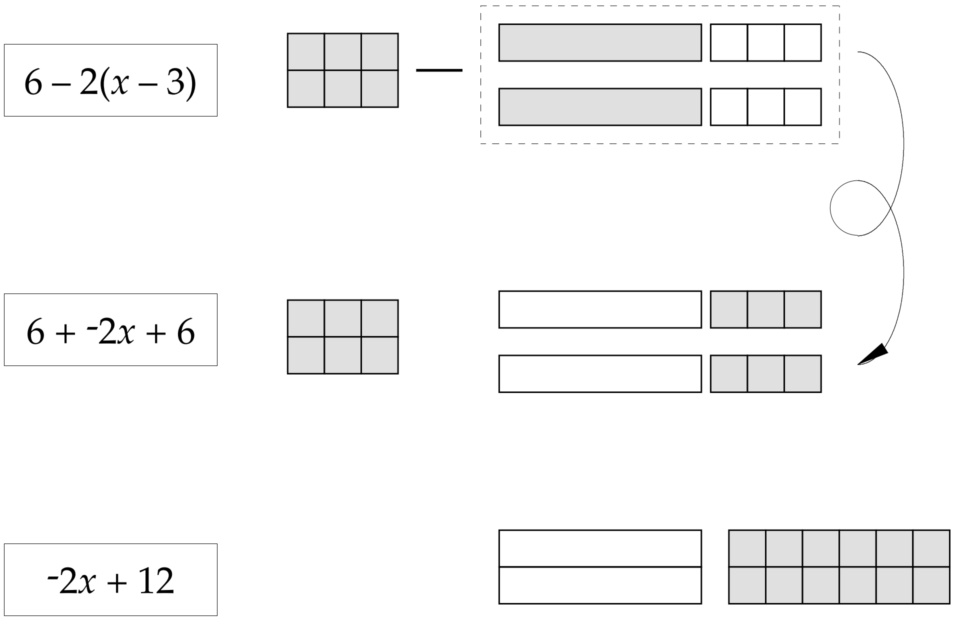
${}^-6x + {}^-15$

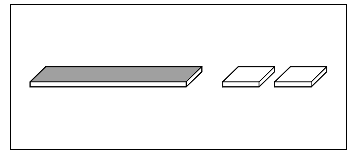


This technique is especially helpful where multiple negative signs are present:

$$\begin{aligned} 6 - 2(x - 3) &= 6 + ({}^-2)(x + {}^-3) \\ &= 6 + {}^-2x + 6 \\ &= {}^-2x + 12 \end{aligned}$$

Here is how this process is shown with the chips:





Summary

To simplify expressions containing parentheses:

- **Rewrite subtractions as additions.**
- **Carry out multiplications using the distributive property.**
- **Combine similar terms.**

Exercises

Simplify the following expressions:

1. $5(x - 3) + 2(3x + 1)$

2. $3 - (x + 5)$

3. $3 - 2(x - 5)$

4. $3 - 2(-x - 5)$

5. $-x - 3x + 4(5 + x)$

Simplify the expressions, then evaluate:

6.	$6x - 2(x + 4)$	$(x = 1)$
7.	$6 - (x - 5) - 3x$	$(x = {}^-1)$
8.	$x + 0(196x - 235)$	$(x = 256)$
9.	$2(3x + 2) - 5(3 + x) + 11$	$(x = {}^-17)$

Section 5
Expressions Containing Fractions

Fractions and Unknowns

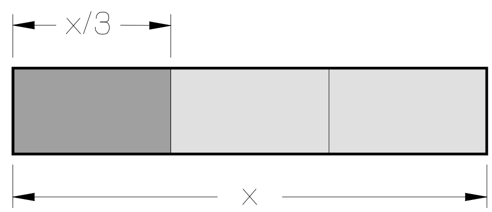
When we wish to represent part of an unknown, we use the same fractional notation that we use with everyday numbers:

$$\frac{1}{3} \text{ of } 6 \text{ means } \frac{1}{3} \cdot 6$$

$$\frac{1}{2} \text{ of } 7 \text{ means } \frac{1}{2} \cdot 7$$

$$\text{So } \frac{1}{2} \text{ of } x \text{ means } \frac{1}{2} \cdot x = \frac{1}{2} \cdot \frac{x}{1} = \frac{1 \cdot x}{2} = \frac{x}{2}$$

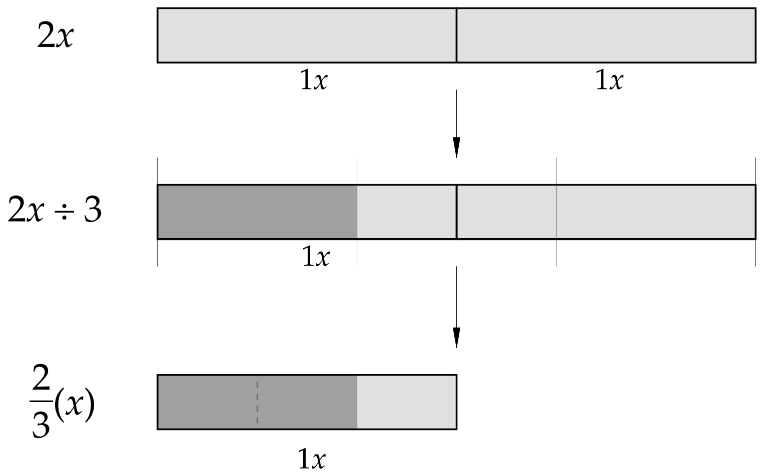
One-third of x would look like this:



There are often several ways to show these fractional unknowns:

Meaning	Alternative Notations		
$3/5$ of x	$\frac{3}{5} \cdot x$	$\frac{3}{5}x$	$\frac{3x}{5}$
$2/3$ of x	$\frac{2}{3} \cdot x$	$\frac{2}{3}x$	$\frac{2x}{3}$
$-2/3$ of x	$-\frac{2}{3} \cdot x$	$-\frac{2}{3}x$	$\frac{-2x}{3}$

If the alternatives seem to represent different quantities, here is a demonstration of the reasons why $^{(2x)}\!/_3$ is equal to $^2\!/_3(x)$:



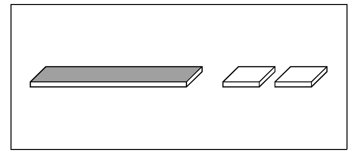
Simplifying Expressions with Fractional Unknowns

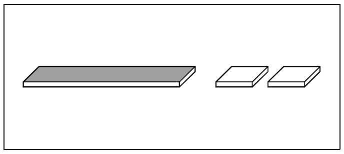
Fractional unknowns are simplified in the same way as numbers. When adding, find common denominators and then combine:

$$\begin{aligned}
\frac{2x}{3}+\frac{3x}{4} &= \frac{2x}{3}\cdot\frac{4}{4}+\frac{3x}{4}\cdot\frac{3}{3}\\
&= \frac{2x\cdot 4}{3\cdot 4}+\frac{3x\cdot 3}{4\cdot 3}\\
&= \frac{8x}{12}+\frac{9x}{12}\\
&= \frac{8x+9x}{12}\\
&= \frac{17x}{12}
\end{aligned}$$

When multiplying, use the same technique that we used with regular fractions:

$$\begin{aligned}
\frac{2x}{3}\cdot\frac{3}{4} &= \frac{2x\cdot 3}{3\cdot 4}\\
&= \frac{6x}{12}\\
&= \frac{6}{12}\cdot x\\
&= \frac{1}{2}x\\
&= \frac{x}{2}
\end{aligned}$$





Exercises

Simplify these expressions. Find common denominators as needed.

1. $\frac{x}{2}+\frac{x}{4}$

2. $\frac{x}{2}-\frac{3x}{4}$

3. $\frac{1}{2}\cdot\frac{x}{3}\cdot\frac{1}{4}$

4. $\frac{1}{2}\div\frac{3}{x}$

5. $\frac{1}{2}(2x+4)$

6. $\frac{1}{2}\left(2x+\frac{1}{2}\right)$

7. $\frac{1}{2}(6x)$

8. $\frac{2}{3}\cdot\frac{3x}{2}$

9. $\frac{3}{4}\cdot\frac{4x}{3}$

10. $12\left(\frac{x}{2}+\frac{x}{3}+\frac{5x}{6}\right)$

11. $\frac{2x}{3}+\frac{x}{5}$

12. $\frac{3x}{2}+\frac{2x}{3}$

13. $\frac{5x}{2}-x$

14. $\frac{5x}{2}-\frac{x}{3}$

15. $\frac{2}{3}\left(\frac{x}{6}\right)$

16. $\frac{3}{5}\left(\frac{2x}{9}\right)$

17. $\frac{1}{3}(6x-5)$

18. $\frac{3}{5}(10x+5)$

Section 6
Properties of Expressions

Properties of Numbers

All of the associative, commutative, and distributive properties described in the previous chapter are true for expressions as well as for numbers. Because the unknowns represent numbers, there is usually no need to state separate properties for expressions.

Most of the following ideas are so intuitive that we have already used them without noticing anything new. It may be helpful, however, to restate some of the properties in a more formal manner.

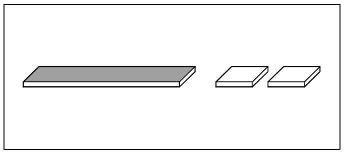
Commutative and Associative Properties

These properties concern the *order* and *grouping* of numbers or terms and are most useful in combining similar terms. For example, consider this illustration from a previous section:

$6x \cdot (^-5)$	$= (6 \cdot x) \cdot (^-5)$	
	$= (x \cdot 6) \cdot (^-5)$	Commutative Prop of Mult
	$= x \cdot 6 \cdot (^-5)$	
	$= x \cdot [6 \cdot (^-5)]$	Assoc. Prop. of Mult
	$= x \cdot (^-30)$	
	$= {^-30} \cdot x$	Comm. Prop. of Mult.
	$= {^-30}x$	

Because the parts of an expression that are being added or multiplied can be rearranged in different orders and groupings, we can more easily combine similar terms.

> **Commutative and Associative Properties**
>
> When *adding* or *multiplying* terms in an expression, the *order* or *grouping* of terms may be changed without affecting the value of the expression.



Distributive Properties

We have easily decided that

$$3x + 4x = 7x$$

More formally, this can be justified by the distributive property:

$$\begin{aligned} 3x + 4x &= 3(x) + 4(x) \\ &= (3 + 4)(x) \\ &= (7)(x) = 7x \end{aligned}$$

We have also been using this property in subtraction problems:

$$\begin{aligned} 3x - 4x &= (3 - 4)x \\ &= ({}^{-}1)x = -x \end{aligned}$$

We can use the idea in division problems and with fractions:

$$\begin{aligned} \frac{6x + 4}{2} &= \frac{6x}{2} + \frac{4}{2} \\ &= 3x + 2 \end{aligned}$$

$$\begin{aligned} \frac{6x - 4}{2} &= \frac{6x}{2} - \frac{4}{2} \\ &= 3x - 2 \end{aligned}$$

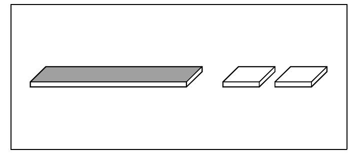
Expressions have the same properties as numbers because expressions *represent* numbers.

Distributive Properties

$$\mathbf{3x + 4x = (3 + 4)x = 7x}$$
$$\mathbf{3x - 4x = (3 - 4)x = {}^{-}1x}$$
$$\mathbf{\frac{6x - 4}{2} = \frac{6x}{2} - \frac{4}{2}}$$

or for any numbers a, b, and c, (c not zero)

$$\mathbf{ax + bx = (a + b)x}$$
$$\mathbf{ax - bx = (a - b)x}$$
$$\mathbf{\frac{ax - b}{c} = \frac{ax}{c} - \frac{b}{c}}$$



Properties of One and Negative One

For multiplication, one is the identity element. One times any number is the original number. With expressions, the property is of course the same:

$$1 \cdot 3x = 3x$$

$$1 \cdot (^-5x) = {}^-5x$$

$$(7x + 2) \cdot 1 = 7x + 2$$

$$1 \cdot x = 1x = x$$

We have also seen from the POSITIVE AND NEGATIVE NUMBERS chapter that multiplying $^-1$ times any number results in the opposite of that number:

$$\begin{aligned} {}^-1 \cdot 3x &= {}^-1 \cdot 3 \cdot x \\ &= (^-1 \cdot 3)x \\ &= (^-3)x \\ &= {}^-3x \end{aligned}$$

Finally, it is most important to understand that multiplying $^-1$ times x is $-x$:

$$^-1 \cdot x = -x$$

Properties of One

$$\mathbf{(1)(3x) = 3x}$$
$$\mathbf{(1)(x) = x}$$
$$\mathbf{(^-1)(x) = -x}$$

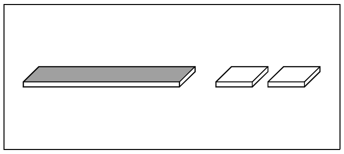
Properties of Zero

When zero is multiplied times an unknown quantity, we are taking an unknown number of zeros. The result is always zero:

$$0 \cdot 3x = 0$$

$$^-3x \cdot 0 = 0$$

$$(3x + 6) \cdot 0 = 0$$



Adding zero to any expression does not change the value of the expression:

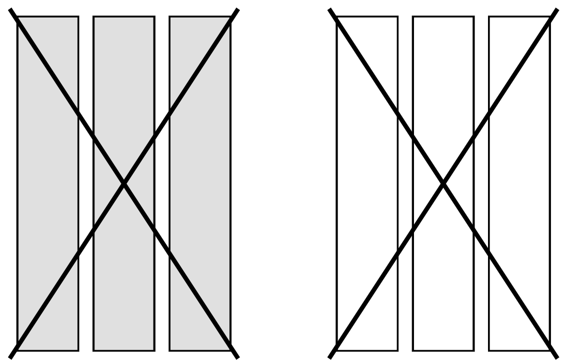
$$0 + 3x = 3x$$

$$^-3x + 0 = {}^-3x$$

$$0 + (4x - 73) = 4x - 73$$

If we add opposites together, they will cancel to zero:

$$3x + {}^-3x = 0$$



A more formal proof of this fact uses the distributive property:

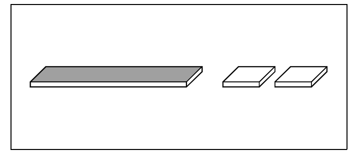
$$\begin{aligned} 3x + {}^-3x &= (3)x + ({}^-3)x \\ &= (3 + {}^-3)x \\ &= (0)x \\ &= 0 \end{aligned}$$

The opposite of any number of x-bars is the same number of -x bars.

Properties of Zero

$$\mathbf{0 + x = x}$$
$$\mathbf{(0)(x) = 0}$$
$$\mathbf{ax + {}^-ax = 0}$$

Order of Operations With Multiple Parentheses



When expressions have multiple levels of parentheses then, as before, we begin simplifying starting from the innermost parentheses and working our way out. For example, let's simplify

$$7x + 3[2x - 5(x - 3 - 2x)]$$

$7x + 3[2x - 5(- x - 3\ )]$	First we *combine like terms* inside the innermost
rentheses.	
$7x + 3[2x + 5x + 15]$	Then we *multiply through* the innermost parentheses. Now the inner parentheses are gone.
$7x + 3(7x + 15)$	Next we combine like terms inside the remaining parentheses.
$7x + 21x + 45$	Then we multiply through the remaining parentheses.
$28x + 45$	When all parentheses are gone, combine like terms (if necessary) in the remaining expression.

As with other expressions having unknowns and numbers, the final result usually has two terms (one with a letter and one without) which cannot be combined.

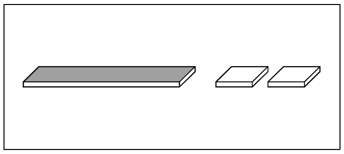
Exercises

Simplify the following expressions. Justify each step by referring to the appropriate property.

Example: $2(3x + 4) - 6x$

Solution:

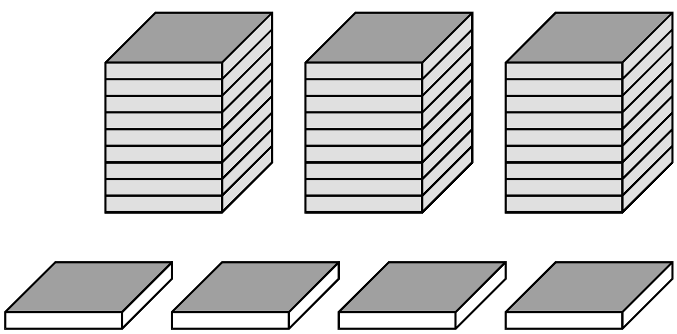
$2(3x + 4) - 6x$	$= 2(3x) + 2(4) - 6x$	Distributive
	$= 6x + 8 - 6x$	
	$= 8 + 6x - 6x$	Commutative
	$= 8 + 0$	Inverses
	$= 8$	



1. $5(x + 6) + x$
2. $2 - (3 + 2x)$
3. $-x(3) + 3x + 22 + x$
4. $9y - 3y + (6)(-y)$
5. $\frac{1}{3}x + \frac{2}{3}x$
6. $2 - 5(3 + x)$
7. $^-2 - 6(3 - x)$
8. $4x + 2x + {}^-2x + {}^-4x$
9. $\frac{6x + 2}{2} - 1$
10. $\frac{^-3x - 6}{^-3} + x$
11. $\frac{8x}{2} + \frac{9x}{3}$
12. $\frac{12x + 6}{6} + x + 1$
13. $(2 - 1)(3x)$
14. $-x + 4(x - 3)$
15. $\frac{4x + 4}{4}$
16. $\frac{6x - 6}{6}$
17. $\frac{6x + 1}{6}$
18. $(^-1)(5x)$
19. $(5)(7x)$
20. $(7x)(5)$
21. $(^-7x)(^-5)$
22. $(7x)(^-5)$
23. $(0)(3)(12)(6x)$
24. $24(3 - 3)(6x)$
25. $\frac{(6x)(0)}{6}$
26. $5[3x - 2(x + 7)]$
27. $4[3(2 - 3x) + 6x]$
28. $7 - [2x + 5(6 - 3x)]$
29. $^-3x + 4[6 + 3(2x - 9)]$
30. $4x + 2[5 - 2(3x - 7)]$

Chapter 7

Equations



Section 1
Introduction to Equations

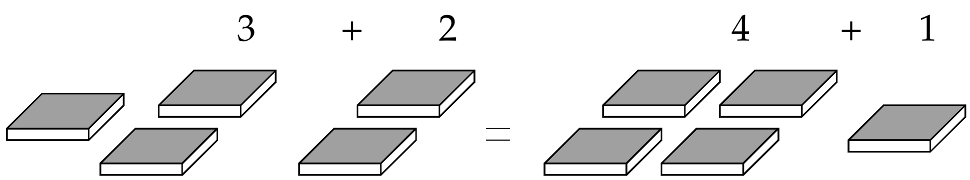
Equations

An **equation** is a number statement which says that two quantities are exactly the same. The symbol **=** (equals) is used between the quantities to show that the amount on the left is the same as the amount on the right. For example:

$$3 + 2 = 4 + 1.$$

Both the numbers on the left and the numbers on the right can be combined to give 5. So the equation really says

$$5 = 5 \quad \text{or} \quad \text{“Five equals Five”}$$



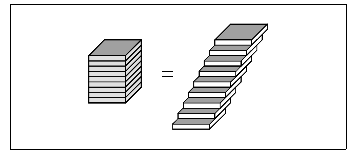
This is obvious if we know all of the numbers on both sides of the equation, but what if the equation has unknowns? When unknowns are included, we can use the fact that both sides are equal to find out the missing amount.

Equations versus Expressions

In the previous chapter, we worked with expressions that involved unknowns. *An expression is a quantity, while an equation is a statement that two quantities are equal.*

Expressions	Equations
$3x + 6$	$x + 3 = 5$
$^{-}17$	$2x - 3 = 15$
$3(5x - 4) + 17x$	$3(5x - 4) + 17x = 20$

An expression can stand for many different amounts, depending on what we choose for the unknown; in an equation, the unknown can only stand for numbers that make both sides of the equation equal. Here is a summary of the differences between equations and expressions:



	Expressions	Equations
Quantities	One quantity	Two amounts that are equal
Equals Sign	No	Yes
Meaning of Unknown	Many choices	Values that make the statement true

Exercises

Decide whether each item is an expression or an equation:

1. $x + 3$

2. $2(x - 5) + 7$

3. $0 = 0$

4. $2(x - 5) = 2$

5. $2(x - 5) = 2(x - 5)$

6. $\frac{1}{6}$

7. $\frac{3x}{5} = 12$

8. $3(x + 5) - 16x + 23$

9. $^{-}1$

10. 0

11. $0 = 0$

12. $3x + 2 = {}^{-}1$

13. $\frac{3x + 12}{x + 16}$

14. $x = y$

15. $y = 1$

Section 2
The Equation Game

Introduction

This game will help you to understand the meaning of equations and the methods by which they can be solved. As in many of the other sections of this book, you may find that you can easily discover the techniques of solving equations; in fact, you may already know a great deal about the subject.

A sample of this game was presented in the INTRODUCTION. We will now give more detailed rules and examples.

The Rules of the Game

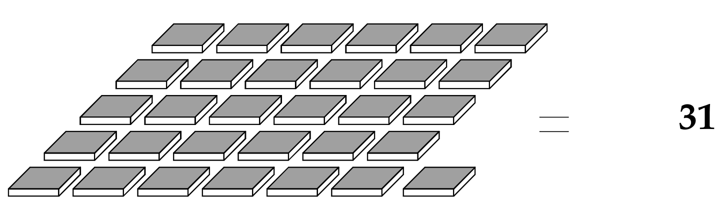
The game can be played alone or with a partner. You can pose equations and solve them yourself, or you can set up equations for your partner to solve.

- **Begin by counting out any number of chips and writing the total in large numerals on one-half of a clean sheet of paper. A number between 10 and 50 chips works best.**

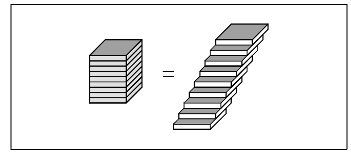
- **Divide up most (but usually not all) of the chips into a small number of stacks which are exactly equal in height.** *If you are playing alone, do not count the number of chips in a stack; you can tell if the stacks are equal by feeling the height of the chips.* **Place these stacks on the other half of the paper with the remaining chips arranged singly next to the stacks.**

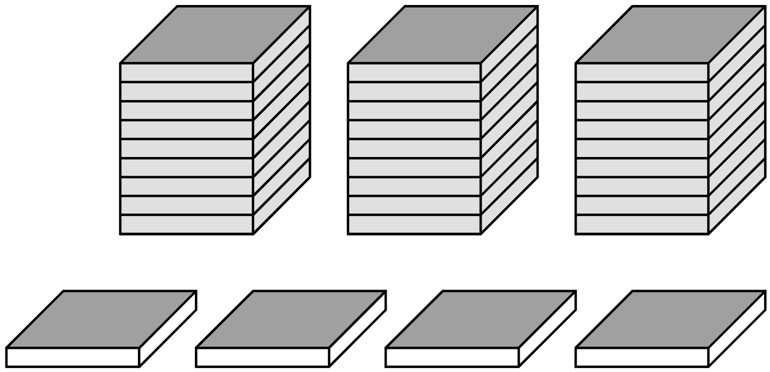
- **Without counting, determine how many chips are in a stack. We know that the total number of chips (stacks and single chips) equals the number written on the paper; use this information to discover how many chips are in a stack. Check the result by counting chips in the stack. If you are not correct, check that the stacks were the same height and that the total number of chips is correct.**

For example, count out 31 chips and write **31** on the paper.



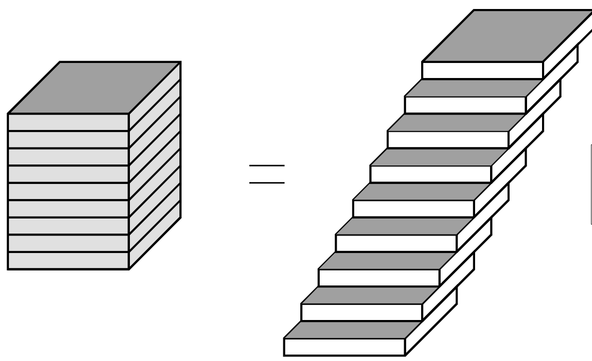
Lay out 4 chips singly and arrange the other chips into 3 equal stacks.





= 31

Calculate the number of chips in each stack. Thirty-one (31) chips minus the 4 extra gives 27 chips, and 27 divided into 3 equal stacks is 9. Check your answer by counting the chips in a stack.

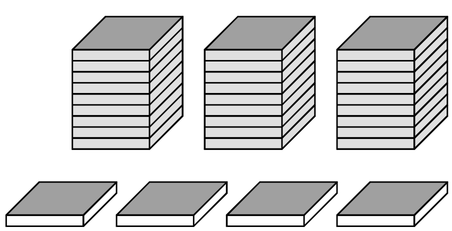


$x = 9$

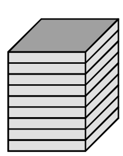
This process is called **solving an equation**. We write the equation as

$$3x + 4 = 31$$

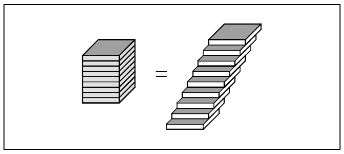
where $\boldsymbol{x}$ is a stack, $\boldsymbol{3x}$ is 3 stacks, and **4** is the 4 single chips.



= 31



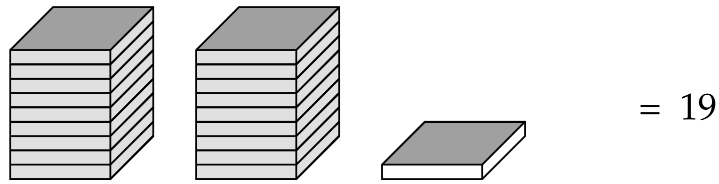
= 9



Exercises

Here are some sample equation games to play. Set up the chips, calculate the solution, and check your answer by counting the chips in a stack.

Example: $2x + 1 = 19$
Solution: $x = 9$



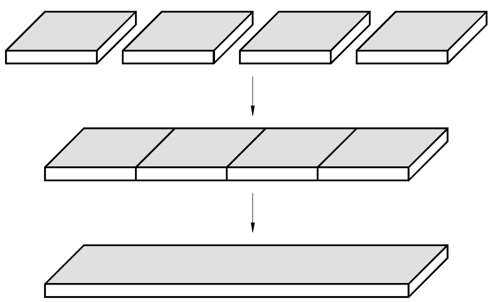
1. 23 chips: 4 stacks and 3 singles. $(4x + 3 = 23)$
2. 17 chips: 2 stacks and 1 single. $(2x + 1 = 17)$
3. 35 chips: 4 stacks and 3 singles.
4. 12 chips: 1 stack and 5 singles.
5. 29 chips:
   (You arrange stacks and singles. All stacks are the same height.)
6. 32 chips: 3 stacks and 11 singles.
7. 23 chips: 2 stacks and 5 singles.
8. 27 chips: 4 stacks and 7 singles.
9. 27 chips: 6 stacks and 3 singles.
10. 27 chips: 4 stacks and 3 singles.
11. 18 chips: 3 stacks and 3 singles.
12. 41 chips: 3 stacks and 5 singles.
13. 32 chips: 4 stacks and 4 singles.
14. 51 chips: 5 stacks and 6 singles.
15. 40 chips: 3 stacks and 1 single.
16. 47 chips: 2 stacks and 5 singles.
17. 38 chips: 5 stacks and 3 singles.
18. 50 chips: 6 stacks and 2 singles.
19. 35 chips: 4 stacks and 3 singles.
20. 29 chips: 3 stacks and 2 singles.

Section 3
Equations Using Unknowns

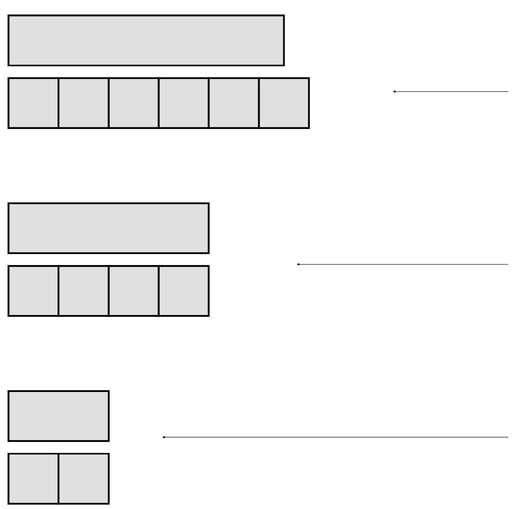
Using the Bar as *x*

As we learned in the last chapter, we can also represent an unknown amount with the long bar found in your packet.

Instead of a stack of chips in a pile, the bar represents a group of chips in a line:



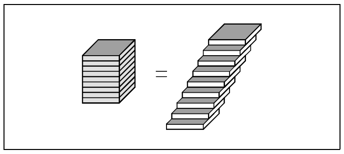
The bar *represents any unknown number of chips; you can imagine that it changes length in each example:*



Actual Proportions: The bar is 5 ½ units long.

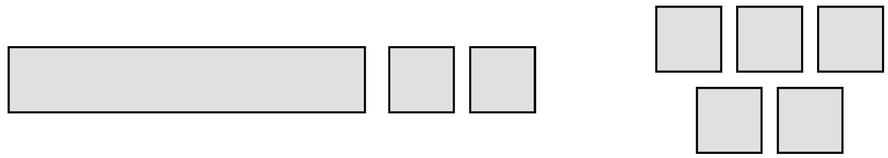
But it may stand for a quantity of 4 units.

Or it may stand for 2.



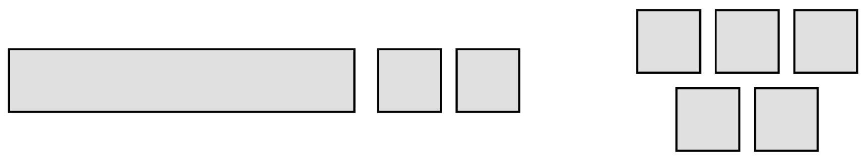
Equations Using Bars

In the following equation, what number does the bar represent?

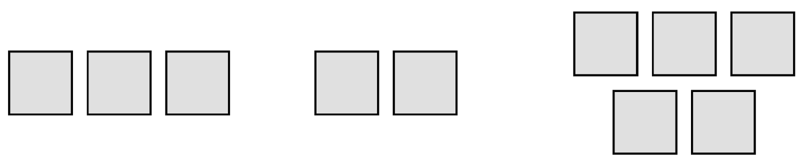


$$x + 2 = 5$$

We are trying to find out what number of chips are needed to replace the bar so that both sides of the equation are equal. The answer is 3, so the bar represents three chips:



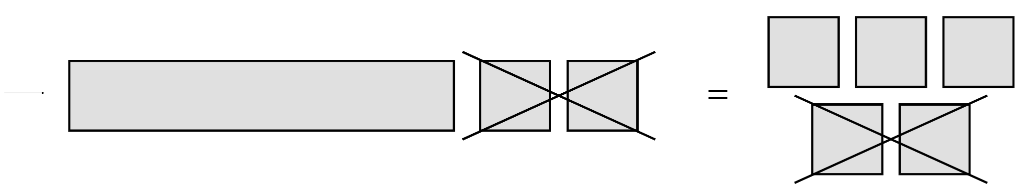
$$x + 2 = 5$$



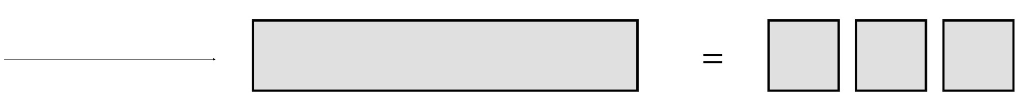
$$3 + 2 = 5$$

The best way to find the answer is to take chips away from each side until one side has only the bar left. In this case, we take 2 chips away and the bar must then be equal to 3 chips.

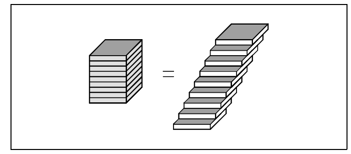
Take 2 chips away from each side.



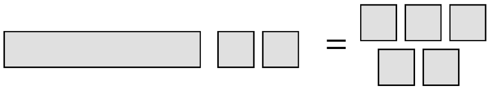
This leaves the bar equal to 3.



To check, replace the bar with 3 chips and make sure that there are equal numbers of chips on both sides.

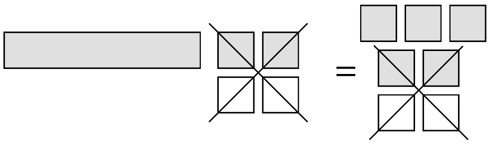


Another way to solve this equation is to add 2 negative chips to each side. Here is the process along with the algebra symbols that we will use:



$x + 2 = 5$

Represent the equation using a bar (unknown) and unit chips.



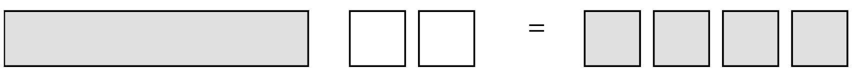
$x + 2 + {}^{-}2 = 5 + {}^{-}2$

Add $^{-}2$ to both sides.

$x = 3$

This leaves the solution: the value for the unknown.

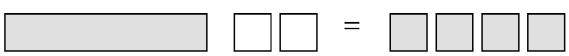
Here is a slightly different example:



$y + ({}^{-}2) = 4$

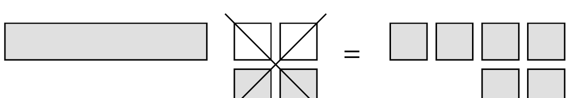
In order to find the value of the unknown (called ***y*** for variety) we look for a number that, when combined with $^{-}2$, becomes 4. The answer is $^{+}6$.

To solve this more easily, we can work to isolate the *y* bar by *adding* 2 positive chips to each side. This will cancel the negative chips and will help us discover the answer:



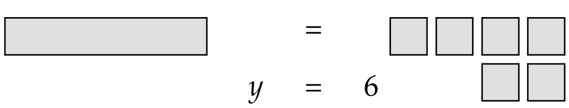
$y - 2 = 4$

Represent the equation using a bar (unknown) and unit chips.



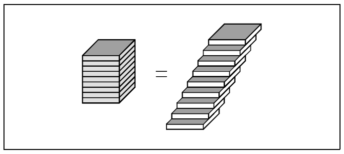
$y - 2 + 2 = 4 + 2$

Add $^{+}2$ to both sides.



$y = 6$

This leaves the solution: $y = 6$



Remember that

$$6 + {}^{-}2 = 6 - 2$$

You can use either form, but with chips it is often easier to represent the idea of adding ${}^{-}2$.

To find the solution:

- **"Isolate" the unknown by adding unit chips to both sides so that the chips other than the bar are cancelled out.**
- **Use positive chips to cancel negative chips, and negative chips to cancel positive chips.**
- **You are done when you have the bar alone, equal to a number of chips.**

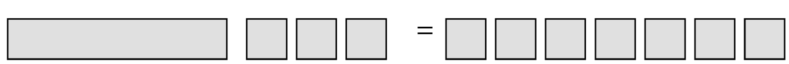
To cancel out units:

Add the Opposite

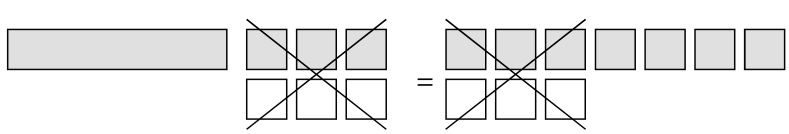
Exercises

Practice on these examples. Use the chips and also write out the algebra symbols for each problem.

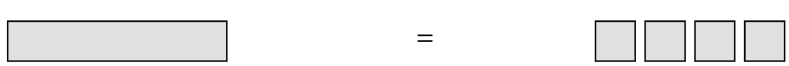
Example: $x + 3 = 7$
Solution:



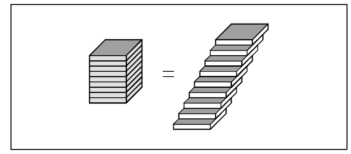
$x + 3 \quad = \quad 7$



$x + 3 - 3 \quad = \quad 7 - 3$



$x = 4$



1. $x + {}^-4 = 5$

2. $x + {}^-2 = {}^-3$

3. $y + 5 = 2$

4. $n - 4 = {}^-1$

5. $y + 2 = 2$

6. $x - 7 = 5$

7. $11 + x = 12$

8. $3 + y = {}^-13$

9. $1 + y = 0$

10. $x - 12 = 11$

11. $x + 12 = {}^-3$

12. $y - ({}^-3) = 5$

13. $y + ({}^-3) = 5$

14. ${}^-2 + x = 13$

15. $x + 0 = 0$

16. $x - 5 = 12$

17. $x + 5 = 12$

18. $y - 2 = {}^-3$

19. $y + 2 = {}^-3$

20. $5 + x = {}^-2$

21. $7 + y = 4$

22. $n + 6 = 5$

23. $n - 3 = 5$

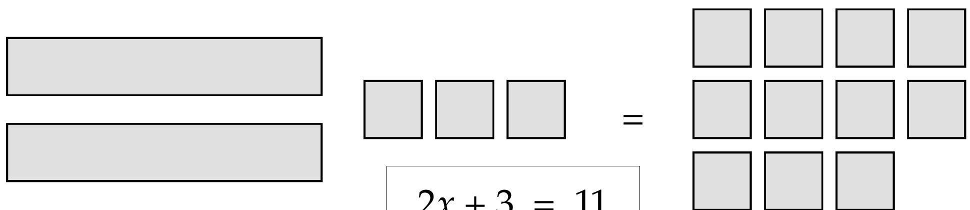
24. $x - 7 = {}^-7$

25. $x + 7 = {}^-7$

Section 4
Equations with Multiples of Unknowns

More than One Unknown

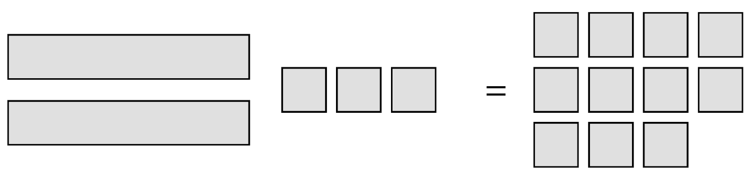
An equation may be more complicated than those that we have looked at thus far. For example, an equation may contain more than one x:



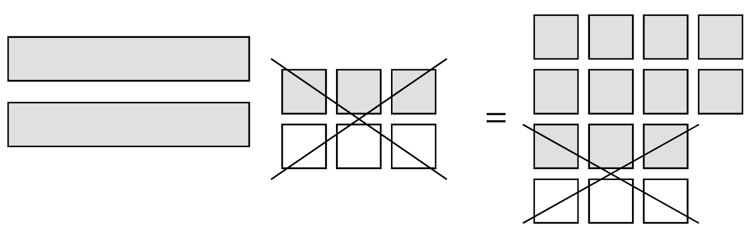
$$2x + 3 = 11$$

The first step in solving the equation is the same as for the simpler equations—we add $^-3$ to each side to isolate the unknowns. This gives:

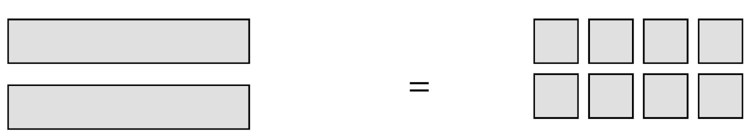
$$2x + 3 = 11$$

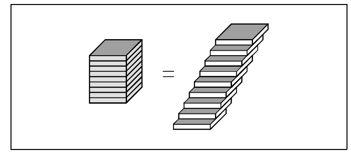


$$2x + 3 - 3 = 11 - 3$$

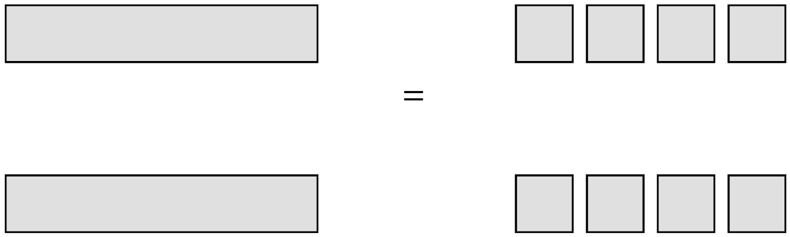


$$2x = 8$$

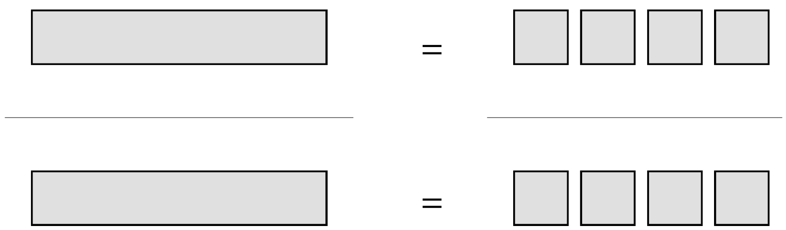




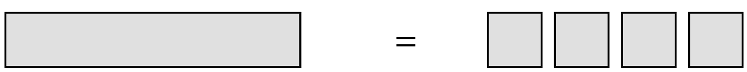
Although we now know what $2x$ is, we would like to know the value of x itself. Because the two sides are equal, we can split up the x bars into 2 groups and the unit chips into 2 groups.



If the two sides are equal, then we can match up half of one side with half of the other side:



$$\frac{1}{2}(2x) = \frac{1}{2}(8)$$

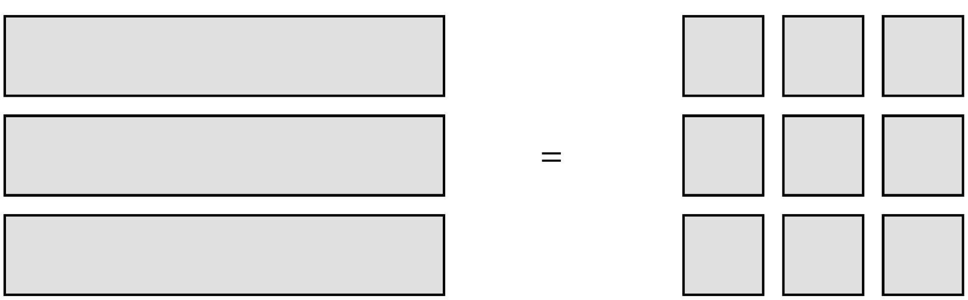


$$x = 4$$

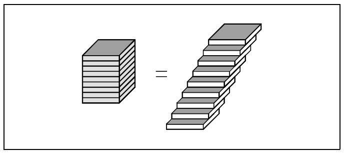
Our solution is 4.

Now we will do another example. Consider

$$3x = 9$$

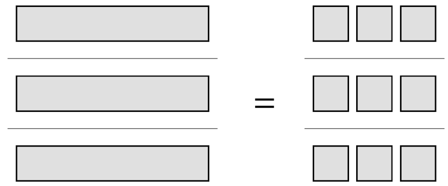


$$3x \quad = \quad 9$$

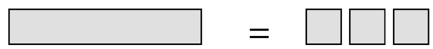


We divide each side into 3 groups and match up one group on each side giving x equal to 3:

$$\frac{1}{3}(3x) = \frac{1}{3}(9)$$

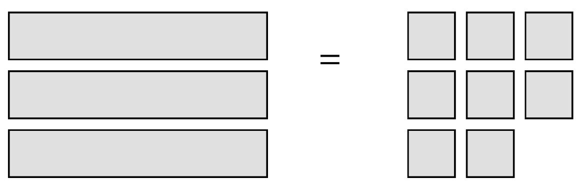


$$x = 3$$



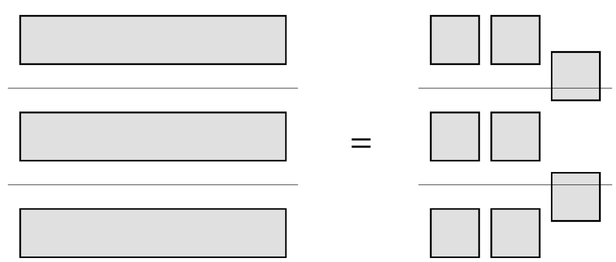
Equations that Result in Fractions

Sometimes an equation will result in a situation where the chips cannot be divided evenly into the desired number of groups. In these cases, the answer will contain a fraction. For example, $3x = 8$:

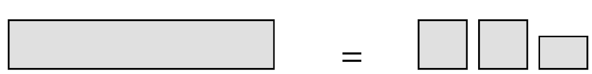


When we divide both sides into thirds, some chips on the right side must be cut to get 3 equal parts. The result is that $x = 2\,\tfrac{2}{3}$:

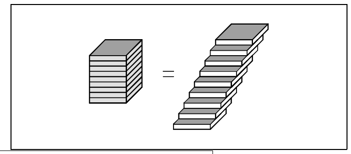
$$\frac{1}{3}(3x) = \frac{1}{3}(8)$$



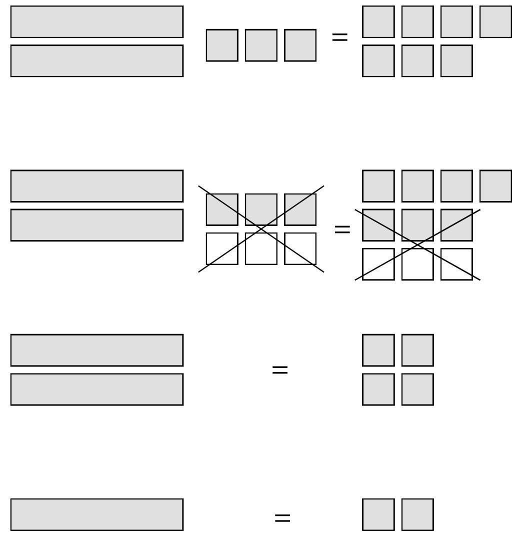
$$x = 2\frac{2}{3} = \frac{8}{3}$$



Exercises



Use chips to solve these equations. Write out the algebra steps for each problem:



$$2x + 3 = 7$$

$$2x + 3 - 3 = 7 - 3$$

$$2x = 4$$

$$x = 2$$

Example: $2x + 3 = 7$

Solution:

1. $3x + 5 = 17$

2. $3x + 4 = {}^{-}17$

3. $4x - 3 = 5$

4. $5x + 2 = 11$

5. $2y - 9 = {}^{-}5$

6. $6n - 2 = 3$

7. $2b + 5 = 5$

8. $5x + 1 = 11$

9. $2 + 3x = 35$

10. $3x - 2 = 8$

11. $6 + 3y = 21$

12. $0 + 2x = 0$

13. ${}^{-}3 + 5x = {}^{-}13$

14. ${}^{-}2 + 4x = {}^{-}10$

15. $3y - 12 = 12$

16. $7n + 7 = 8$

17. $3x + 1 = 5$

18. $5x = 3$

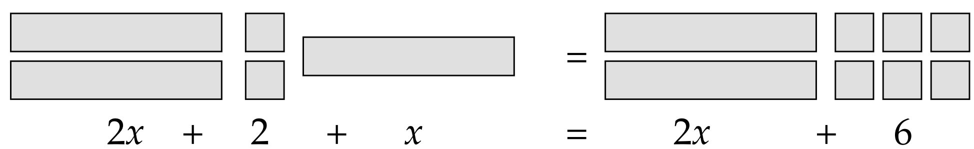
19. $2x - 1 = 0$

Section 5
Unknowns in More than One Term

Keeping the Equation Balanced

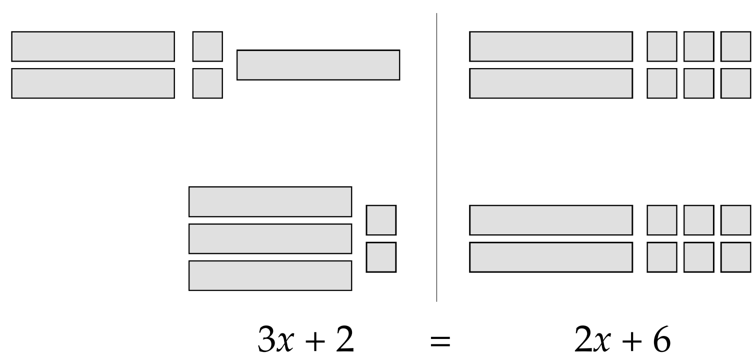
An equation may have unknowns in several places—on one side of the equation or on both sides. Consider the following equation:

$$2x + 2 + x = 2x + 6$$

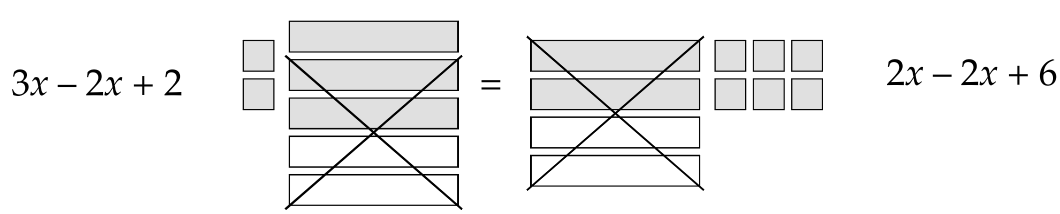


In an equation like this it is important to notice the position of the equals sign because it separates the left and right sides. The equation is like a balance and the amount on the left exactly balances the amount on the right.

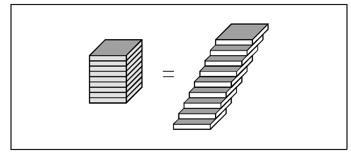
Our first step is to combine like terms on each side:

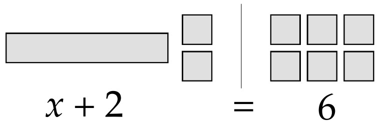


The next step is to add or remove unknowns from both sides. *We must add or remove equally on both sides or the equations will not remain balanced.* Since the right side has less unknowns, we can cancel out these by adding two negative bars to both sides:

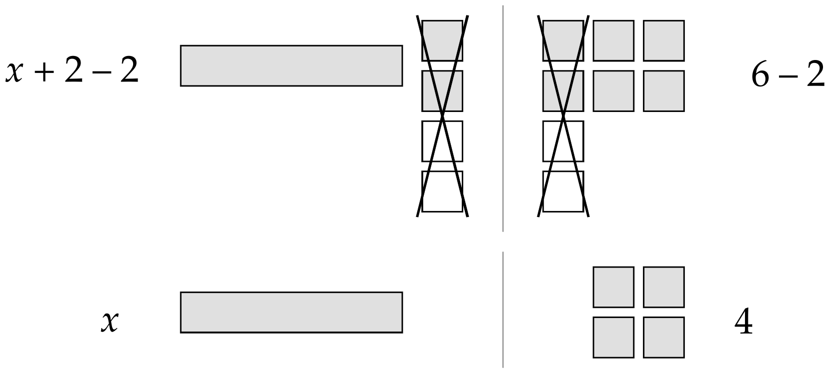


This leaves us with unknowns on one side only. When we combine similar terms, we are left with:



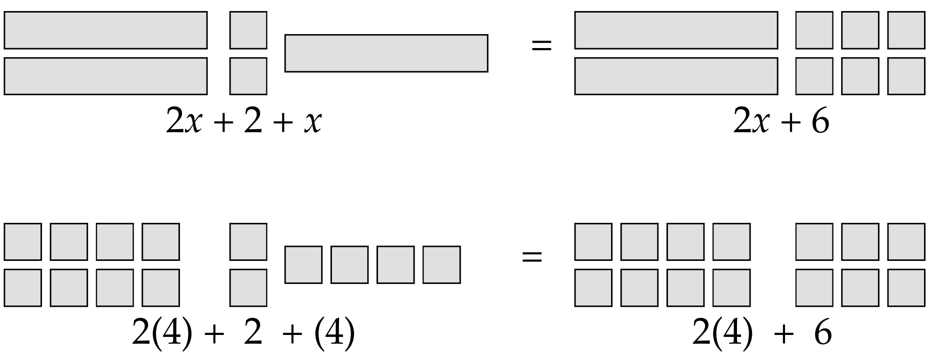


From this point, we can solve the equation exactly as before:



$x = 4$

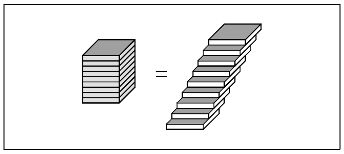
Our solution is that $x = 4$. To check our answer, we replace each x bar on both sides of the equation with 4 chips and then confirm that both sides have a balanced (equal) number of chips:



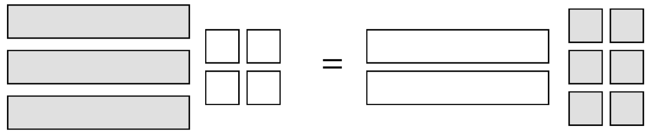
$14 = 14$

To summarize these steps:

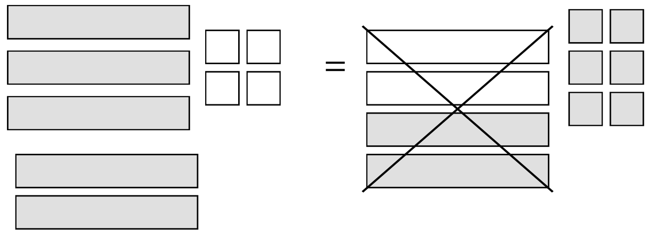
- **Combine similar terms on each side of the equation.**
- **Eliminate the unknowns from one side by adding the opposite type of bars. Add negatives to eliminate positives, and add positives to eliminate negatives.**
- **Add positive or negative chips to cancel out the units and to "isolate" the unknown.**
- **Multiply both sides by ½, ⅓, etc. to match up a single unknown with the correct number of chips.**



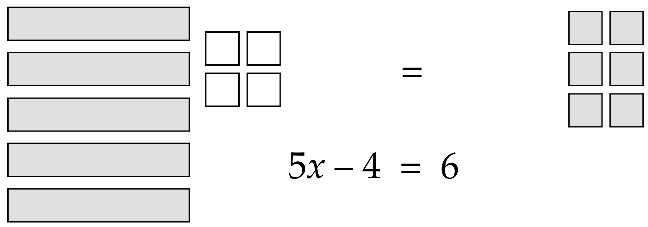
Here is another example:



$$3x - 4 = {}^{-}2x + 6$$

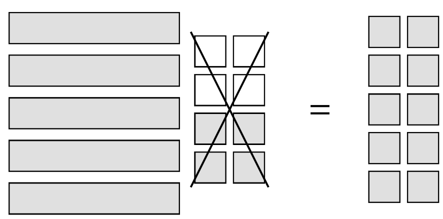


$$3x + 2x - 4 = {}^{-}2x + 2x + 6$$

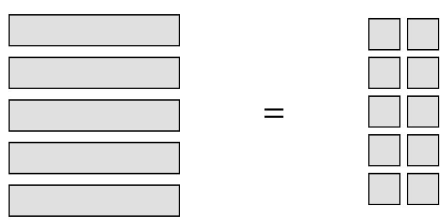


$$5x - 4 = 6$$

We can now solve as before:

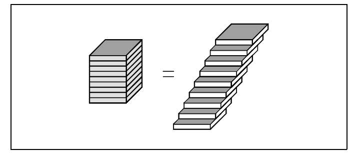


$$5x - 4 + 4 = 6 + 4$$



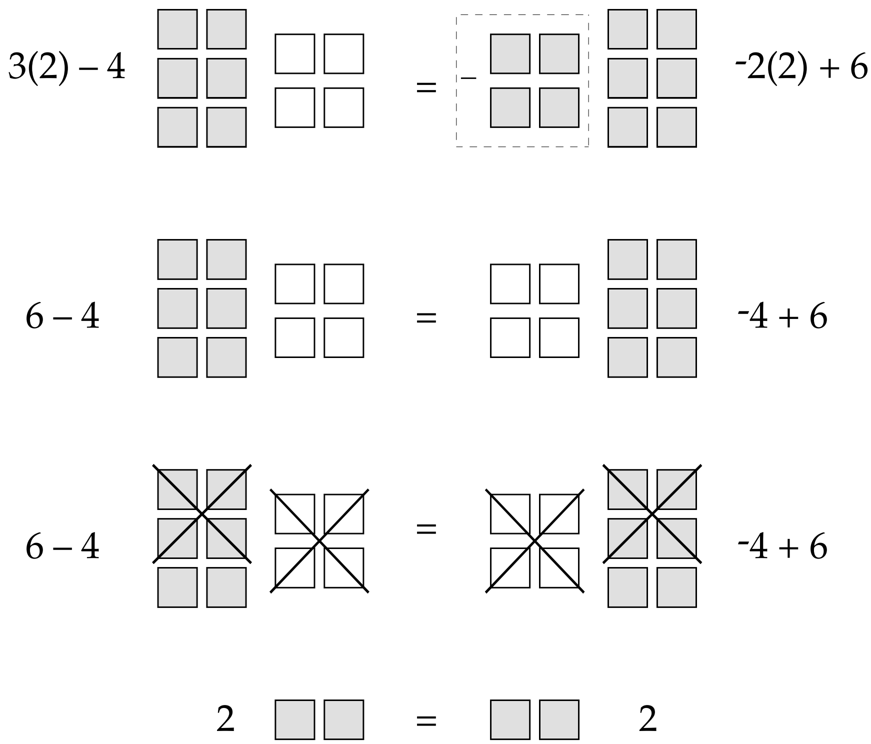
$$5x = 10$$

$$x = 2$$



To check our result of $x = 2$, we replace the x's with 2 chips and the -x's with $^-2$ chips:

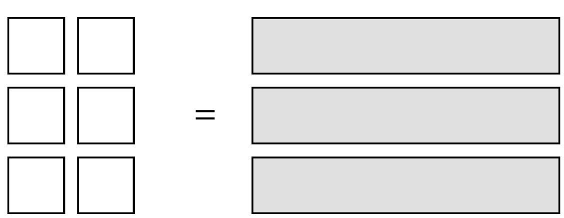
$$\begin{aligned} 3x - 4 &= {}^-2x + 6 \\ 3(2) - 4 &= {}^-2(2) + 6 \\ 6 - 4 &= {}^-4 + 6 \\ 2 &= 2 \end{aligned}$$



$$2 = 2$$

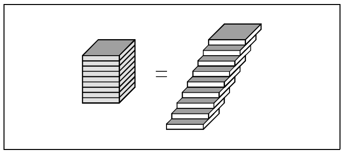
Unknowns on the Right Side of the Equation

When we isolate the unknowns, the unknowns may be on the right side of the equation instead of on the left.



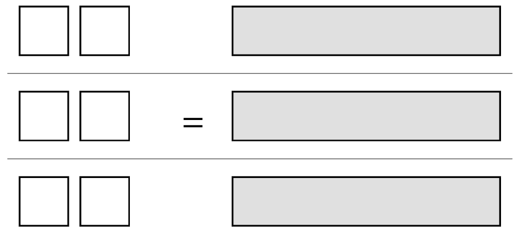
$$^-6 = 3x$$

Because

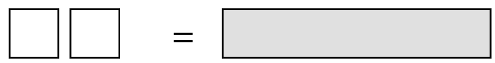


$$^{-}6 = 3x \quad \text{has the same meaning as} \quad 3x = {}^{-}6$$

we do not have to swap the sides of the equation. Instead, we continue solving in the usual way:



$$\frac{1}{3}(^{-}6) = \frac{1}{3}(3x)$$



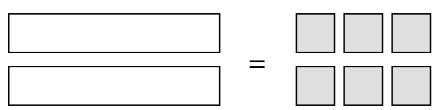
$$^{-}2 = x$$

Negative Unknowns

In the final step of solving an equation, we may be left with negative unknowns:

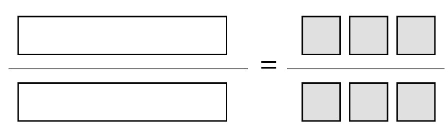
We can use our usual method to isolate the negative x by multiplying both sides by one-half:

$$^{-}2x = 6$$

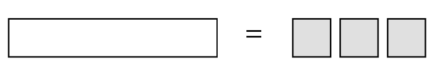


But now we have the value of the *opposite* of x instead of the value of x itself. It is clear that if the opposite of x is 3, then x is $^{-}3$. We can show this physically

$$\frac{1}{2}(^{-}2x) = \frac{1}{2}(6)$$

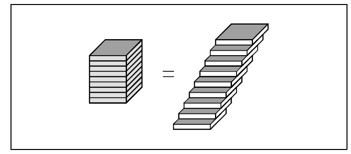


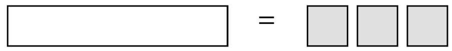
$$-x = 3$$



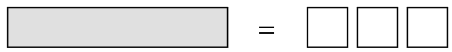
by flipping all of the chips on both sides:

This keeps our equation balanced, because *if two quantities are equal, their opposites are also equal.* With symbols, it is often written as:





$-x = 3$



$x = {}^{-}3$

$$-x = 3$$
$$^{-}1(-x) = {}^{-}1(3)$$
$$x = {}^{-}3$$

Multiplying both sides by negative one can be shown as flipping the chips on both sides.

Exercises

Do these problems using the chips. Write out the steps.

1. $3x + 5 - x = x - 6$

2. $2x - 4 + x = -x + 8$

3. $^{-}2y - 2 + y = 2y + 7$

4. $6 - 3n = n + 5 - 3$

5. $4y - 3 = {}^{-}3 - 6y$

6. $1x + 2x + 3x = 4x + 4$

7. $4 - z - 2z - 3z = {}^{-}20$

8. $6x = 2x - 12$

9. $7x - 5x + x = 14 + x$

10. $10 + x = {}^{-}12x + 5x - 6$

11. $9y + 2 = 6y - 4$

12. $-x = 5$

13. $-x = {}^{-}3$

14. $^{-}7x = {}^{-}3x$

15. $^{-}2x + 6 = x - 9$

16. $-x + 6 = -2x + 2$

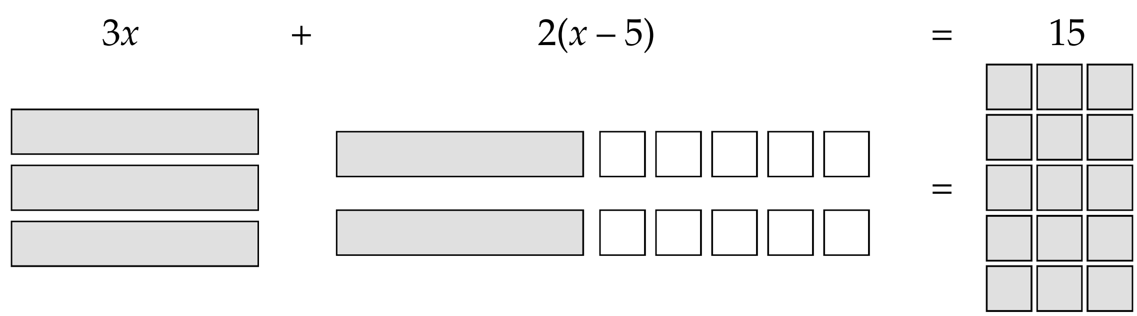
17. $-x + 1 = 5x + 1$

Section 6
Equations with Parentheses

Using the Distributive Property

Some equations may contain complicated expressions including parentheses. For example:

$$3x + 2(x - 5) = 15$$



Notice that the 2 is not a term. It cannot be eliminated by adding $^{-}2$ to both sides. In most cases, it is necessary to use the distributive property to multiply out the expression **$2(x - 5)$** so that we can combine terms and proceed with the solution of the equation:

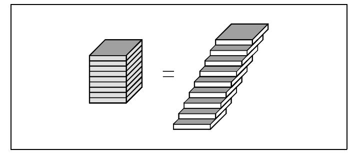
$$\begin{aligned} 3x + (2 \cdot x) - (2 \cdot 5) &= 15 \\ 3x + 2x - 10 &= 15 \\ 5x - 10 &= 15 \\ 5x &= 25 \\ x &= 5 \end{aligned}$$

Consider the equation:

$$5 + 3(x + 2) = 2(x + 1) + 12$$

Again, we cannot eliminate any of the parts of $3(x + 2)$ or $2(x + 1)$ until we multiply out these expressions using the distributive property:

$$
\begin{aligned}
5 + 3(x + 2) &= 2(x + 1) + 12 \\
5 + 3x + (3 \cdot 2) &= (2 \cdot x) + (2 \cdot 1) + 12 \\
5 + 3x + 6 &= 2x + 2 + 12
\end{aligned}
$$



From here on, the procedure is the same as in the previous section:

$$
\begin{aligned}
5 + 3x + 6 &= 2x + 2 + 12 \\
11 + 3x &= 2x + 14 \\
11 + 3x - 11 &= 2x + 14 - 11 \\
3x &= 2x + 3 \\
3x - 2x &= 2x + 3 - 2x \\
x &= 3
\end{aligned}
$$

Subtraction and the Distributive Property

When the product of two amounts is *subtracted* in an equation, we must be careful to use signed numbers correctly. Consider the equation:

$$5 - 2(x + 1) = 1$$

This is not the same as:

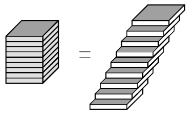
$$5 - 2x + 1 = 1 \quad \text{(Why not?)}$$

Instead, rewrite the subtraction as addition:

$$
\begin{aligned}
5 - 2(x + 1) &= 1 \\
5 + {}^{-}2(x + 1) &= 1 \\
5 + ({}^{-}2 \cdot x) + ({}^{-}2 \cdot 1) &= 1 \\
5 + {}^{-}2x + {}^{-}2 &= 1 \\
3 + {}^{-}2x &= 1 \\
{}^{-}2x &= {}^{-}2 \\
x &= 1
\end{aligned}
$$

If there are two subtractions, you may want to rewrite both as addition:

$$
\begin{aligned}
10 - 3(2x - 5) &= 19 \\
10 + {}^{-}3(2x + {}^{-}5) &= 19 \\
10 + ({}^{-}3 \cdot 2x) + ({}^{-}3 \cdot {}^{-}5) &= 19 \\
10 + {}^{-}6x + 15 &= 19 \\
{}^{-}6x + 25 &= 19 \\
{}^{-}6x &= {}^{-}6 \\
x &= 1
\end{aligned}
$$



Summary

We can now add an initial step to our plan from the previous section:

- ***Use the distributive property to complete any multiplications of expressions in parentheses.***

- **Combine similar terms on each side of the equation.**

- **Eliminate the unknowns from one side by adding the opposite type of bars. Add negatives to eliminate positives, and positives to eliminate negatives.**

- **Add positive or negative chips to cancel out the units and to "isolate" the unknown.**

- **Multiply both sides by ½, ⅓, etc. to match up a single unknown with the correct number of chips.**

Exercises

Solve these equations using chips and algebra symbols:

1. $4(x + 1) - 3 = 3(x - 2) + 13$

2. $5 - 2(x - 2) = 5(x + 1) + 4$

3. $3(2x + 1) - 2(x - 1) = 21$

4. $6(1 - x) + 3 = {}^{-}3x - 3$

5. $3(3 + 2x - 1) = 2x - 1(3 - 2x) + 9$

6. $5(x + 1) - 4x = 2$

7. $2x + 3(x - 2) = 9x + 2$

8. $7y - 6(y - 1) + 3 = 9$

9. $3 - y(1 + 2) = {}^{-}2y - 5$

10. $2(3x + 1) - 5(x + 2) = 1 - 10$

11. $3(7 - 2x) = 14 - 8(x - 1)$

12. $1 - 3(x + 4) = {}^{-}5x - 5$

13. $1 - 1(x - 1) = x$

14. $5x + 6x + 3(x - 2) = {}^{-}6 + x$

15. $3x + 2(3x - 1) = 6x$

16. $5(x - 2) - 3x = 6$

17. $2x - 9 = 3(2 - x)$

18. $5 - 2(1 - x) = 3(x - 4)$

19. $2(4x + 3) - 3(x - 2) = 3x + 8$

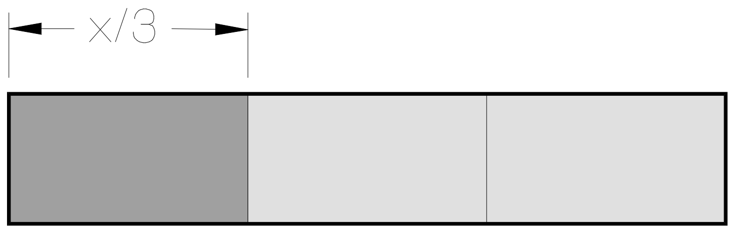
20. $8x - 3(2x + 5) = x - 4$

Section 7
Equations with Fractions or Decimals

Simplifying Equations with Fractions

If fractions occur in an equation, there is an easy technique for creating an equivalent equation without fractions. For example:

$$\frac{x}{3} + 4 = x$$



First, think of $x/3$ as x divided into 3 pieces, or one-third of x.
Although equations of this type can be solved in the usual way by subtracting parts of x from both sides, it is usually easier (with symbols or chips) to multiply both sides of the equation by a number so that the resulting equation has no fractions. *Multiplying both sides of an equation by the same number creates an equivalent equation with the same solution as the original equation.*

$$\frac{x}{3} + 4 = x$$

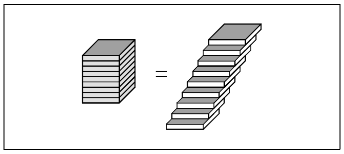
$$3\left(\frac{x}{3} + 4\right) = 3(x)$$

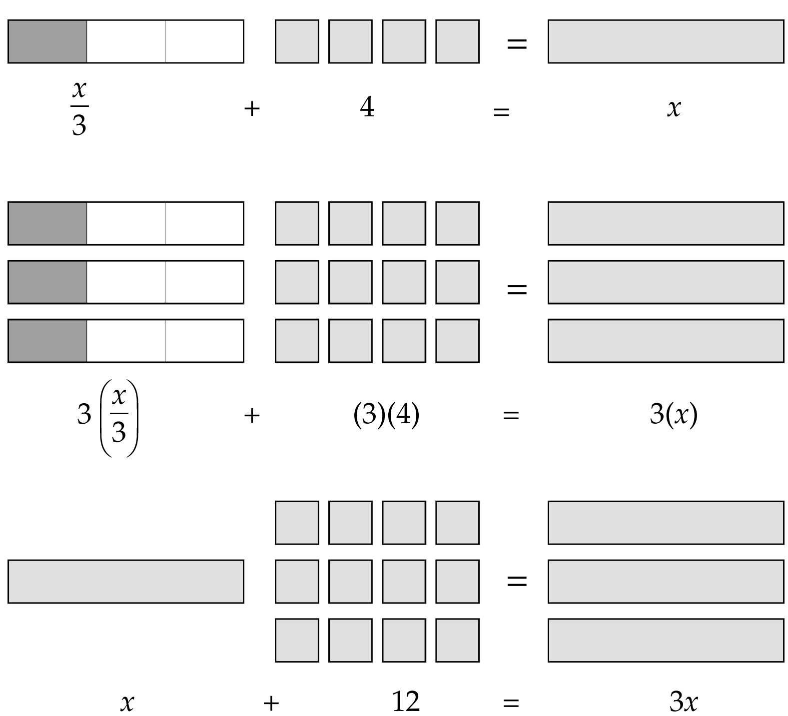
$$3\left(\frac{x}{3}\right) + 3(4) = 3x$$

$$\frac{3x}{3} + 12 = 3x$$

$$x + 12 = 3x$$

Notice that you must multiply 3 times each term on both sides.





After this point, the steps are the same as in previous sections:

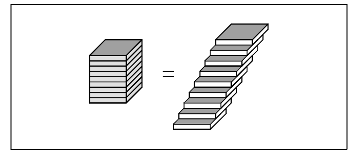
$$
\begin{aligned}
x + 12 &= 3x \\
x + 12 - x &= 3x - x \\
12 &= 2x \\
\frac{1}{2}(12) &= \frac{1}{2}(2x) \\
\frac{12}{2} &= \frac{2x}{2} \\
6 &= x
\end{aligned}
$$

How do you choose the number that you will use to multiply? If x is divided by 3, multiply by 3. If x is divided by 4, multiply by 4. If we triple $x/3$ (one-third of x), we will get one x.

If the equation has more complicated fractions, we still multiply by a number which will cancel the denominators. For example:

$$\frac{2}{3}x + 2 = \mathrm{x}$$

It is still useful to multiply both sides by a number, and we use the same number as in the previous example:



$$
\begin{aligned}
\frac{2}{3}x + 2 &= x \\
3\left(\frac{2}{3}x + 2\right) &= 3(x) \\
3\left(\frac{2}{3}x\right) + 3(2) &= 3(x) \\
\left(\frac{3}{1} \cdot \frac{2}{3}\right)x + 6 &= 3x \\
\frac{6}{3}x + 6 &= 3x \\
2x + 6 &= 3x \\
2x + 6 - 2x &= 3x - 2x \\
6 &= x
\end{aligned}
$$

Equations with Several Fractions

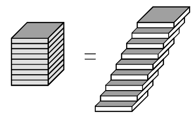
An equation may contain fractions with *different* denominators. We will still multiply both sides by a number, but this time we will use a number that will eliminate all of the fractions.

The number we want will have to be divisible by all of the denominators, or it will not "cancel" when it is multiplied times each term. The lowest number that is divisible by a group of numbers is called the **least common multiple** or **least common denominator**. Here is an example of the steps:

$$\frac{x}{3} + \frac{x}{4} = \frac{x}{2} + 1$$

The least common denominator of 3, 4, and 2 is 12.

$$
\begin{aligned}
12\left(\frac{x}{3} + \frac{x}{4}\right) &= 12\left(\frac{x}{2} + 1\right) \\
12\left(\frac{x}{3}\right) + 12\left(\frac{x}{4}\right) &= 12\left(\frac{x}{2}\right) + 12(1) \\
\frac{12x}{3} + \frac{12x}{4} &= \frac{12x}{2} + 12 \\
4x + 3x &= 6x + 12 \\
7x &= 6x + 12 \\
7x - 6x &= 6x - 6x + 12 \\
x &= 12
\end{aligned}
$$



Equations with Decimals

Decimal numbers such as .1 and 3.034 are often called decimal fractions because they represent fractions with denominators of 10, 100, 1000, etc. Because decimals are really fractions, we solve equations with decimals in the same way that we solve equations with fractions.

Consider this equation:

$$.3x + .2 = 1.7$$

We find a number (the least common multiple) that we can use to multiply times both sides to eliminate the decimals. The correct choice is to multiply by 10, because the equation could be written as:

$$\frac{3}{10}x + \frac{2}{10} = \frac{17}{10}$$

Multiplying by 10 will eliminate the decimals and will result in a new equation that is easier to solve:

$$\begin{aligned} 10(.3x + .2) &= 10(1.7) \\ (10 \cdot .3x) + (10 \cdot .2) &= 17 \\ \mathbf{3x + 2} &\mathbf{= 17} \end{aligned}$$

We can now solve the equation in the usual way:

$$\begin{aligned} 3x + 2 &= 17 \\ 3x + 2 - 2 &= 17 - 2 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

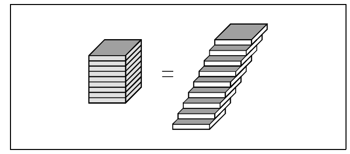
Equations may also contain decimals with different numbers of decimal places. Again, we multiply both sides by the power of 10 (10, 100, 1000, etc.) that will eliminate decimals from all of the numbers. Consider this equation:

$$.03x + .7 = x - 3.18$$

There are three numbers with decimal points. Two of them (.03 and 3.18) have two decimal places and one (.7) has one decimal place. We need to multiply by 100 to eliminate all of the decimal places:

$$\begin{aligned} 100(.03x + .7) &= 100(x - 3.18) \\ 3x + 70 &= 100x - 318 \\ 3x + 388 &= 100x \\ 388 &= 97x \\ 4 &= x \end{aligned}$$

To check our answer:



$$\begin{aligned} .03x + .7 &= x - 3.18 \\ .03(4) + .7 &= (4) - 3.18 \\ .12 + .7 &= .82 \\ .82 &= .82 \end{aligned}$$

To review, we chose 100 as our number to multiply because it was the least common multiple. We could have rewritten the original equation to show why this is true:

$$.03x + .7 = x - 3.18$$

is also:

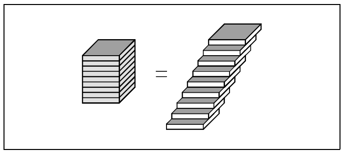
$$\frac{3}{100}x + \frac{7}{10} = x - \frac{318}{100}$$

The common denominator for 10 and 100 is clearly 100.

Fractions and Decimals: How to Multiply

With both fractions and decimals, we multiply both sides of the equation by the least common multiple. With fractions, we look at the denominators and choose the least common denominator. With decimals, we look at the number of decimal places and we multiply by the appropriate power of 10 (10, 100, 1000 …).

For this equation:	Multiply by:	Reason:
$\frac{x}{6} + \frac{x}{8} = 7$	24	Common denominator of 6 and 8
$\frac{x}{3} + \frac{3x}{4} = \frac{x}{6} + 11$	12	Common denominator of 3, 4, and 6
$.02 + .13x = .15$	100	Maximum of 2 decimal places
$3 + .001x = 3.1$	1000	Maximum of 3 decimal places



It is important to understand that you do not have to multiply these equations, but it is usually easier to do so. If you do not multiply both sides, you can solve the equations by subtracting and dividing with the fractions or decimals:

$$
\begin{aligned}
.03x + .7 &= x - 3.18 \\
.03x + .7 + 3.18 &= x - 3.18 + 3.18 \\
.03x + 3.88 &= x \\
.03x + 3.88 - .03x &= x - .03x \\
3.88 &= .97x \\
\frac{3.88}{.97} &= \frac{.97}{.97}x \\
4 &= x
\end{aligned}
$$

Summary

Now we can add one more step to our list:

- **Use the distributive property to complete any multiplications of expressions in parentheses.**
- ***If fractions or decimals are present, multiply both sides of the equation by the least common multiple (least common denominator).***
- **Combine similar terms on each side of the equation.**
- **Eliminate the unknowns from one side by adding the opposite type of bars. Add negatives to eliminate positives, and positives to eliminate negatives.**
- **Add positive or negative chips to cancel out the units and to "isolate" the unknown.**
- **Multiply both sides by ½, ⅓, etc. to match up a single unknown with the correct number of chips.**

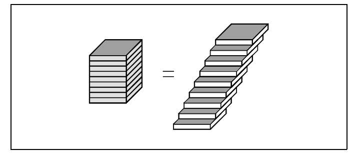
Exercises

Solve for x.

1. $\frac{x}{12} + 1 = x - 21$

2. $\frac{x}{2} = x + 4$

3. $\frac{x}{2} = {}^{-}1$



4. $\frac{x}{2} + \frac{x}{3} = \frac{5}{6}$

5. $\frac{x}{3} + x = 7$

6. $x - \frac{x}{4} = 9$

7. $\frac{x}{3} + \frac{x}{2} = \frac{x}{4} + \frac{7}{2}$

8. $\frac{2x}{3} + 2 = \frac{2}{3}$

9. $\frac{1}{5}x + 3 = 6$

10. $\frac{3}{4}x + 1 = x - 2$

11. $3 - \frac{2}{3}x = 4x - 5$

12. $6.9 + x = 3.3$

13. $.3 + 2x = 3.9$

14. $.2x + 3.1 = 3.9$

15. $.02 + .13x = .15$

16. $3 + .001x = 3.1$

17. $.2x + .8 = x - 4$

18. $3x + 4x = 6.8 + .2x$

19. $.002x = 0$

20. $\frac{3}{5}x + .2 = .1x + 10.2$

(What is the common denominator for 5 and 10?)

Section 8
Special Solutions

When the Variable Disappears

Some equations may contain the same number of x's on both sides. This may be obvious:

$$3x + 7 = 3x + 7$$

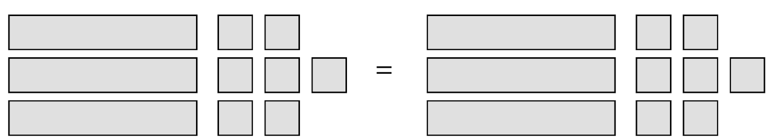
or it may occur after you have combined similar terms:

$$\begin{aligned} 3x + 2(x + 1) &= 5(x + 1) \\ 3x + 2x + 2 &= 5x + 5 \\ 5x + 2 &= 5x + 5 \end{aligned}$$

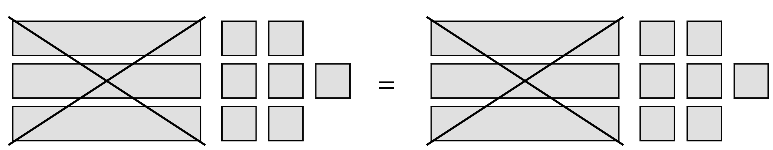
If you proceed in the usual way by subtracting x's, you will get strange results. In the first case:

$$\begin{aligned} 3x + 7 &= 3x + 7 \\ 3x + 7 - 3x &= 3x + 7 - 3x \\ 7 &= 7 \end{aligned}$$

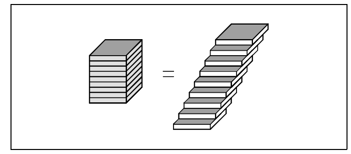
$3x + 7 = 3x + 7$



$3x + 7 - 3x = 3x + 7 - 3x$

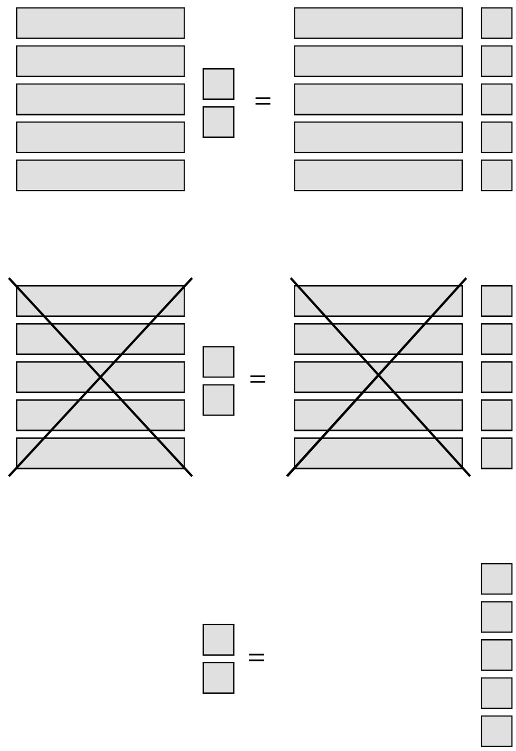


$7 = 7$



In the second case:

$$5x + 2 = 5x + 5$$
$$5x + 2 - 5x = 5x + 5 - 5x$$
$$2 = 5$$



The two cases above have a different meaning:

- **7 = 7 Since this is a true statement, the equation will be true for any x. You can choose any value for x, and the equation is still true. The equation is true for all x.**

- **2 = 5 Since this statement is false, the equation is false for any x. We substitute any value for x, but the equation will always turn out to be false. There is no solution.**

Exercises

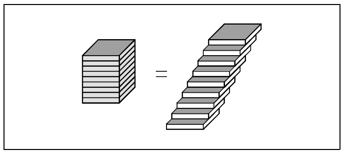
Solve for x. Determine if there is a solution, if there is no solution, or if the equation is true for all values.

1. $2x + 3x = 5x$

2. $4x + 3x = 7x + 1$

3. $2(x + 1) = 8$

4. $2(x + 1) - 2 = 3(x + 4) - (x + 12) + 1$



5. $3(3 + 2x - 1) = 4x - 1(3 - 2x) + 9$

6. $2(x + 3) - 2x = 5$

7. $2x + 3 = 3$

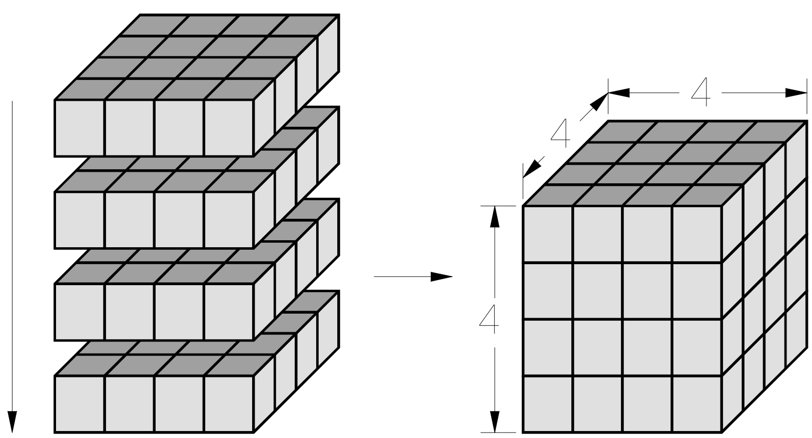
8. $^-3(x - 1) = x + 4(2 - x) - 5$

9. $1 + x - (x - 1) = 2$

10. $3(x + 1) = 3x + 1$

Chapter 8

Powers and Roots



Section 1
Introduction to Exponents

Exponents and Repeated Multiplication

Multiplication is often defined as repeated addition:

$$5 + 5 + 5 = 3 \cdot 5$$

$$8 + 8 + 8 + 8 + 8 + 8 + 8 = 7 \cdot 8$$

For repeating large numbers of additions, this notation becomes essential:

$$\mathbf{300 \cdot 5 =}$$

$$\begin{gathered}
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+ \\
5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5+5
\end{gathered}$$

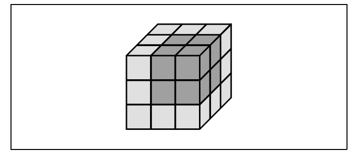
$$\mathbf{= 1500}$$

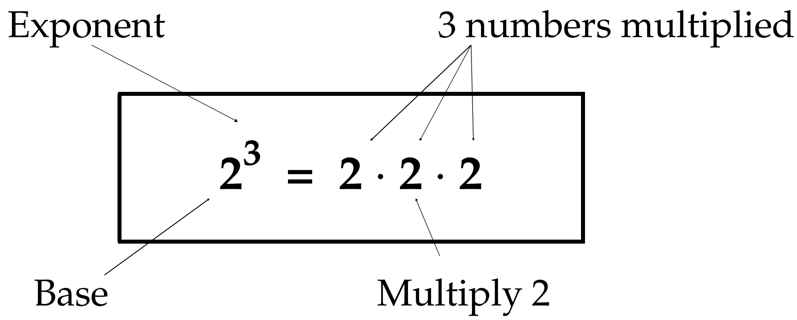
When we need to repeat *multiplications*, we invent a new notation to save time and space:

$$3 \cdot 3 \cdot 3 \cdot 3 = 3^4$$

$$2 \cdot 2 \cdot 2 = 2^3$$

The raised number is called a **power** or **exponent**—it stands for the number of quantities that we multiply. The number that is being multiplied is called the **base**. We read $\mathbf{2^3}$ as "Two to the third power." The operation of raising a number to a power is called **exponentiation**.





Notice that this is a new operation. *2^3 is not the same as 2 times 3.*

Symbols and the Order of Operations

What is the meaning of:

$$2 \cdot 3^2 \text{ ?}$$

Two operations are indicated—multiplication and exponentiation. Which comes first?

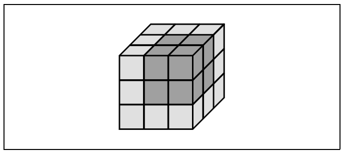
$$2 \cdot (3^2) \quad \text{or} \quad (2 \cdot 3)^2 \text{ ?}$$

$$\begin{aligned} 2 \cdot (3^2) &= 2 \cdot (3 \cdot 3) \\ &= 2 \cdot 9 \\ &= 18 \qquad \textbf{(We agree that this is } \textbf{\textit{correct}}\textbf{)} \end{aligned}$$

$$\begin{aligned} (2 \cdot 3)^2 &= 6^2 \\ &= 6 \cdot 6 \\ &= 36 \qquad \textbf{(We agree that this is } \textbf{\textit{incorrect}}\textbf{)} \end{aligned}$$

The two alternatives have different answers! To avoid confusion and to save time, we agree that the first meaning is correct. *Exponentiation happens before multiplication and addition, unless parentheses indicate otherwise.*

$$2x^2 \quad \text{means} \quad 2 \cdot (x^2)$$



When there is no sign for an operation between two quantities, the meaning is the same as before—multiplication.

$$x^3 y^2 \quad \text{means} \quad x^3 \cdot y^2$$

$$(x-3)^2\,(y+6)^3 \quad \text{means} \quad (x-3)^2 \cdot (y+6)^3$$

Exponentiation happens *before* addition, subtraction, or negative signs:

$$\mathbf{-x^2} \quad \text{means} \quad \mathbf{-(x^2)}, \quad \textit{not} \quad (-x)^2$$

$$\mathbf{3 + x^2} \quad \text{means} \quad \mathbf{3 + (x^2)}, \quad \textit{not} \quad (3 + x)^2$$

$$\mathbf{3 - x^2} \quad \text{means} \quad \mathbf{3 - (x^2)}, \quad \textit{not} \quad (3 - x)^2$$

Exercises

Write each quantity as a multiplication problem, then calculate the answer:

1. 5^4
2. 4^5
3. 2^6
4. 10^3

Write each multiplication using powers:

5. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
6. $7 \cdot 7 \cdot 7$
7. $32 \cdot 32 \cdot 32 \cdot 32$
8. $0 \cdot 0$
9. $3 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 5$
10. $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$

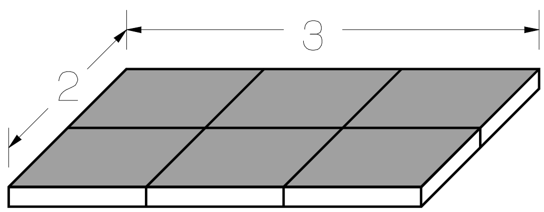
Calculate each answer:

11. $(-2)^3$
12. $(-2)^4$
13. $3 - 2^4$
14. $(3 - 2)^4$

Section 2
Squares

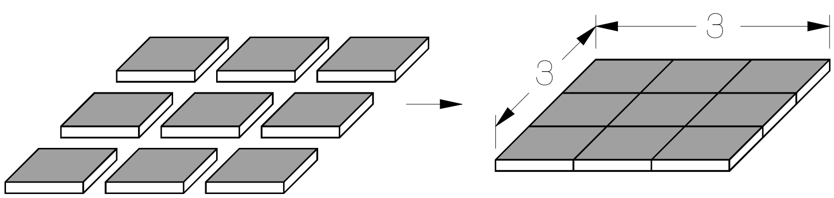
Squares and Second Powers

This section will present a visual explanation of raising numbers to a power of 2. In past chapters, we have considered multiplication of two numbers as the formation of rectangles:



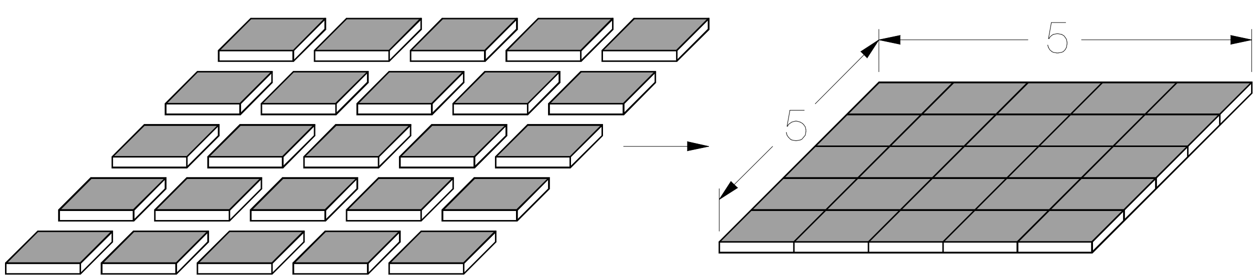
$$2 \cdot 3 = 6$$

Because $\mathbf{3^2}$ means $\mathbf{3 \cdot 3}$, raising numbers to the second power forms a square:

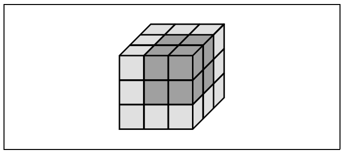


$$3^2 = 3 \cdot 3 = 9$$

When we raise 5 to the second power, we get $\mathbf{5^2}$ or $\mathbf{5 \cdot 5}$:



$$5^2 = 5 \cdot 5 = 25$$



This geometric property leads us to call 3^2 "three squared."

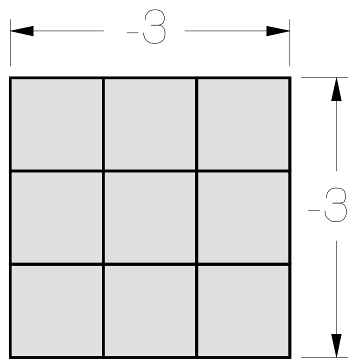
Raising a quantity to the second power

Make a square using the quantity for the length and width. The result is the area (number of unit chips) inside the square.

Squares of Negative Numbers

We are already familiar with the meaning of multiplying two negative numbers. The square of a negative number is always positive; we have to "imagine" the two negatives in the original multiplication:

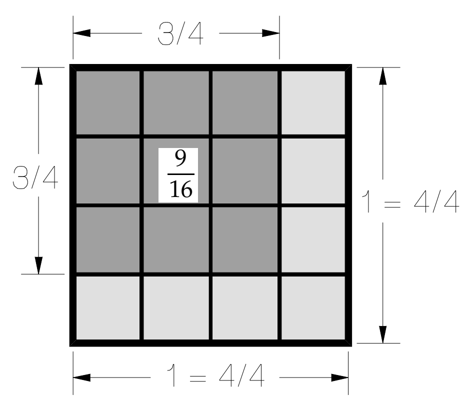
The result is $^{+}9$



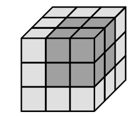
Squares Involving Fractions

Raising ¾ to the second power has the same meaning as raising a whole number to the second power—we build a square ¾ long by ¾ wide:

The result is $\frac{9}{16}$



Exercises



Use pictures or chips to illustrate and to answer these problems:

1. $(^-7)^2$

2. $(^-1)^2$

3. $\left(\frac{2}{3}\right)^2$

4. $\left(\frac{3}{5}\right)^2$

5. $\left(-\frac{4}{3}\right)^2$

Complete the operations of multiplication and exponentiation:

6. 15^2

7. $2^2 \cdot 7^2$

8. $(2 \cdot 7)^2$

9. $1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2$

10. $52 \cdot 1^{15}$

Write each number as the square of another number:

Example: 169
Solution: $169 = 13^2$

11. 121

12. 225

13. 10,000

14. 81

15. 144

Complete the multiplication and exponentiation:

16. $3 + (^-7)^2$

17. $3 - 7^2$

18. $3 \cdot 4^2$

19. $4 + 4^2 + 3^2$

20. $3^2\,(2^3)$

Section 3
Cubes

Cubes and Third Powers

If powers of 2 represent squares, what is the visual meaning of raising a number to a power of 3? Start by considering

$$4^3$$

This means the repeated multiplication of 4:

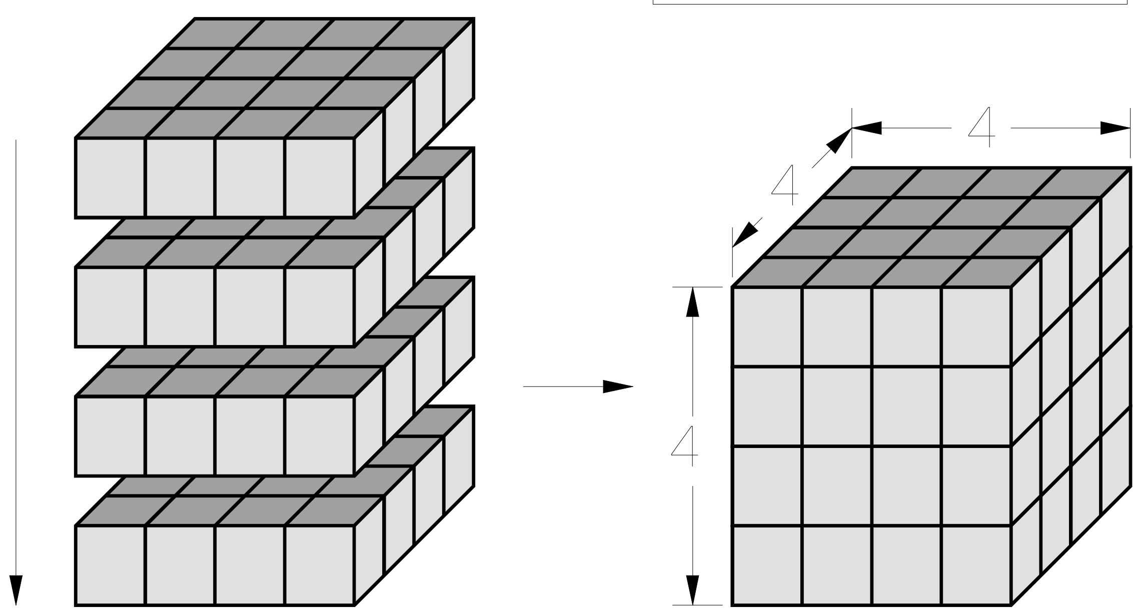
$$4 \cdot 4 \cdot 4$$

If we think of this as:

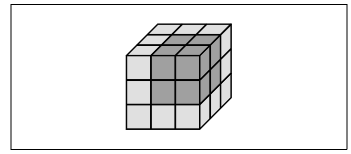
$$4 \cdot (4 \cdot 4)$$

then it is 4 squares, each 4 by 4. Using cubes, we can rearrange these to form a 4 by 4 by 4 cube:

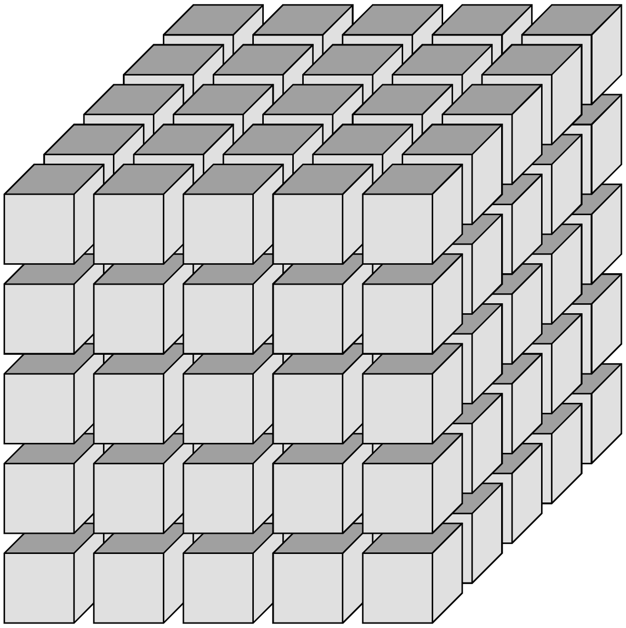
$$4^3 = 4 \cdot 4 \cdot 4 = 64$$



We read the symbol 4^3 as "four to the third power," "four to the power of three," or "four cubed." Visually, when we raise a number to the third power, we are building a larger cube composed of smaller unit cubes. The result of the multiplication is found by counting the number of unit cubes:



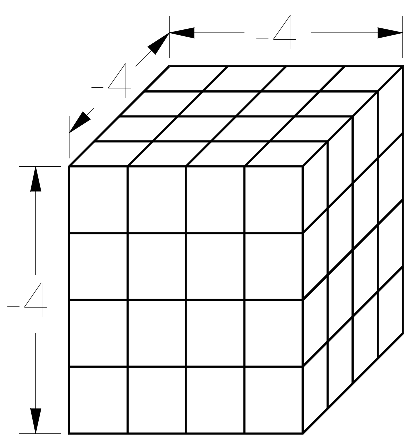
$$5^3 = 5 \cdot 5 \cdot 5 = 125$$

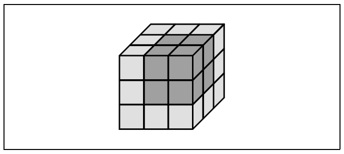


Cubes of Negative Numbers

We have already learned that the product of three negative numbers is negative. Therefore the cube of a negative number is also negative:

$$({}^-4)^3 = ({}^-4) \cdot ({}^-4) \cdot ({}^-4) = {}^-64$$

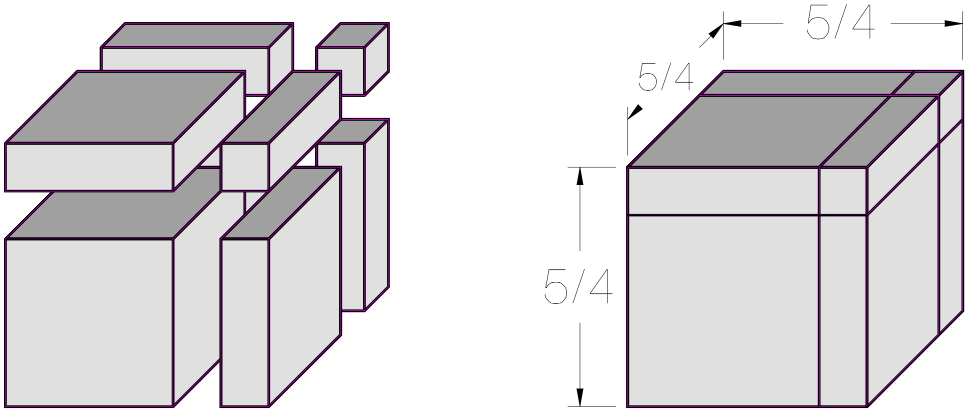




Cubes of Fractions and Mixed Numbers.

As with squares, there is no special difficulty with visualizing the cube of a fraction—for $(5/4)^3$, we build a cube that is $5/4$ on each side:

$$\left(\frac{5}{4}\right)^3 = \frac{5}{4}\cdot\frac{5}{4}\cdot\frac{5}{4} = \frac{125}{64}$$



Exercises

Draw a sketch and calculate:

1. $2^3 =$

2. $1^3 =$

Calculate the answer:

3. 7^3

4. 21^3

5. $\left(\frac{7}{3}\right)^3$

6. $2^2 \cdot 2^3$

7. 0^3

8. $291^3 \cdot 0^3$

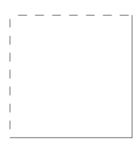
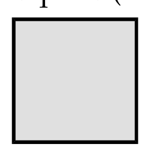
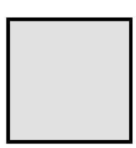
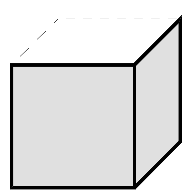
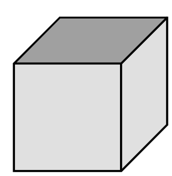
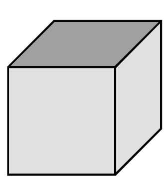
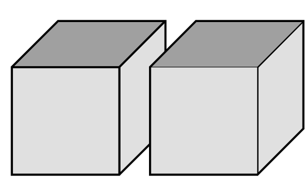
9. $\left(\frac{7}{3}\right)^3 \cdot 3^3$

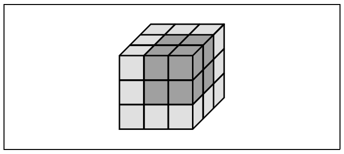
10. $3^2 \cdot 3^2$

Section 4
Higher Powers

Powers Greater Than Three

Our visual models become more difficult after the power of three. For each additional step from 1 to 2 to 3, we extended the model in another direction:

Start with:	Make 2	Connect to form:
Point (no dimensions)		Line (1 dimension)
Line (1 dimension)		Square (2 dimensions) 
Square (2 dimensions) 		Cube (3 dimensions) 
Cube (3 dimensions) 		?? (4 dimensions)

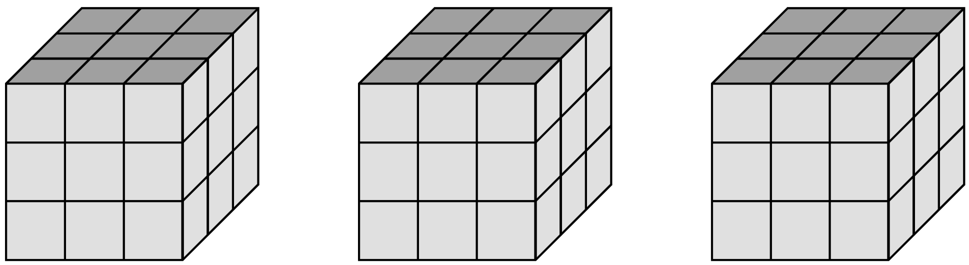


Because we cannot easily visualize a 4th dimension, we will stop at this point. It may be useful with some topics, however, to consider a picture of the 4th power of a number as a group of cubes:

$$\begin{aligned} 3^4 &= 3 \cdot 3 \cdot 3 \cdot 3 \\ &= 3 \cdot (3 \cdot 3 \cdot 3) \\ &= 3 \cdot (3^3) \end{aligned}$$

This would be 3 cubes:

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$



Levels of Exponents

We may need to understand the meaning of a more complex expression such as:

$$(2^3)^4$$

What does this mean? The outside exponent of 4 indicates that we are to multiply four of the quantity in parentheses:

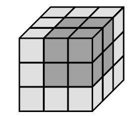
$$(\ )^4 = (\ ) \cdot (\ ) \cdot (\ ) \cdot (\ )$$

$$(2^3)^4 = (2^3) \cdot (2^3) \cdot (2^3) \cdot (2^3)$$

Even in complex expressions, the exponents have the same meaning—repeated multiplication. After multiplying the above expression out, we will have:

$$\begin{aligned} (2^3)^4 &= (2^3) \cdot (2^3) \cdot (2^3) \cdot (2^3) \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \\ &= 2^{12} \end{aligned}$$

Exercises



Complete the operations and simplify to one number:

1. $3^5 + 2^4$

2. $3^5 \cdot 3^4$

3. $(2^2)^2$

4. $(2 \cdot 3)^3$

5. $(2^3)^2 \cdot 5^2$

6. $(3^2)^2$

7. $(5^2)^2$

8. 7^4

9. $(^{-}1)^5$

10. $(^{-}1)^{36}$

11. $[(2^2)^2]^2$

12. $3^3 \cdot 2^2$

13. 2^{10}

14. 2^7

15. $(^{-}2)^{10}$

16. $^{-}(2)^{10}$

17. $5^3 + 6$

18. $1 + 5^3$

19. $1 - 5^3$

20. $(^{-}5)^3$

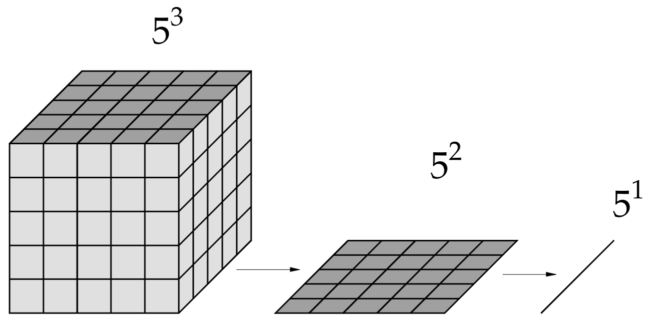
Section 5
Other Exponents: Negative Numbers, Zero, and One

The Power of One

What is the meaning of one as an exponent? When we raise a number to the power of one, we have the number only once. This means that *any number raised to the power of one is equal to itself*:

$$5^3 = (5) \cdot (5) \cdot (5)$$
$$5^2 = (5) \cdot (5)$$
$$5^1 = (5)$$

The progression of powers from 3 to 2 to 1 can be visualized in this manner: In order to extend this sequence, it will be helpful to think of every group of



multiplications as including a multiplication by the number 1; the 1 does not change the value.

$$5^3 = (1) \cdot (5) \cdot (5) \cdot (5)$$
$$5^2 = (1) \cdot (5) \cdot (5)$$
$$5^1 = (1) \cdot (5)$$

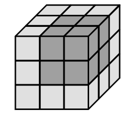
The Power of Zero

What would be a sensible meaning for the power of zero?

$$5^0 = ? \qquad 3^0 = ?$$

From our discussion above, we can extend our idea to zero:

$$5^3 = (1) \cdot (5) \cdot (5) \cdot (5) \quad \text{(3 fives)}$$
$$5^2 = (1) \cdot (5) \cdot (5) \quad \text{(2 fives)}$$
$$5^1 = (1) \cdot (5) \quad \text{(1 five)}$$
$$5^0 = (1) \quad \text{(0 fives)}$$



Using the exponent to represent how many numbers to multiply, *the zero power must mean that we do not multiply any numbers at all.* For positive integers as bases, *any number raised to the power of zero is one.*

What would be the meaning of a negative exponent?

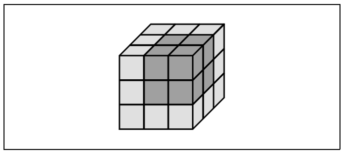
Negative Numbers as Exponents

Consider our familiar decimal system of place value. As we move to the left, each place or column is 10 times as large as the one before. As we move to the right, each column is $\frac{1}{10}$ as large; we divide the previous column by 10:

Place	**3**	**2**	**1**				
Value	1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
Exponent	10^3	10^2	10^1				

Because the place-values are multiples of 10, they can be represented by powers of 10 as shown above. If we add a column labeled "0" and use our new definition of $10^0 = 1$, the ones column will make sense.

Place	**3**	**2**	**1**	**0**			
Value	1000	100	10	**1**	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
Exponent	10^3	10^2	10^1	$\mathbf{10^0}$			



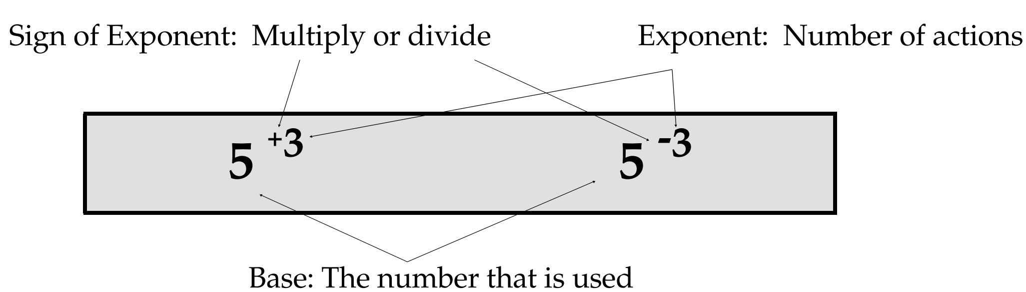
Finally, let's extend our system to the right and label columns as -1 (10^{-1}), -2 (10^{-2}), and so forth. This will preserve the pattern of multiplying by 10 when moving to the left and dividing by 10 when moving to the right:

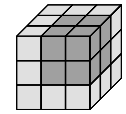
Place	**3**	**2**	**1**	**0**	**-1**	**-2**	**-3**
Value	1000	100	10	**1**	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
Exponent	10^3	10^2	10^1	$\mathbf{10^0}$	$\mathbf{10^{-1}}$	$\mathbf{10^{-2}}$	$\mathbf{10^{-3}}$

By this scheme, it seems sensible to define *negative* powers as *dividing* one by the base and *positive* powers as *multiplying* one by the base:

Exponent	Action (to 1)	Examples
Positive	Multiply	$10^3 = 1 \cdot 10 \cdot 10 \cdot 10 = 1000$
Zero	Nothing	$10^0 = 1 \cdot \_ = 1$
Negative	Divide	$10^{-2} = 1 \div 10 \div 10 = \frac{1}{100} = \frac{1}{10^2}$

It is important to notice that the *negative sign in the exponent* ***does not*** *mean that negative numbers are being multiplied or that the answer is negative*. Instead, it means that the base is on the *bottom* of the fraction.





Now we can consider the meaning of

$$0^0$$

Is there a reasonable meaning for this expression? If we use our idea of the progression of powers, we immediately run into a problem when we use zero as a base. As we move to the left, each number is zero times the one before; as we move to the right, we cannot divide by zero, so there is no clear answer. For this and many other reasons, *we leave 0^0 as* ***not defined.***

Place	**3**	**2**	**1**	**0**			
Value ?	0	0	0	**0?**	$\frac{1}{0}$	$\frac{1}{(0)(0)}$	$\frac{1}{(0)(0)(0)}$
Exponent	0^3	0^2	0^1	**0^0 ?**	**?**	**?**	**?**

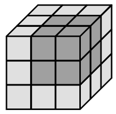
Summary

- **We can think of starting each exponential expression with the number one.**
- ***Positive Exponents*** **indicate that 1 is being multiplied by the base number several times. The exponent tells us how many times.**
- ***Negative Exponents*** **indicate that the starting number of 1 is being divided by the base number several times. The exponent tells how many times. Because a fraction indicates division, we often show these divisions as the denominator of a fraction.**
- ***Zero Exponents*** **indicate that we begin with 1 and then multiply by the base zero times (not at all). The result is 1.**

$$x^3 = 1 \cdot x \cdot x \cdot x$$

$$x^{-3} = \frac{1}{x \cdot x \cdot x} = \frac{1}{x^3}$$

$$x^0 = 1 \qquad \text{for } x \neq 0$$



- **Raising *zero to the zero power* is not defined.**

Zero to the Zero Power

0^0 is not defined

Exercises

Evaluate these expressions:

1. 999^1

2. 999^0

3. 6^{-3}

4. 1^{-5}

5. 5^{-1}

6. $(5^{-1}) \cdot 5^2$

7. $(10^{-2}) \cdot 10^2$

8. 4^{-3}

9. 5^{-4}

10. 1^{-1}

11. 5^0

12. $(^{-}3)^0$

13. 0^0

14. 273^1

15. 273.6^0

16. $(^{-}3)^{-4}$

17. $(^{-}3)^{-2}$

18. $2 + 2^{-1}$

19. $3 + 3^{-2}$

20. $16 + 4^{-2}$

Section 6
Properties of Powers

Introduction

In this section, we will examine some of the properties that allow us to restate exponential expressions. All of the properties have a clear basis; it is not necessary to memorize any of them. As you gain an understanding of these properties, you will find that you will remember them easily.

Multiplying with the Same Base

Consider the expression

$$3^5 \cdot 3^4$$

From the symbols alone, it is difficult to tell if we can combine powers or rewrite terms. Is the answer

$$3^{20} \ ?$$

$$9^{20} \ ?$$

$$3^9 \ ?$$

$$6^9 \ ?$$

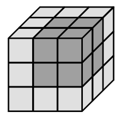
The best way to find out is to ask "What does it mean?"

$$\begin{aligned} 2^3 \cdot 2^4 &= (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) \\ &= (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \\ &= 2^7 = 128 \end{aligned}$$

or

$$2^{(3 + 4)}$$

Once we replaced the powers of 2 with their meaning in terms of multiplication, it was clear that *when multiplying two quantities with the same base raised to a power, the exponents add.* We add exponents because we are summing up the total number of factors.



To check, we calculate the value of $2^3 \cdot 2^4$ and compare this to the value of 2^7:

$$2^3 \cdot 2^4 = 8 \cdot 16 = 128$$

$$2^7 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = 128$$

Here are some other examples:

$$\begin{aligned} 2^3 \cdot 25 &= (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \\ &= (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \\ &= 2^{(3+5)} \\ &= 2^8 \end{aligned}$$

$$\begin{aligned} x^3 \cdot x^5 &= (x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x \cdot x) \\ &= (x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x) \\ &= x^{(3+5)} \\ &= x^8 \end{aligned}$$

Multiplying with the same base

$$x^a x^b = x^{a+b}$$

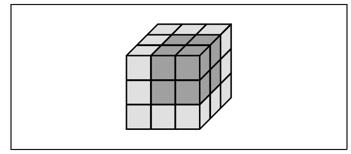
Dividing with the Same Base

A similar property exists when we divide two quantities where the same base is raised to a power. Consider:

$$\frac{2^4}{2^3}$$

Again, if we think about the meaning of this expression, it will be easy to discover the property:

$$\begin{aligned} \frac{2^4}{2^3} &= \frac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} \\ &= \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} \cdot \frac{2}{1} \\ &= 1 \cdot 2 = 2 \end{aligned}$$



This suggests the idea that

$$\frac{2^4}{2^3} = 2^{(4-3)}$$
$$= 2^1$$
$$= 2$$

We subtract exponents because we are counting the number of factors that remain after cancelling to one. *When dividing two quantities where the same base is raised to a power, we subtract the bottom exponent from the top exponent.*

Here are some other examples:

$$\frac{3^4}{3^2} = \frac{3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3}$$
$$= \frac{3 \cdot 3}{3 \cdot 3} \cdot \frac{3 \cdot 3}{1}$$
$$= 3^{(4-2)}$$
$$= 3^2$$

$$\frac{x^4}{x^2} = \frac{x \cdot x \cdot x \cdot x}{x \cdot x}$$
$$= \frac{x \cdot x}{x \cdot x} \cdot \frac{x \cdot x}{1}$$
$$= x^{(4-2)}$$
$$= x^2$$

Dividing with the same base

$$\frac{x^a}{x^b} = x^{a-b}$$ **where x is not zero**

Zero and Negative exponents—Again

Let us return to our earlier definition that any non-zero number raised to the zero power is one. Using our latest property, look at

$$\frac{3^5}{3^5}$$

This is one, because any number divided by itself is one. By our property,

$$\begin{aligned} \frac{3^5}{3^5} &= \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} \\ &= 3^{(5-5)} \\ &= 3^0 = 1 \end{aligned}$$

This is another reason that 30 must be equal to one. For 00, consider a similar example:

$$\begin{aligned} \frac{0^5}{0^5} &= 0^{(5-5)} \quad ? \\ &= 0^0 \qquad ? \end{aligned}$$

We cannot cancel out the quantities because we cannot divide by zero. This is another reason to decide that 0^0 is not defined.

We can think of a given quantity as if it were the result remaining from a fraction where the numerator and the denominator both had the same base. We see the result after common factors have cancelled.

- **If the exponent of the result is positive, there were more factors in the numerator.**
- **If the exponent of the result is negative, there were more factors in the denominator.**
- **If the exponent of the result is zero, there were equal numbers of factors that cancelled to 1.**

The property shows that our definition of negative exponents is sensible.

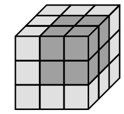
$$\begin{aligned} \frac{3^2}{3^5} &= \frac{3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} \\ &= \frac{3 \cdot 3}{3 \cdot 3} \cdot \frac{1}{3 \cdot 3 \cdot 3} \\ &= \frac{1}{3 \cdot 3 \cdot 3} = \frac{1}{3^3} \\ &= 3^{-3} \end{aligned}$$

By our property of subtracting exponents, this is

$$\frac{3^2}{3^5} = 3^{(2-5)} = 3^{-3}$$

Again, the property confirms our previous definition.

Two Levels of Exponents



When evaluating exponential expressions, we often encounter quantities like these:

$$(2^3)^4$$

$$(3^5)^2$$

$$(x^2)^3$$

By examining the meaning of these expressions, we can discover another useful property. First, as we discussed previously, raising a quantity to a power has the same meaning even if the quantity contains exponents:

$$(\quad)^4 = (\quad)\cdot(\quad)\cdot(\quad)\cdot(\quad)$$

$$(2^3)^4 = (2^3)\cdot(2^3)\cdot(2^3)\cdot(2^3)$$

If we expand this further, we see we have 4 groups, each containing 3 two's. The total number of two's is 3 · 4 or 12:

$$\begin{aligned}(2^3)^4 &= (2^3)\cdot(2^3)\cdot(2^3)\cdot(2^3)\\ &= (2\cdot 2\cdot 2)\cdot(2\cdot 2\cdot 2)\cdot(2\cdot 2\cdot 2)\cdot(2\cdot 2\cdot 2)\\ &= 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\\ &= 2^{12}\\ &= 2^{(3\cdot 4)}\end{aligned}$$

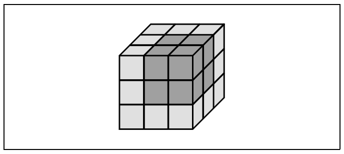
When a base is raised to a power, and the expression is again raised to a power, the result is the base raised to the product of the powers.

Two levels of exponents

$$(x^a)^b = x^{ab}$$

A Product Raised to a Power

If a product of two quantities is raised to a power, we can find another way to write the resulting expression:



$$\begin{aligned}(2 \cdot 3)^2 &= (2 \cdot 3) \cdot (2 \cdot 3)\\ &= 2 \cdot 3 \cdot 2 \cdot 3\\ &= (2 \cdot 2) \cdot (3 \cdot 3)\end{aligned}$$

Because the factors 2 and 3 both occur twice, the associative and commutative properties allow us to rearrange the numbers; the result is two of each.

This will clearly hold true for any quantities and any power:

$$\begin{aligned}(x \cdot y)^3 &= (x \cdot y) \cdot (x \cdot y) \cdot (x \cdot y)\\ &= (x \cdot x \cdot x) \cdot (y \cdot y \cdot y)\\ &= x^3 \cdot y^3\end{aligned}$$

Any product raised to a power can be restated as the product of each factor raised to a power. Note that this pattern occurs because of the specific situation—there is no general "distributive" law that allows us to always take something on the outside and apply it to the inside.

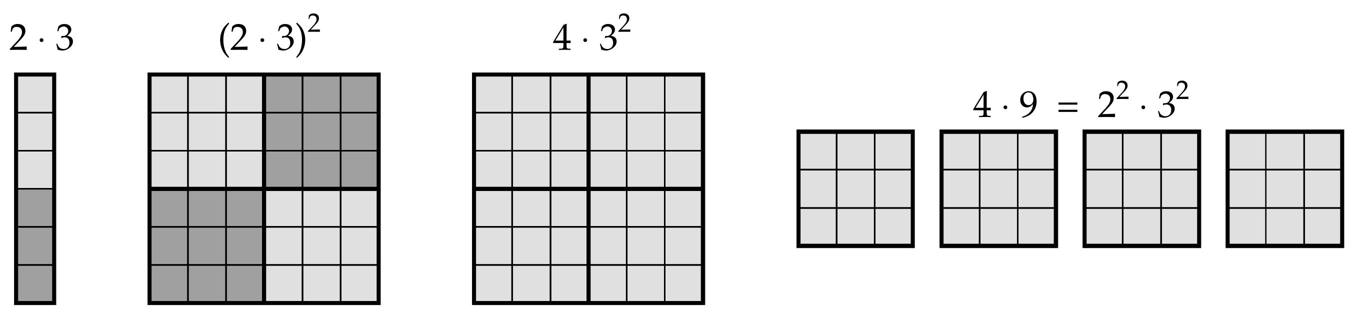
A product raised to a power

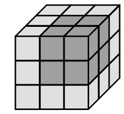
$$(xy)^a = x^a y^a$$

The picture of a simple example—

$$(2 \cdot 3)^2 = 2^2 \cdot 3^2$$

may help us to understand the meaning of this property. We start with the left side—a square that is 2·3 or 6 units on each side. We then show that this is the same as the right side—4 (or 2^2) groups of 9 (or 3^2):





Fractions and Exponents

There are two final properties involving fractions that we will find useful to discuss. Consider an expression where we are raising a fraction to a power:

$$\left(\frac{2}{3}\right)^2$$

We evaluate this by using the meaning of the exponent 2:

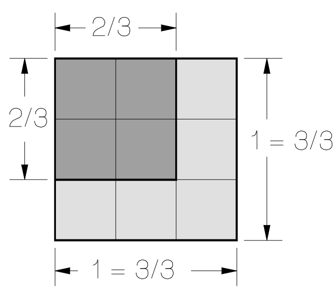
$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{2 \cdot 2}{3 \cdot 3} = \frac{4}{9}$$

When we square a fraction, we actually square both the top and bottom of the fraction. *When we raise a fraction to a power, we raise both the numerator (top) and the denominator (bottom) to the same power.*

A Fraction Raised to a Power

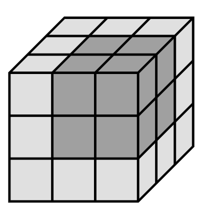
$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

One way of showing this visually is as follows:



$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$

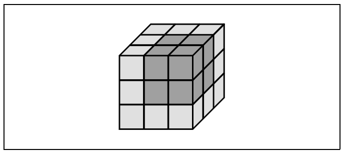
In three dimensions, here is $(2/3)^3$:



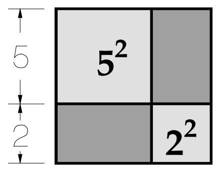
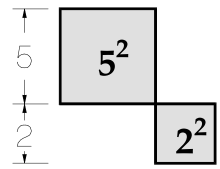
$$\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$$

Common Errors

The material in this section—properties of powers—is difficult for many students. Most errors result from attempting to memorize patterns of symbols instead of working to understand the concepts involved. *Properties can be learned as facts about real things rather than as meaningless patterns of symbols.*



Here are some of the common errors. Each is an attempt to apply a pattern of symbols to an inappropriate situation:

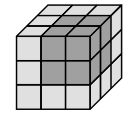
Error (False)	**Picture** (Why it's not true)	**Looks like:** (True)
$(5+2)^2$ *does not equal* 5^2+2^2	$(5+2)^2$  5^2+2^2 	$(5 \cdot 2)^2 = 5^2 \cdot 2^2$
$\frac{3^4}{4^3}$ *does not equal* $\left(\frac{3}{4}\right)^{4-3}$	**??** Top and bottom must have the same base to combine in this way.	$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$ $\frac{x^a}{x^b} = x^{a-b}$
$3^5 \cdot 2^6$ *does not equal* $(3 \cdot 2)^{5+6}$	**??** Both factors must have the same base to combine in this way.	$x^a y^a = (xy)^a$ $x^a x^b = x^{a+b}$

Summary

The properties we have learned, like any rules or shortcuts, are difficult to remember and use correctly unless you know where they come from. Starting with basic definitions, you can derive the properties for yourself any time that you need them.

Properties of Exponents	
$x^1 = x$	$\frac{x^a}{x^b} = x^{a-b}$ (*x* is not zero)
$x^{-a} = \frac{1}{x^a}$	$(x^a)^b = x^{ab}$
$x^0 = 1$ (*x* is not zero)	$(xy)^a = x^a y^a$
$x^a x^b = x^{a+b}$	$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

Exercises



Use the properties in this section to simplify the expressions:

1. $a^3 \cdot a^7$

2. $\dfrac{x^{17}}{x^3}$

3. $(a^3)^6$

4. $(a^0)^{16}$

5. $2^3 \cdot 5^1$

6. $\dfrac{2^5 \cdot 5^{-1}}{2^4 \cdot 3^1}$

7. $\left(\dfrac{3}{5}\right)^3$

8. $\dfrac{x^3 y^{-5}}{x^2 y^2}$

9. $2^3 \cdot 3^2 \cdot 5^{-2}$

10. $\dfrac{x^3 x^1}{(x^3)^2}$

Decide whether each equation is true or false. If it is true, why?

11. $x^5 \cdot y^{-2} = (xy)^3$

12. $(a^3)^3 = a^9$

13. $3^5 \cdot 3^3 = 3^8$

14. $(3^5)^5 = 3^{25}$

15. $\dfrac{2^5}{2^{-2}} = 2^3$

16. $2^5 \cdot 2^{-2} = 2^3$

17. $(3^2 \cdot 5^3)^0 = 1125$

18. $\dfrac{4^0}{3^3} = \dfrac{1}{27}$

19. $\dfrac{a^2}{b^5} = \dfrac{1}{b^3}$

20. $(15)^4 = 3^4 \cdot 5^4$

Section 7
Simplifying Expressions

Using the Properties

Expressions often contain many levels of exponents and many different fractions, multiplications, etc. It is easy to combine and simplify expressions if we use the appropriate properties one at a time. For example:

$$\begin{aligned}(x^2y^3)^5 &= (x^2)^5 \cdot (y^3)^5 \\ &= x^{(2 \cdot 5)} \cdot y^{(3 \cdot 5)} \\ &= x^{10} \cdot y^{15} \\ &= x^{10}y^{15}\end{aligned}$$

It is sometimes helpful to temporarily ignore the quantity inside of a pair of parentheses if this makes the use of the properties more easily apparent:

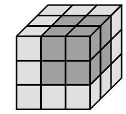
$$\begin{aligned}(x^2y^3)^5 &= (\quad)^5 \cdot (\quad)^5 \\ &= (x^2)^5 \cdot (y^3)^5 \\ &= x^{(2 \cdot 5)} \cdot y^{(3 \cdot 5)} \\ &= x^{10} \cdot y^{15} \\ &= x^{10}y^{15}\end{aligned}$$

When several different properties apply, it is often possible to simplify an expression in several ways; one way may be faster or easier, but it is not important which way we choose. For example:

$$\left(\frac{x^3}{x^2}\right)^5$$

If we begin by raising each part of the fraction to the 5^{th} power, it looks like this:

$$\begin{aligned}\left(\frac{x^3}{x^2}\right)^5 &= \frac{x^{3 \cdot 5}}{x^{2 \cdot 5}} \\ &= \frac{x^{15}}{x^{10}} \\ &= x^{(15 - 10)} \\ &= x^5\end{aligned}$$



If we simplify the fraction inside of the parentheses first, then the process is somewhat easier:

$$\left(\frac{x^3}{x^2}\right)^5 = (x^{(3-2)})^5$$
$$= (x^1)^5$$
$$= x^5$$

Here is the same problem done by cancelling common factors. This is a demonstration of why the rules work:

$$\left(\frac{x^3}{x^2}\right)^5 = \left(\frac{x \cdot x \cdot x}{1 \cdot x \cdot x}\right)^5$$
$$= \left(\frac{x}{1}\right)^5$$
$$= x^5$$

Whenever possible, simplify fractions and quantities in parentheses before raising quantities to a power.

Properties and Negative Exponents

We will now discover if the properties of the previous section will apply to quantities with negative exponents. For each property, we can evaluate the expression in two ways: first using the rule directly, and second, using the definition of negative exponents. For example:

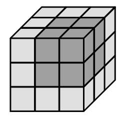
$$x^2 \cdot x^{-1}$$

By the property:

$$x^2 \cdot x^{-1} = x^{(2+-1)}$$
$$= x^{(2-1)}$$
$$= x^1$$

By the definition of negative exponents:

$$x^2 \cdot x^{-1} = x^2 \cdot \frac{1}{x^1}$$
$$= \frac{x^2}{x^1} = \frac{x \cdot x}{1 \cdot x}$$
$$= x^{(2-1)}$$
$$= x^1$$



We can see that both methods give the same answer; we can also see how adding a negative exponent gives the same result as subtracting a positive exponent. Here is a second example:

$$(x^2)^{-3}$$

By the property:

$$\begin{aligned}(x^2)^{-3} &= x^{(2 \cdot -3)} \\ &= x^{-6}\end{aligned}$$

By the definition of negative exponents:

$$\begin{aligned}(x^2)^{-3} &= \frac{1}{(x^2)^3} \\ &= \frac{1}{x^{(2 \cdot 3)}} \\ &= \frac{1}{x^6} \\ &= x^{-6}\end{aligned}$$

Again, the two methods give the same results.

Negative Exponents in the Denominator.

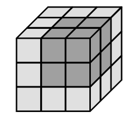
Consider the expression:

$$\frac{1}{x^{-2}}$$

Because a fraction can also represent a division problem, we can evaluate it like this:

$$\begin{aligned}\frac{1}{x^{-2}} &= 1 \div x^{-2} \\ &= 1 \div \frac{1}{x^2} \\ &= 1 \cdot \frac{x^2}{1} \\ &= x^2\end{aligned}$$

This will be true for all negative powers that are factors in the denominator.



Of course, we also know that a quantity in the numerator (top) of the fraction raised to a negative power can be rewritten as a positive exponent on the denominator (bottom) of the fraction:

$$\frac{z^{-3}}{1} = \frac{1}{z^{3}}$$

These two ideas can be used in the same fraction. If you wish, you can now rewrite all factors with negative exponents by inverting these factors and using all positive exponents. For example:

$$\frac{a^{-3}x^{-2}y^{5}}{z^{-3}b^{3}} = \frac{1}{a^{3}} \cdot \frac{1}{x^{2}} \cdot \frac{z^{3}}{1} \cdot \frac{y^{5}}{b^{3}}$$

$$= \frac{z^{3}y^{5}}{a^{3}x^{2}b^{3}}$$

Quantities having positive exponents do not change, but quantities with negative exponents are written in an inverted manner and the exponents become positive. Fractions having negative exponents (around the whole fraction) are inverted and the exponent becomes positive:

$$\left(\frac{a}{x}\right)^{-2} = \frac{a^{-2}}{x^{-2}} = \frac{x^{2}}{a^{2}} = \left(\frac{x}{2}\right)^{2}$$

- **A quantity with a negative exponent:**

  In the numerator—may be rewritten with a positive exponent in the denominator.

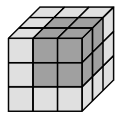
  In the denominator—may be rewritten with a positive exponent in the numerator.

Format and Symbols

Many students are not sure whether it is necessary to change all factors with negative exponents into factors with positive exponents. For the purposes of this book, there are two acceptable methods; it is not important which way you do it, as long as you are consistent:

$$x^{-6} \quad \text{or} \quad \frac{1}{x^{6}}$$

$$x^{2}y^{-6} \quad \text{or} \quad \frac{x^{2}}{y^{6}}$$



Being consistent means that you show the result in one of two ways:

- **Method 1:** ***With fractions, using only positive exponents*****. Each expression with x or y is shown on the top or the bottom of the fraction, whichever will result in a positive exponent.**

  Examples: $\frac{x^3}{y^2}$, $\frac{8}{x^3 y^2}$

- **Method 2:** ***Without fractions, using both positive and negative exponents.***

  Examples: $x^3 y^{-2}$, $8x^{-3} y^{-2}$

Preferred		**Not Preferred**
Method 1	**Method 2**	
$\frac{x^3}{y^2}$	$x^3 y^{-2}$	$\frac{y^{-2}}{x^{-3}}$
$\frac{128}{x^2}$	$128x^{-2}$	$\frac{x^{-2}}{2^{-7}}$
$\frac{1}{8x^3}$	$\frac{1}{8} x^{-3}$	$2^{-3}x^{-3}$
$\frac{a^3 b^2 c^5}{x^2 y^3 z^4}$	$a^3 b^2 c^5 x^{-2} y^{-3} z^{-4}$	$\frac{x^{-2} y^{-3} z^{-4}}{a^{-3} b^{-2} c^{-5}}$

Other forms may be called "not preferred," but they are not wrong. It is convenient to agree upon standard forms so that we will be able to compare our results with others. Even if you prefer one form over the other, it is still useful to practice both formats. Our agreement on the final form is given in the following summary:

Summary

To simplify complicated expressions:

- **Apply each property separately.**
- **If the properties seem confusing, return to the most basic definitions and work through every step.**
- **Simplify fractions and expressions inside of parentheses first, before raising the expressions to a power.**
- **Use the properties with negative as well as positive exponents.**
- **If desired, factors with negative exponents can be rewritten with positive exponents by using the ideas that:**

$$x^{-a} = \frac{1}{x^{a}}$$

$$\frac{1}{x^{-a}} = x^{a}$$

- **Where negative exponents or fractions occur, write the result consistently with one of these methods: no fractions *or* no negative exponents. Do not mix fractions and negative exponents.**
- **Write small common numbers (8, 16, 25) as integers, not exponential expressions (2^3, 2^4, 5^2).**

Exercises

Evaluate these expressions by combining and simplifying. Write your answers in one of the two forms described above:

1. $x^{-2}y^{-5}x^{3}$

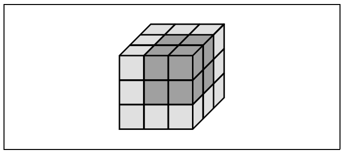
2. $\dfrac{(3xy)^2}{x^{-2}y^{3}}$

3. $(a^{2})^{3}(a^{0})^{2}(a^{-5})$

4. $\dfrac{(2m)^3}{(mn^{5})^2}$

5. $\dfrac{6^2 \cdot 5 \cdot 3^{-3}}{2^4}$

6. $\dfrac{x^{2}y^{-3}z^{5}}{x^{4}y^{-5}z^{3}}$



7. $(2 \cdot 5)^4$

8. $(3^2)^{-3}$

9. $(-2x^{-3})^3$

10. $\dfrac{1}{2^2 \cdot 3^{-2}}$

11. $(x^2)^5$

12. $(x^{-2})^5$

13. $\dfrac{y^2}{(y^3)^4}$

14. $\dfrac{(x^{-9})^0 (x^4)^{-3}}{(x^{-1})^5 (x^5)^{-2}}$

15. $\dfrac{15x^3 y^2}{20x^2 y}$

16. $\dfrac{12x^3 y^6}{12x^3 y^7}$

17. $\dfrac{a^3 b^5 c^7}{a^{-3} b^{-5} c^{-7}}$

18. $\dfrac{1}{a^{-3} b^{-5} c^{-7}}$

19. $\left(\dfrac{(a^3 b^4)^{-2}}{a^3 c^5}\right)^{-1}$

20. $\left(\dfrac{a^3 b^{-2}}{(c^5 a^2)^{-1}}\right)^2$

21. $\left(\dfrac{(x^2 y^{-2})^3}{(x^3 y^2)^{-2}}\right)^2$

22. $\left(\dfrac{a^{-1} x^{-1}}{a^{-1} x^{-1}}\right)^{10}$

Section 8
Roots and Radicals

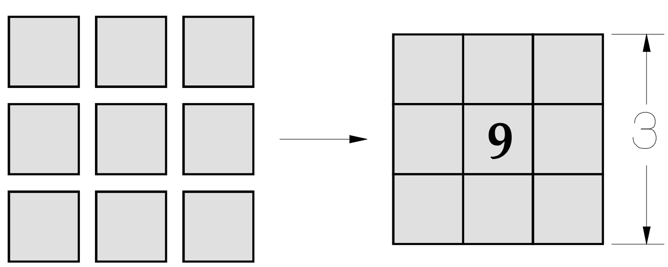
Square Roots

We have defined raising 3 to the second power as follows:

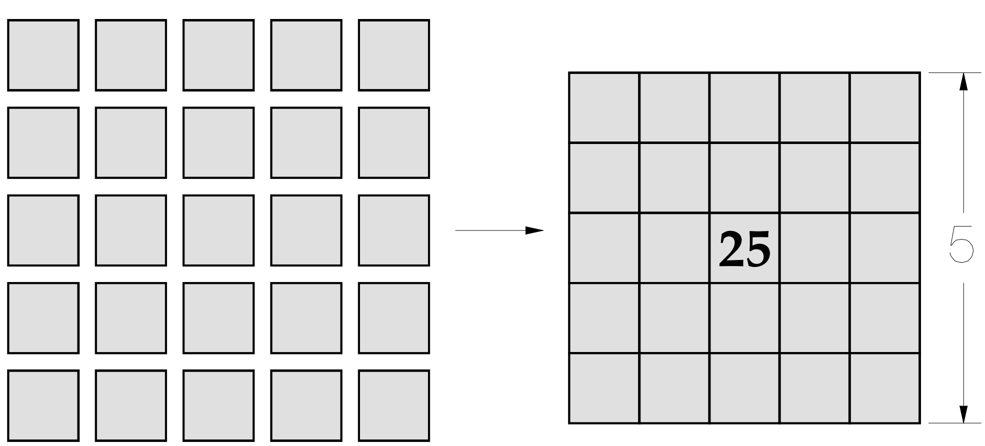
Make a square that is 3 long by 3 wide.
Count the number of squares inside.
This is the result: 9.

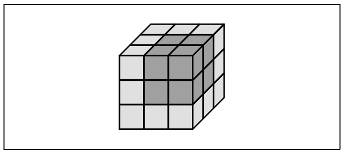
We could also consider the opposite problem:

Count out 9 unit chips.
Form the chips into a square.
Measure the length of the side.
This is the result: 3.



Try it now with 25 chips:





The result is 5. We call this process **taking the square root**. It is indicated as:

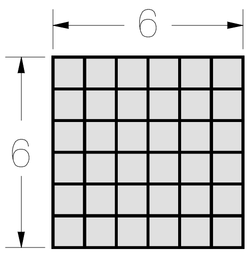
$$\sqrt{25} = 5$$

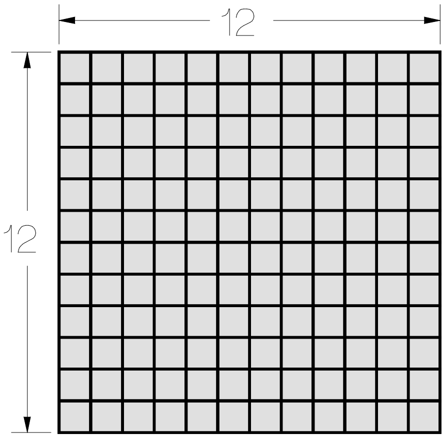
The new symbol $\sqrt{}$ is called a **radical sign**. Here are some more examples:

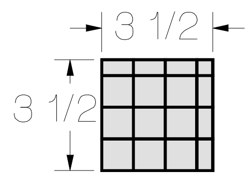
$$\sqrt{144} = 12$$

$$\sqrt{36} = 6$$

$$\sqrt{12.25} = 3.5$$



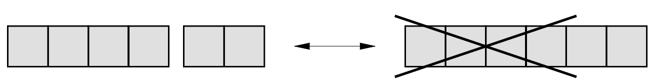




Taking the square root of 25 can also be stated as the question "What positive number can be multiplied times itself to get 25?" Notice that to avoid confusion, we have ignored the possibility of choosing $^-5$ as the answer, even though $(^-5)^2$ is 25. *The square root is always a positive number.*

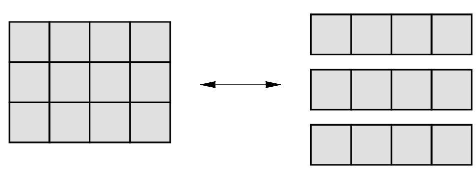
If the difference between the square and the square root seems hard to grasp, you have probably noticed that the two operations are very similar. In fact, they are opposite or inverse operations in the same way that we discussed addition/subtraction and multiplication/division as opposites:

$4 + 2 = 6$



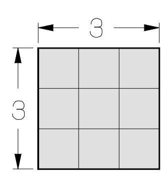
$6 - 4 = 2$

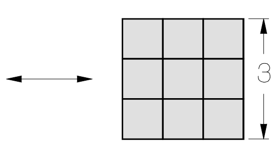
$3 \cdot 4 = 12$



$12 \div 4 = 3$
or
$12 \div 3 = 4$

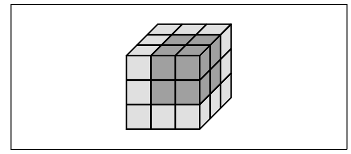
$3^2 = 9$





$\sqrt{9} = 3$

Cube Roots



Again, we can look at raising 2 to the third power as:

Build a cube 2 long by 2 wide by 2 high.

Count up the number of unit cubes inside.

This is the result: 8.

To reverse the process:

Start with 8 unit cubes.

Arrange them into a larger cube.

How large is the side of the cube?

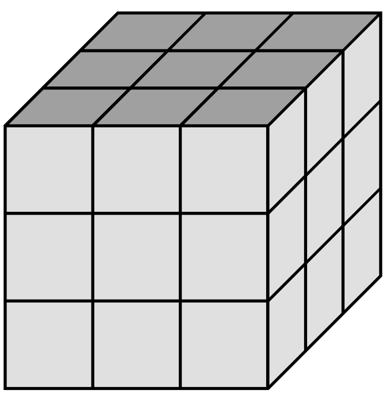
The result is 2.

This process is called **taking the third root or cube root.** Taking the cube root of 8 can also be stated as "What number to the 3^{rd} power gives 8?" The cube root of 8 is indicated by:

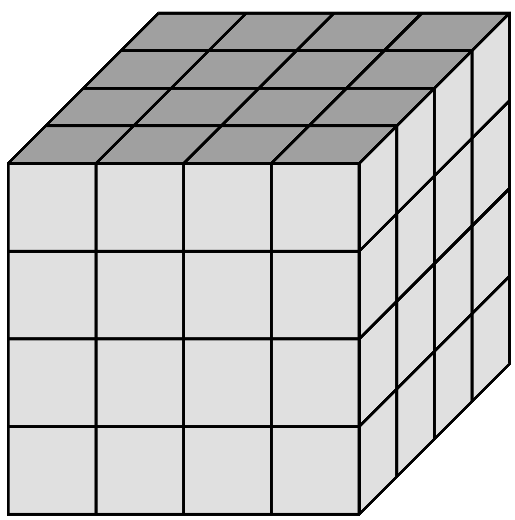
$$\sqrt[3]{8} = 2$$

The small 3 indicates the type of root. Here are some other examples:

$$\sqrt[3]{27} = 3$$

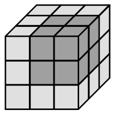


$$\sqrt[3]{64} = 4$$



Other Roots

Just as we agreed to define exponents of 4, 5, or any positive integer, we can define other kinds of roots. For example, the 5^{th} root of 32 is the number that that is raised to the 5^{th} power to give 32. The correct choice is 2.



The following chart shows some other examples:

Symbol	Meaning	Result	Check
$\sqrt[5]{32}$	What number to the 5th power gives 32?	2	$2^5 = 32$
$\sqrt[4]{16}$	What number to the 4th power gives 16?	2	$2^4 = 16$
$\sqrt[4]{81}$	What number to the 4th power gives 81?	3	$3^4 = 81$

Roots with Negative Bases

Is it sensible to define the square or cube root of a negative number?

$$\sqrt{{}^-25} = ?$$

$$\sqrt{{}^-16} = ?$$

$$\sqrt[3]{{}^-8} = ?$$

First, consider the square roots of negative numbers. If we ask the question "What number times itself equals $^-25$," we know from our study of integers that neither negative nor positive numbers will fit; any number times itself results in a positive number or zero. We conclude that *the square root of a negative number is not defined.*

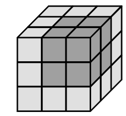
Next, consider cube roots of negative numbers. Our question is "What number to the 3rd power gives a result of $^-8$?" Because three negative numbers multiplied together will result in a negative number, there is a possible answer: $^-2$. Unlike the difficulty with square roots, there is no problem in deciding that *the cube root of a negative number must be negative.* Here are some more examples:

$$\sqrt{{}^-73} = \text{not defined}$$

$$\sqrt[3]{{}^-64} = {}^-4$$

$$\sqrt[5]{{}^-32} = {}^-2$$

Because odd numbers of negatives multiply to give negative results, you can see that *odd-numbered roots of negative numbers have a solution, but even-numbered roots do not.*



Summary

- **Taking roots is the inverse operation of exponentiation.**

- **The radical sign indicates the operation of taking the root. A small raised number indicates the type of root.** ***If there is no number, we agree that the quantity will be a square root.***

- **The square root of a given number is interpreted as taking that number of unit chips, building a larger square, and measuring the length of the side. It is also interpreted as the answer to the question "What number can be multiplied by itself to result in the given number?"**

- **The cube root of a given number is interpreted as taking that number of unit cubes, building a larger cube, and measuring the length of the side. It is also interpreted as the answer to the question "What number raised to the 3^{rd} power will result in the given number?"**

- ***The square root (or any even root) of a negative number is not defined.*** **The cube root (or any odd root) of a negative number will be negative. Any root of a positive number will be positive.**

Exercises

Evaluate these roots. If necessary, simplify the radicals and complete the multiplication or addition.

1. $\sqrt{64}$

2. $\sqrt[3]{125}$

3. $\sqrt[3]{1000}$

4. $\sqrt[3]{1}$

5. $\sqrt{10{,}000}$

6. $\sqrt{1}$

7. $\sqrt{0}$

8. $\sqrt[3]{-1000}$

9. $\sqrt[3]{-125}$

10. $\sqrt{-25}$

11. $\sqrt{25} + \sqrt{36}$

12. $\sqrt{25} \cdot \sqrt{36}$

13. $\sqrt[3]{8} \cdot \sqrt[3]{64}$

14. $\sqrt[3]{8 \cdot 64}$

Section 9
Irrational Numbers

The Square Root of 10

Most of the examples of square roots we have been considering have answers that are positive integers. A positive integer with an integer square root is called a **perfect square.**

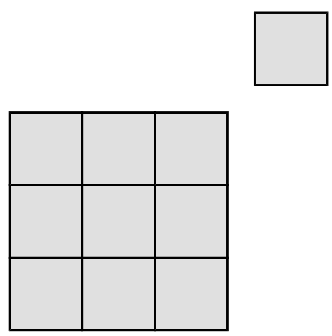
16, 25, 144, and 100 are perfect squares because

$$16 = 4^2,\ 25 = 5^2,\ 144 = 12^2,\ \text{and } 100 = 10^2.$$

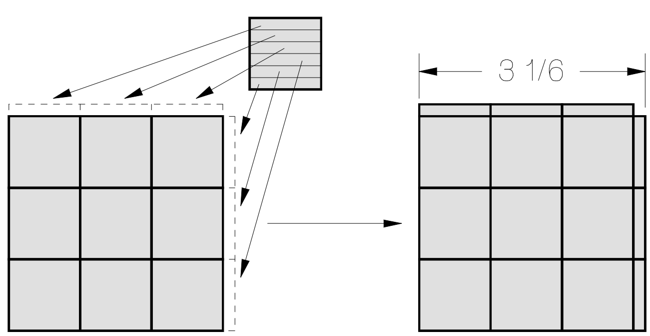
Can we extend the idea of a square root to numbers that are not perfect squares? Let us consider this expression:

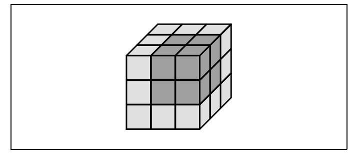
$$\sqrt{10}$$

By our previous definition, we should take 10 unit chips and rearrange them to form a larger square. After we use up 9 chips, we have 1 left over:



If we cut up this chip (you might want to use a paper chip), we can rearrange the pieces to get closer to a square; since we need to add equally to the height and width, 3 + 3 or 6 pieces will work best:

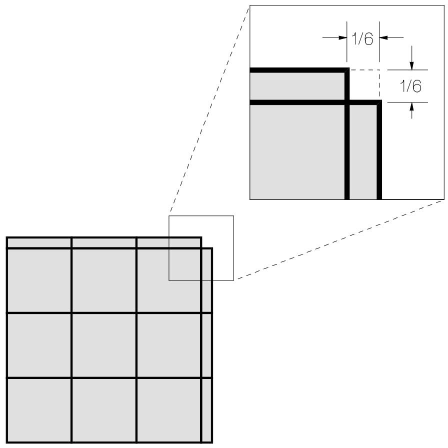




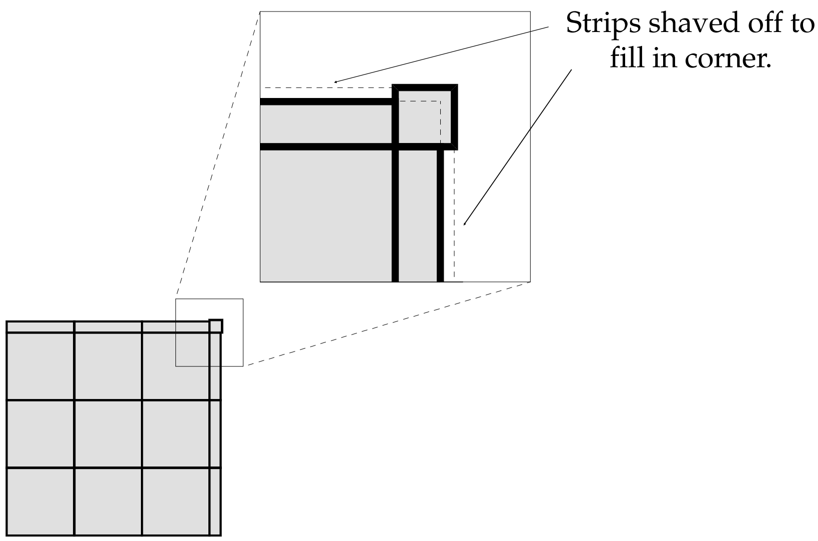
Each piece we added was ⅙ thick, so the figure is now 3⅙ wide and 3⅙ high. To check our work, we convert 3⅙ to a decimal and then square it:

$$\left(3\frac{1}{6}\right)^2 = (3.1667)^2 = 10.03$$

This is a good estimate, but it is not exact. Why not?—because there is a small area in the top right corner that is not filled in:



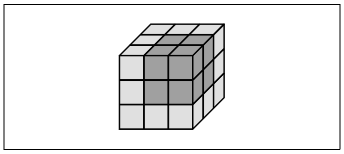
If we wish, we can continue to try to get an exact fit by shaving off a little from the top and right sides; we then use this to fill in the missing corner:



$$\text{Missing area} = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$$

$$\text{Shaved from top and right (6 pieces)} = \frac{1}{36} \div 6 = \frac{1}{216}$$

$$\text{New length of side} = 3 + \frac{1}{6} - \frac{1}{216} = 3.1620$$



To check our work:

$$(3.1620)^2 = 9.998$$

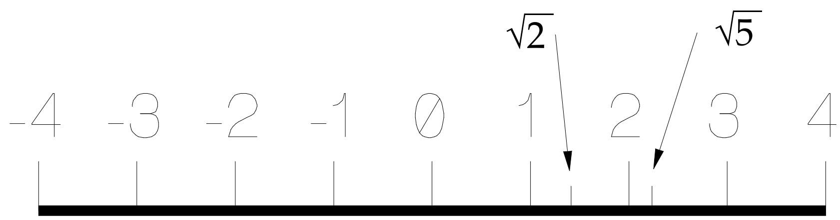
This is even closer, but the answer is now a little too small because we have cut off too much; the corner square now sticks out a little past the sides.

We can see that this process will get us increasingly more accurate answers in terms of sums and differences of fractions, but that we will never get the exact value. *The square root of 10 is not a fraction.*

Fractions and integers are called **rational numbers** because they can be expressed as ratios of integers. If an integer is not a perfect square, we can see that its square root is not a fraction. The square roots of 2, 3, 5, 6, 7, 8, and 10 are not fractions; we call numbers **irrational** if they cannot be represented as fractions.

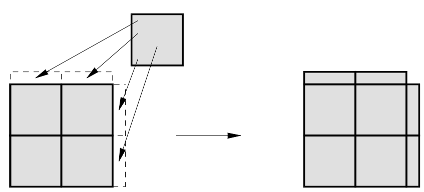
Square roots of positive integers are either integers or irrational numbers.

On a number line, irrational numbers are exact lengths just as integers are exact lengths. We can draw a line that is $\sqrt{2}$ units long just as accurately as we can draw a line that is 2 units long, but we can't write the value of $\sqrt{2}$ as an exact fraction or decimal.

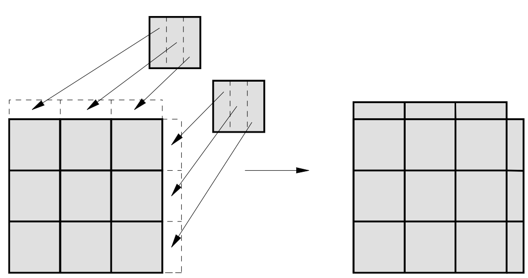


Here are simple estimates of other square roots:

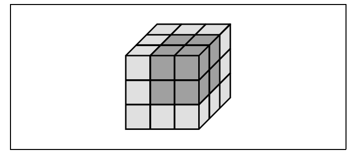
$\sqrt{5}$ is about $2\frac{1}{4}$



$\sqrt{11}$ is about $3\frac{1}{3}$



Summary



- **Perfect squares are integers that have exact integer square roots.**
- **All positive numbers have square roots.**
- **If an integer is not a perfect square, its square root is not a fraction—it is irrational.**

Exercises

Decide whether these square roots are integers or irrational numbers:

1. $\sqrt{17}$

2. $\sqrt{121}$

3. $\sqrt{82}$

4. $\sqrt{1000}$

5. $\sqrt{144}$

Using the method of this section, make a first estimate of these square roots. Square your answer to determine its accuracy:

6. $\sqrt{17}$

7. $\sqrt{8}$

8. $\sqrt{26}$

9. $\sqrt{38}$

10. $\sqrt{6}$

11. $\sqrt{12}$

Section **10**
Properties of Roots

The Root of a Product or Fraction

When we examine the square root of $(4 \cdot 9)$, we can discover a useful property by getting the result in two different ways:

$$\begin{aligned}\sqrt{4 \cdot 9} &= \sqrt{(2 \cdot 2) \cdot (3 \cdot 3)} \\ &= \sqrt{(2 \cdot 3) \cdot (2 \cdot 3)} \\ &= \sqrt{(2 \cdot 3)^2} \\ &= (2 \cdot 3) = 6 \\ &= \sqrt{4} \cdot \sqrt{9}\end{aligned} \qquad \textit{or} \qquad \sqrt{4 \cdot 9} = \sqrt{36} = 6$$

To find the square root of a product, we find the square roots of both factors and then multiply them to get the result. *The square root of the product is the product of the square roots.* Both methods give the same result. We check the property by finding the square root of the product. Here is another example:

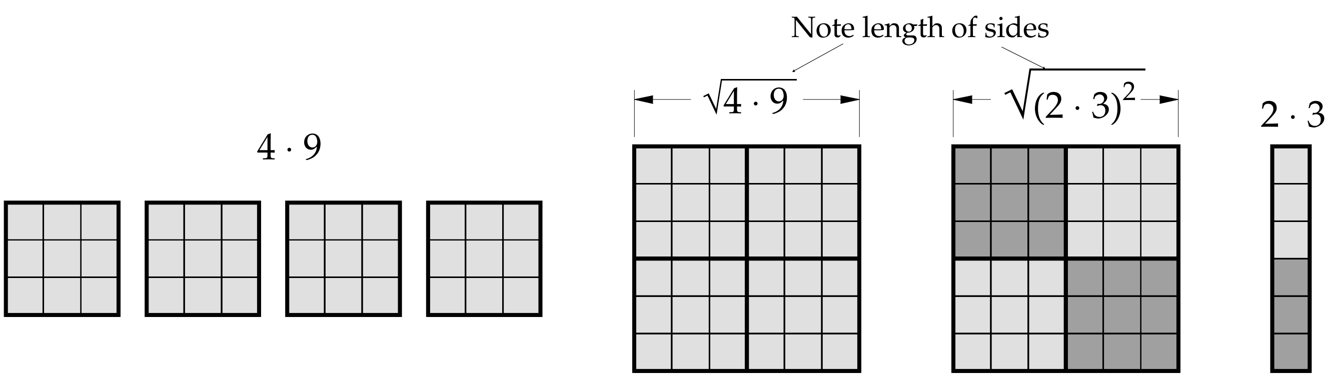
$$\begin{aligned}\sqrt{16 \cdot 25} &= \sqrt{16} \cdot \sqrt{25} \\ &= 4 \cdot 5 \\ &= 20\end{aligned} \qquad \textit{or} \qquad \begin{aligned}\sqrt{16 \cdot 25} &= \sqrt{400} \\ &= \sqrt{20 \cdot 20} \\ &= 20\end{aligned}$$

We can summarize this property as follows:

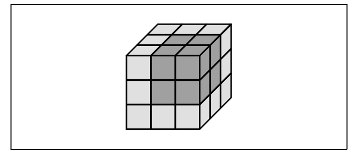
The square root of a product

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

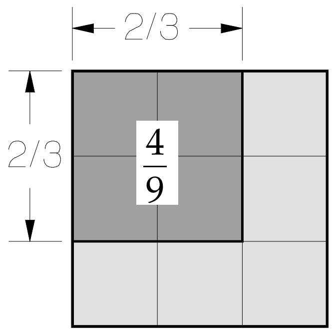
To demonstrate the first example, we begin with 4 groups of 9 $(4 \cdot 9)$ and then take the square root by arranging the groups of 9 in a 2 by 2 square. Each 9 is 3 by 3, so the resulting side (square root) is 2 groups of 3 or $2 \cdot 3$:



When we take the square *root* of a fraction, the same property applies:



$$\sqrt{\frac{4}{9}} = \sqrt{\frac{2}{3} \cdot \frac{2}{3}}$$
$$= \frac{2}{3}$$
$$= \frac{\sqrt{4}}{\sqrt{9}}$$



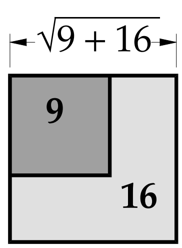
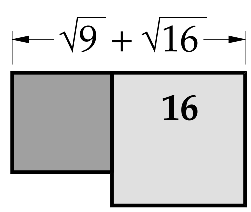
In summary:

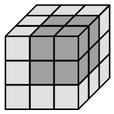
The square root of a fraction

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

Common Errors

Be careful. Taking the square root is a factoring process. The square roots of products and quotients can easily be factored and simplified. The same is *not* true for the square roots of sums.

Error (False)	**Picture** (Why it's not true)	**Looks like:** (True)
$\sqrt{9+16}$ *does not equal* $\sqrt{9}+\sqrt{16}$	 $= 5$  $= 7$	$\sqrt{9 \cdot 16} = \sqrt{9}\sqrt{16}$



Exercises

Use the properties from this section to simplify these expressions:

1. $\sqrt{100 \cdot 16}$

2. $\sqrt{4 \cdot 36}$

3. $\sqrt{9 \cdot 25}$

4. $\sqrt{16 \cdot 49}$

5. $\sqrt{\frac{36}{25}}$

6. $\sqrt{\frac{100}{9}}$

7. $\sqrt{\frac{49}{81}}$

8. $\sqrt{\frac{64}{25}}$

Factor the larger numbers given below into perfect square factors and then simplify:

9. $\sqrt{400}$

10. $\sqrt{2500}$

11. $\sqrt{4900}$

12. $\sqrt{8100}$

Show that these exercises have the same result if you simplify first and then take the square root or if you take the square root first and then simplify:

13. $\sqrt{\frac{25}{100}}$

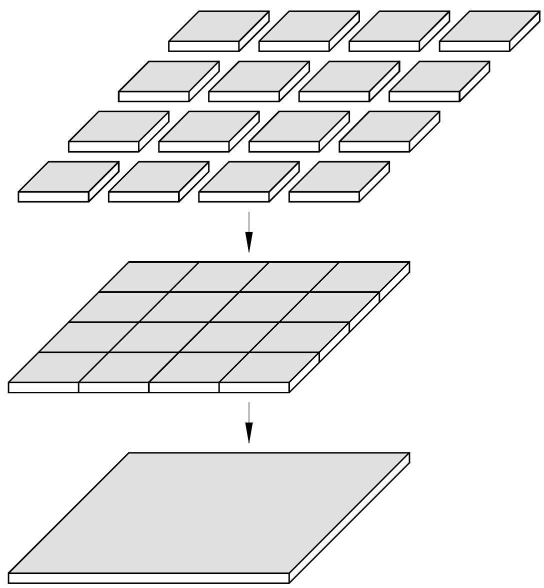
14. $\sqrt{\frac{36}{4}}$

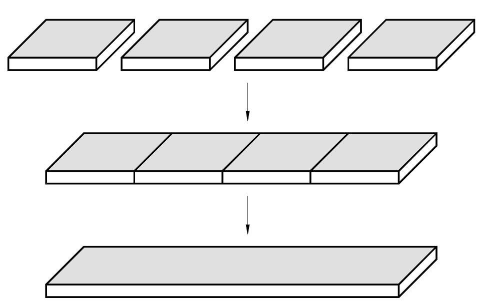
15. $\sqrt{\frac{144}{81}}$

16. $\sqrt{\frac{100}{16}}$

Chapter 9

Polynomials

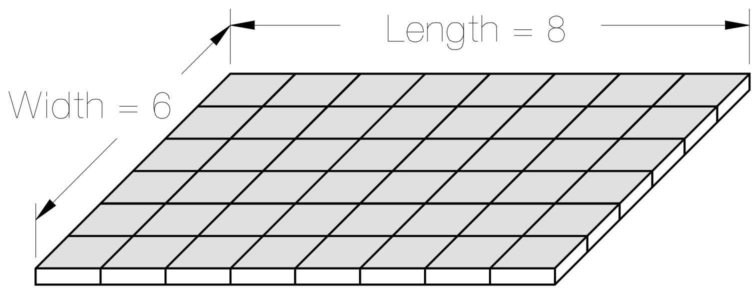




Section 1
Using Unknowns: 1, x, x^2

The Meaning of Multiplication

To do multiplication with unknowns we must remember how we do multiplication with positive and negative numbers. When we multiply numbers we are making rectangles, and the product (the answer to the multiplication problem) is the area of the rectangle:

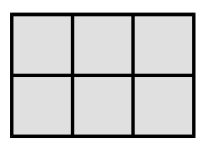


Length × Width = 8 · 6 = 48 units

To get the sign of the answer (product), we start with the colored side up, and then flip the chips once for each negative (-) sign in the problem.

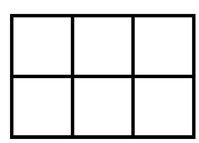
$(+3)\cdot(+2) = 6$

(No Flips)



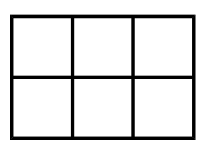
$(+3)\cdot({}^-2) = {}^-6$

(One Flip)



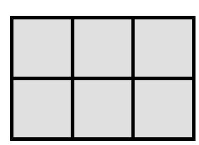
$({}^-3)\cdot(+2) = {}^-6$

(One Flip)

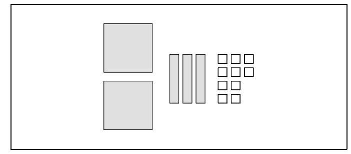


$({}^-3)\cdot({}^-2) = +6$

(Two Flips)

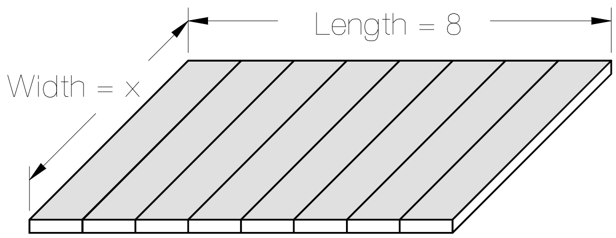


We should note here that the two numbers we are multiplying become the **dimensions** of the resulting rectangle. If just one of these dimensions is negative, then the rectangle ends up white side up (negative). If both dimensions are negative, then the product (the rectangle) will end up positive, with colored side up, just as it does when neither side is negative.



Multiplying With Unknowns

So far, we have been multiplying lengths and widths that are numbers. Can we make areas that have lengths or widths of x? Multiplication will still have the same meaning, but the sides may have dimensions involving x.



Length times Width $= 8 \cdot x = 8x$

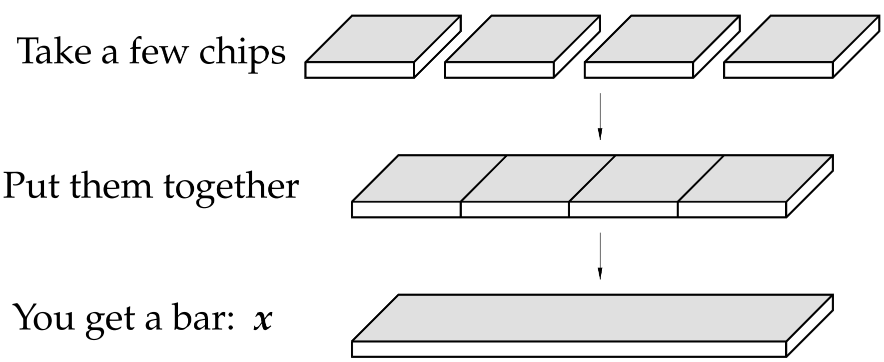
.

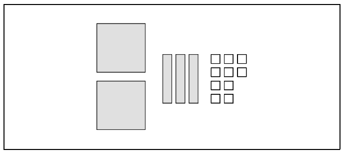
As we begin making rectangles using both numbers and unknowns, the process for determining the sign of the rectangle will remain the same. If just one dimension (side) of a rectangle is negative, the white side is up and the result is negative; if both or neither sides are negative, then the colored side is up and the answer is positive.

When we're using unknowns, we can still think of making rectangles, but now our rectangles will have bars as well as units.

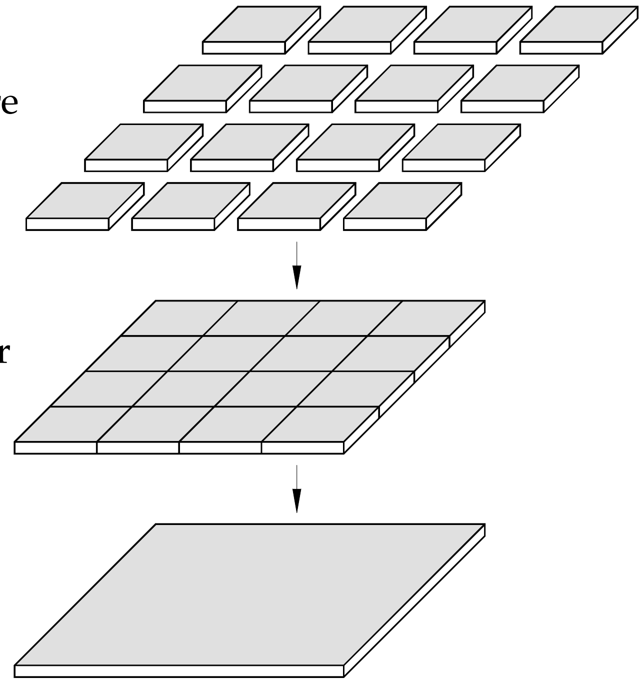
Definition of x and x^2

First, let's define x. Take a few chips and line them up in a row. Imagine that they are joined together in a bar, but then erase the boundaries so that all we see is a bar of unknown length. **This is x.**





Next, we will take a few chips and form a square. If we put the chips together and imagine that we cannot see exactly how many chips there are, we have built a square that is an unknown width and length. This is $x \cdot x$ or x^2:

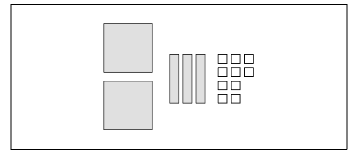


More on x and x^2

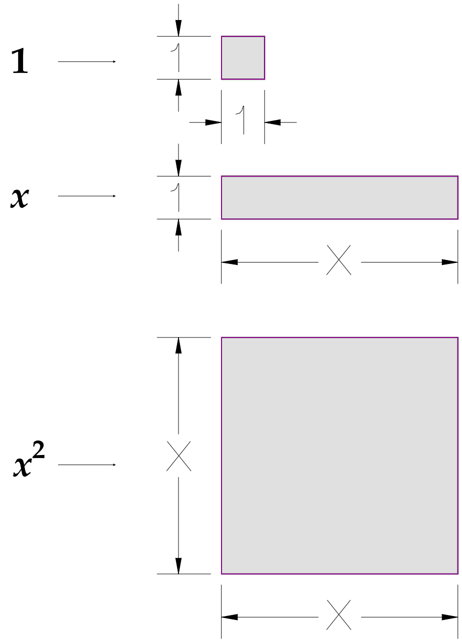
The following table shows the names and sizes of our new pieces. Note that the value of each piece is equal to its area.

Piece	**Value**	**Length** ×	**Width** =	**Area**
Chip or Little Square	**1** (Unit)	1	1	1
Bar	x	x	1	x
Big Square	x^2	x	x	x^2

Some of the pieces have sides in common. The unit and the x both have a side of one. The x and the x^2 both have a side of x. The x does not represent a specific number of chips; it represents *any* unknown number of chips. If you try to match up unit chips along the long (x) side of the x bar, you will find that neither 5 nor 6 nor any number of chips fits exactly.



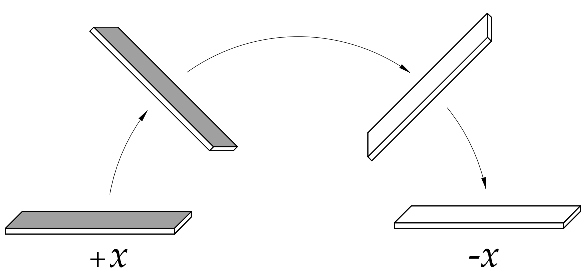
Our set of chips now looks like this:



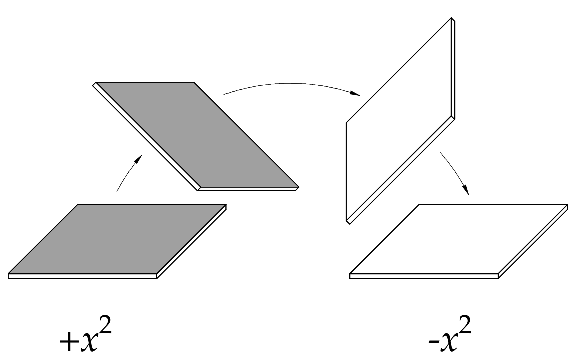
The Opposites of x and x^2

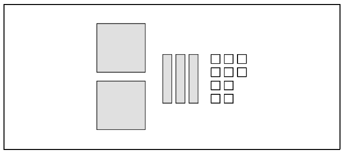
Unknowns can also have opposites. We have already been introduced to the idea that flipping a unit chip to the white side represents $^-1$; now we will put together the opposites of x and x^2.

First, let's review the idea of the opposite of the x bar. This will be written as **-*x*** **a**nd will be called **negative *x*** or **the opposite of *x***.



In the same way, we can construct **$-x^2$**:

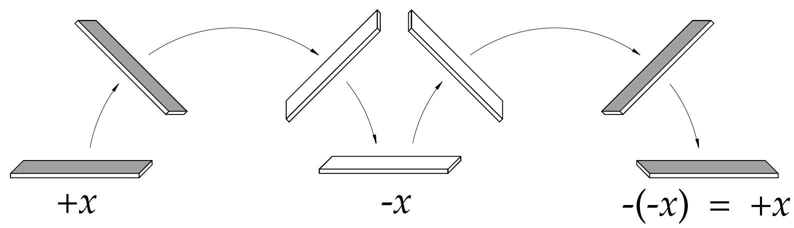




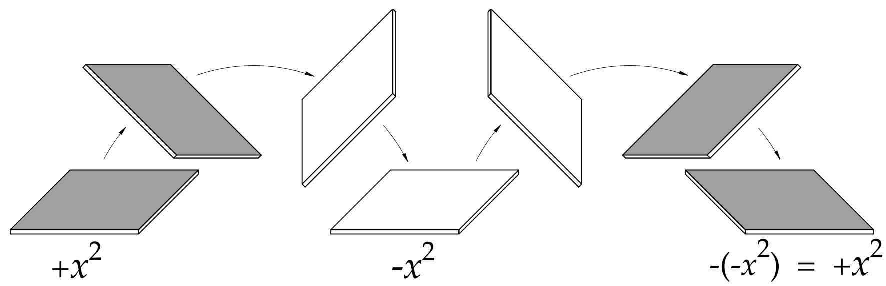
These new pieces behave in the same way as single chips—*Flipping an* $\boldsymbol{x}$ *or* $\boldsymbol{x}^2$ *changes the sign.*

$$-(x) = -x$$
$$-(-x) = +x$$
$$-(x^2) = -x^2$$
$$-(-x^2) = +x^2$$

For $\boldsymbol{x}$:



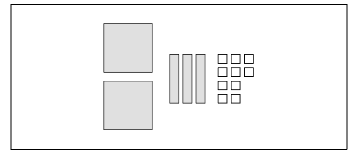
For $\boldsymbol{x}^2$:

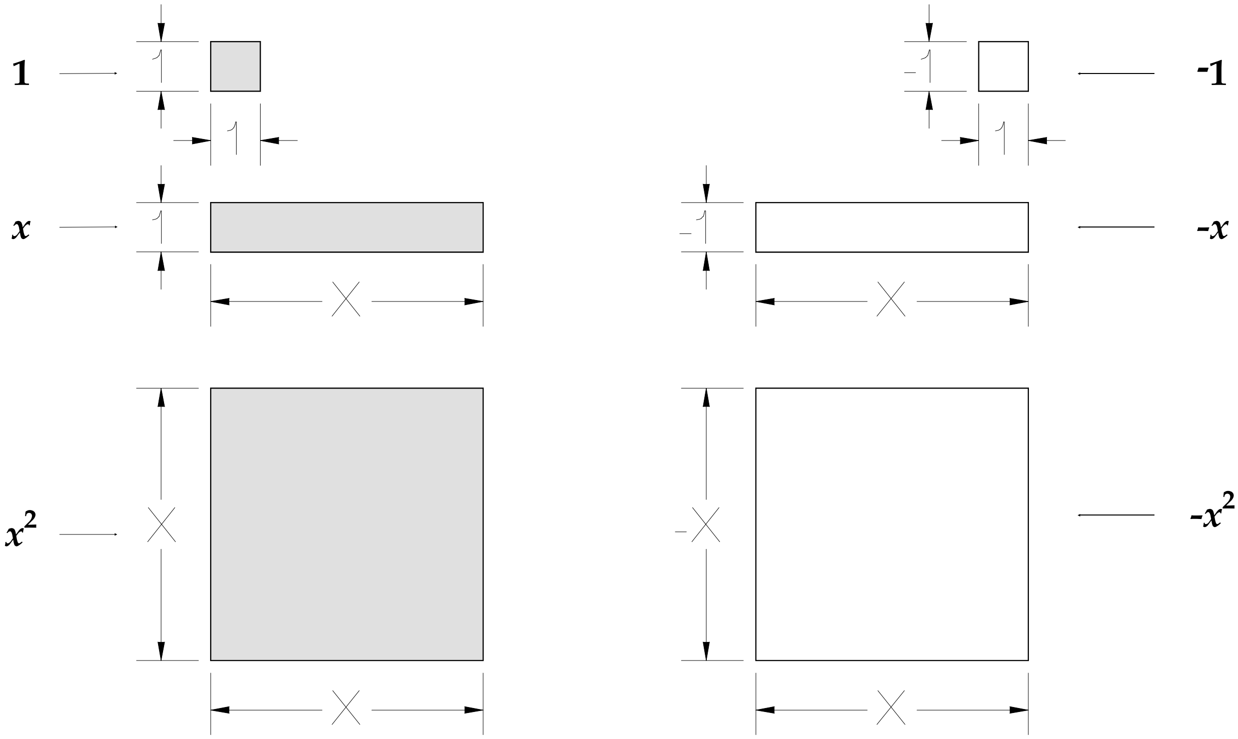


It is best to think of $-x$ and $-x^2$ as the opposites of x and x^2 ($-x$ *is not necessarily a negative number*!). Here is an expanded table of our pieces:

Piece	Value	Length ×	Width =	Area
Chip or Little Square	**1** (Unit)	1	1	1
Negative Chip	**-1** (Unit)	1	-1	-1
Bar	$\boldsymbol{x}$	x	1	x
Opposite Bar	$\boldsymbol{-x}$	x	-1	$-x$
Big Square	$\boldsymbol{x^2}$	x	x	x^2
Opposite Big Square	$\boldsymbol{-x^2}$	x	$-x$	$-x^2$

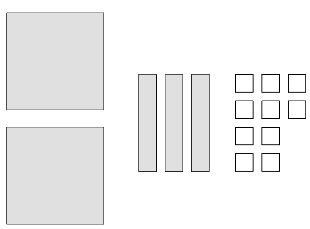
Finally, here are all of the new pieces:



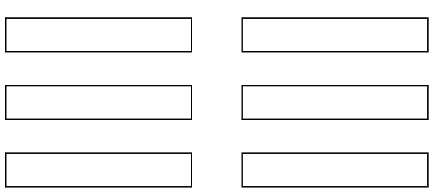


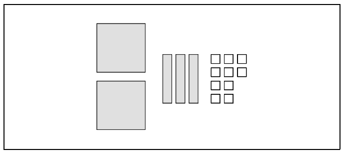
Polynomials

When we have an assortment of pieces such as units, x's, and x^2 chips, we call this a **polynomial**. A polynomial can have many types of pieces

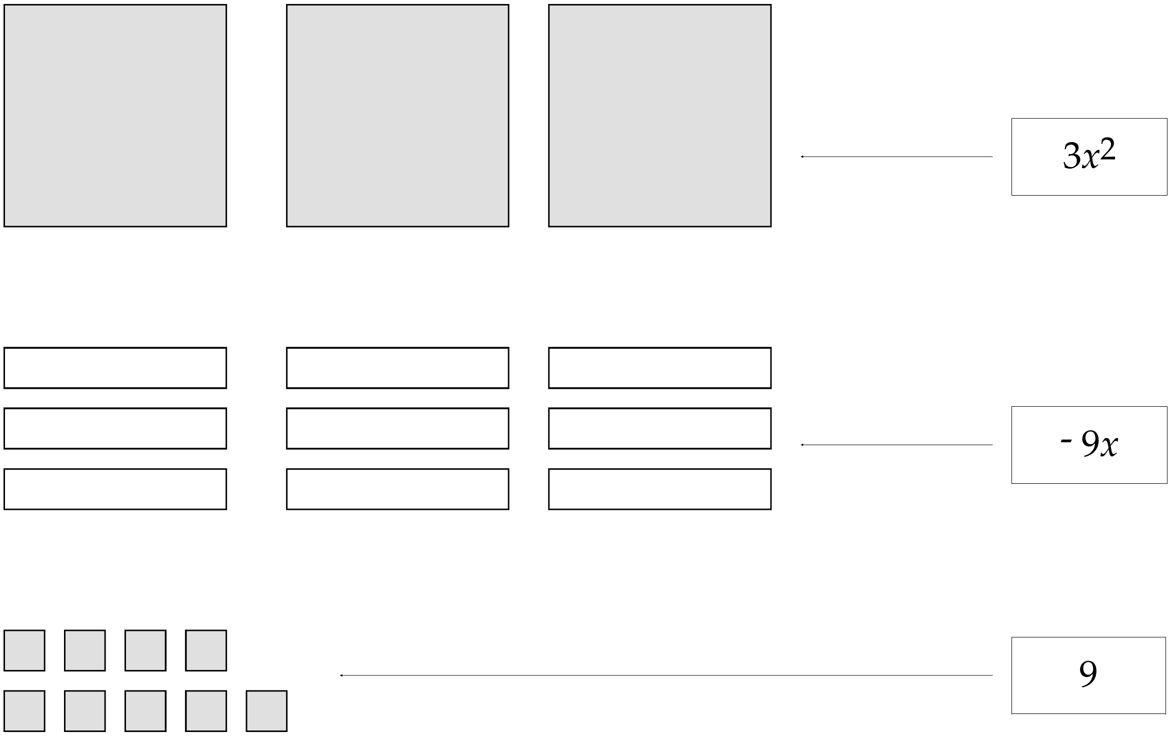


or just one kind of piece.





Each group of like shapes is called a **term**. When we have two bars we have two x's, or $2x$. Three x^2 pieces can be written as a $3x^2$ term. Here are some examples of terms:



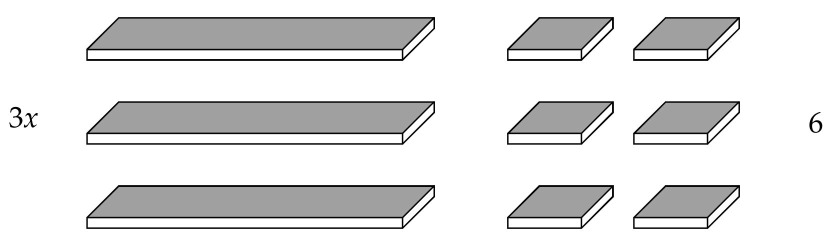
An expresion with *two* terms is called a **binomial**. An expression with *three* terms is called a **trinomial**.

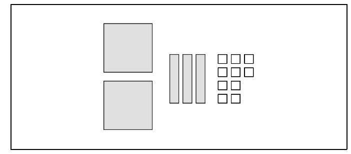
Exercises

Set up the following expressions with chips and identify individual terms:

Example: $3x + 6$

Solution: Terms are $3x$ and **6.**





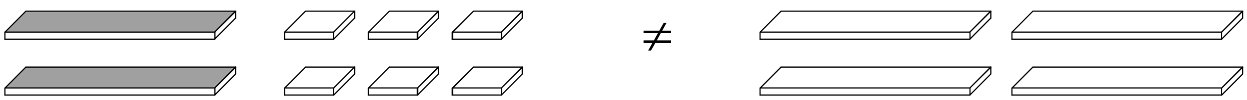
1. $7x$
2. $7x - 2$
3. $4x^2$
4. $3x^2 - 6$
5. $6 - 2x^2$
6. $2x^2 - 3x + 12$
7. $^{-}2x^2 - 5x - 1$
8. $^{-}0x^2$
9. $5 - 3x^2$
10. $2x + 3$
11. $x^2 - 5x + 6$
12. $2x - x^2 + 4$
13. $4x + 3x^2$
14. $2x^2 - 7$
15. $3x^2 - 5x + 2$

Section 2
Adding and Subtracting Polynomials

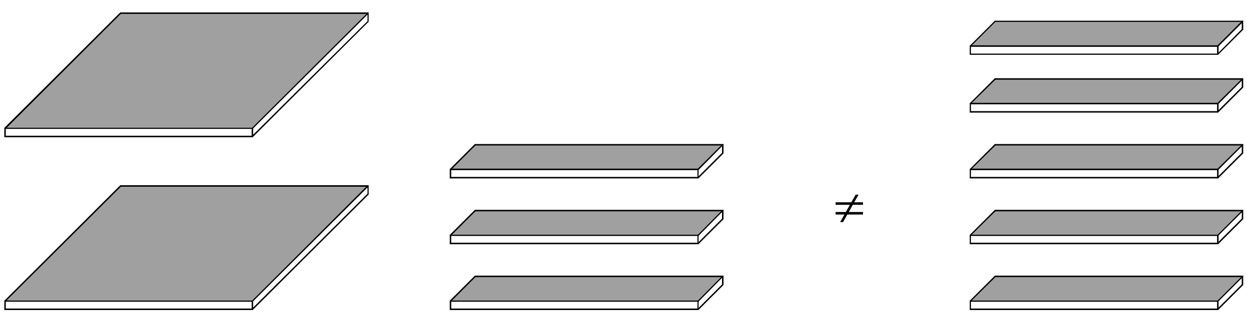
Combining Like Terms

If a polynomial has two separate kinds of pieces (bars and chips), they are not the same size and shape. This means that, in the symbolic language of algebra, we must also have ***two separate terms***; one with ***x***'s (bars) and the other with units (chips). These two terms are made up of ***different pieces*** and therefore they *cannot be combined*.

$2x - 6$ is not $^{-}4x$

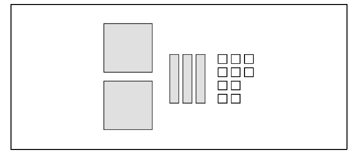


$2x^2 + 3x$ is not $5x$



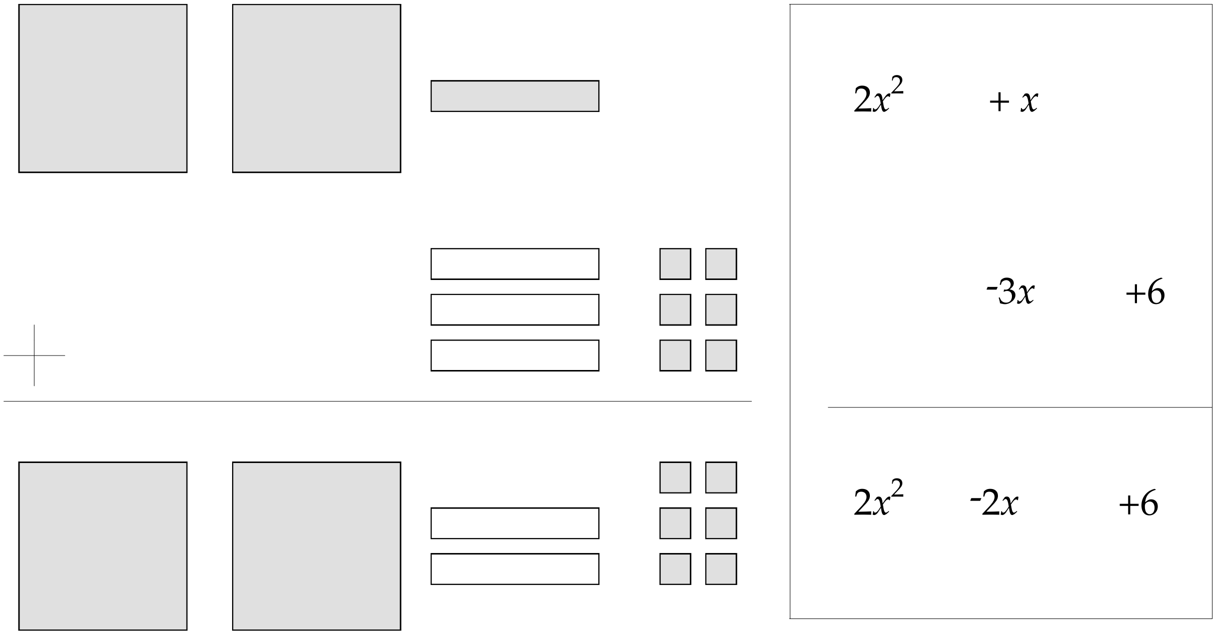
We cannot combine these terms because different shapes cannot be treated as if they are the same; they must be kept separate, ***x*'s** in one term and units in another.

If you use the chips and think of polynomials as groups of shapes, it will be easy to work with them without needing to memorize any rules. Just combine similar shapes.



Adding Polynomials

Adding two polynomials is done in the same way that we add units—we take similar pieces from each polynomial and combine like terms.



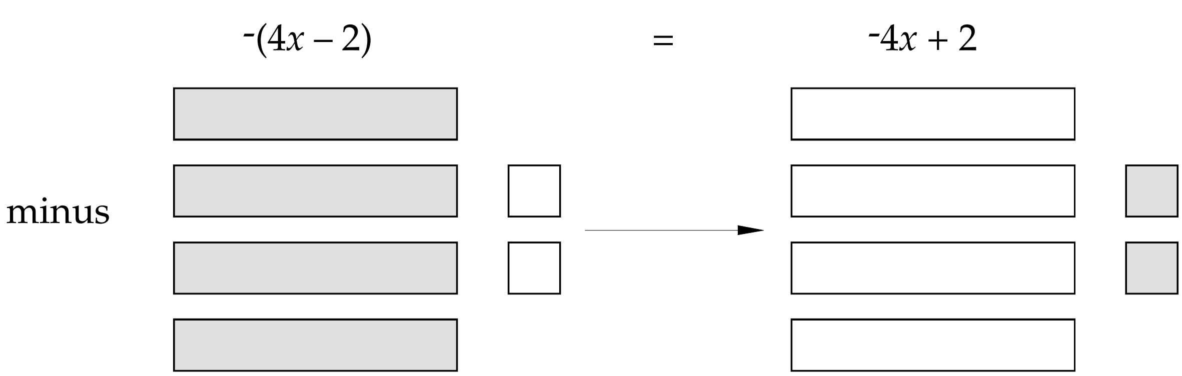
The algebra symbols show like *terms* being combined; the chips show like *pieces* being combined.

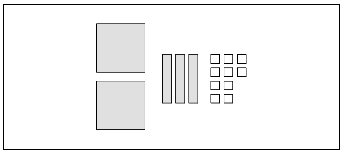
Subtracting Polynomials

To subtract a polynomial, we think of adding the opposite:

$$(3x - 5) - (4x - 2) = (3x - 5) + {}^{-}(4x - 2)$$

Just as with signed numbers, the negative sign means take the opposite, or *flip the chips*.



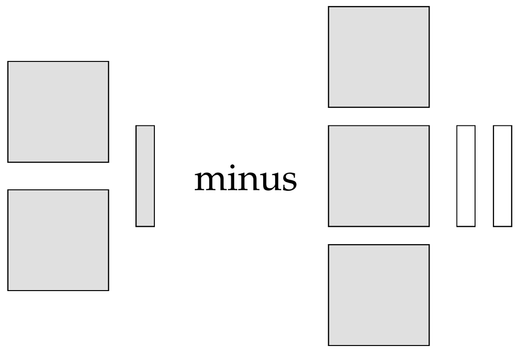


So for each subtraction, write the problem as an addition (flip the subtracted chips) and proceed as usual by combining like terms.

- **identify the two polynomials**
- **subtraction becomes adding the opposite**
- **find the opposite**
- **add**

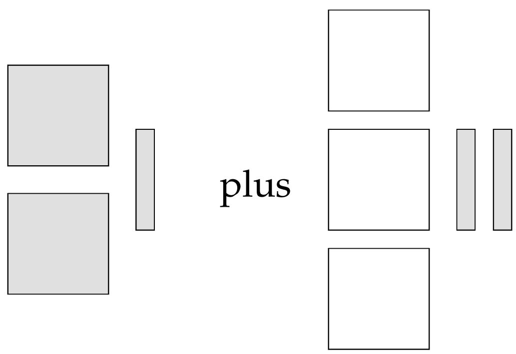
Here is an example of this process:

$(2x^2 + x) - (3x^2 - 2x)$

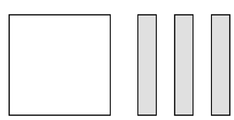


$(2x^2 + x) + {}^{-}(3x^2 - 2x)$

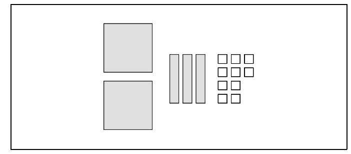
$(2x^2 + x) + {}^{-}3x^2 + 2x$



$-x^2 + 3x$



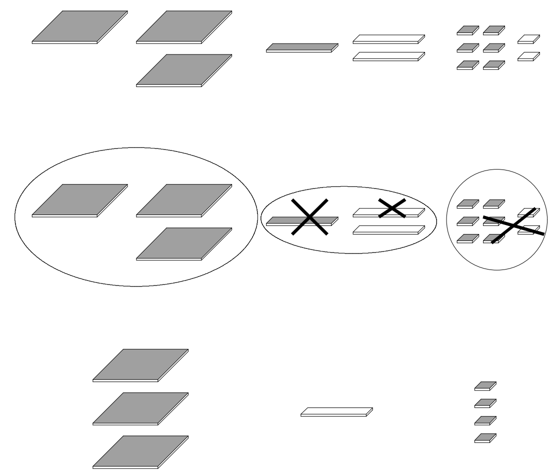
Exercises



Use your chips to set up the following problems. Combine similar shapes (terms).

Example: $x^2 + 2x^2 + x - 2x + 6 - 2$

Solution: $3x^2 - x + 4$



1. $3x - 2x$

2. $^-3x + 5 - 6x$

3. $5 - 4x^2 + x$

4. $2x^2 + x + x^2 + 3x$

5. $^-3x^2 - x^2$

6. $5x + 5 + (^-6x) + (^-3) + 2x^2$

7. $2x + x^2 - 5x - 3$

8. $^-3x^2 + 5 - x - 7$

9. $2x - 5 + x^2 + 3x$

10. $5 - (^-3x) + x + 7$

Use chips to complete these addition problems and write the algebra symbols as well:

11. $(3x - 2) + (5x - 6)$

12. $(x^2 + 3x + 3) + (2x^2 - x)$

13. $(^-2x^2 - x - 1) + (2x^2 + x - 1)$

14. $(2x - 5) + (x^2 + 3x + 2)$

15. $(x^2 - 3x + 1) + (x^2 - 7)$

16. $(^-5x + 3) + (2x^2 - 3x)$

17. $(x^2 + 3x - 2) + (3x^2 - x - 5)$

18. $(^-3x + 5) + (4x^2 - 5)$

Perform the following subtractions:

19. $(3x - 2) - (5x - 6)$

20. $(x^2 + 3x + 3) - (2x^2 - x)$

21. $(^-2x^2 - x - 1) - (2x^2 + x - 1)$

22. $(6x - 2) - (3 - 2x)$

23. $(x^2 + 3x - 1) - (x^2 - 2x + 5)$

24. $(2x + 3) - (x^2 - 5x)$

25. $(2x^2 - 5) - (x^2 + 5x - 6)$

26. $(3x^2 - 5x + 1) - (x^2 - 3x - 2)$

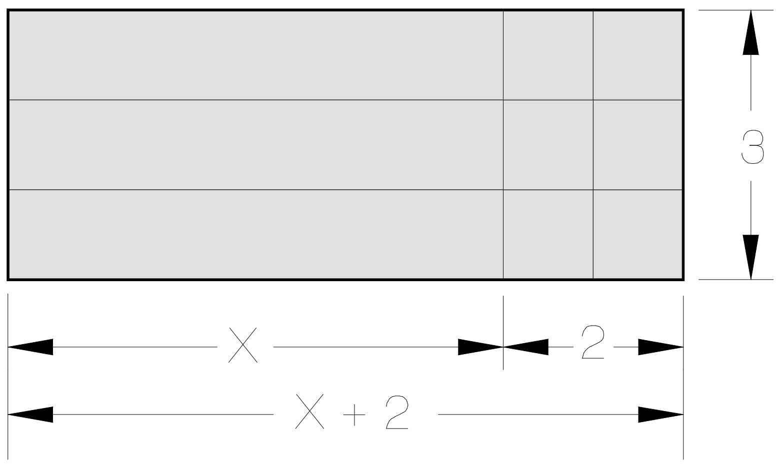
Section 3
Multiplying Polynomials

Multiplying with One Unknown

If we have a product (multiplication) like

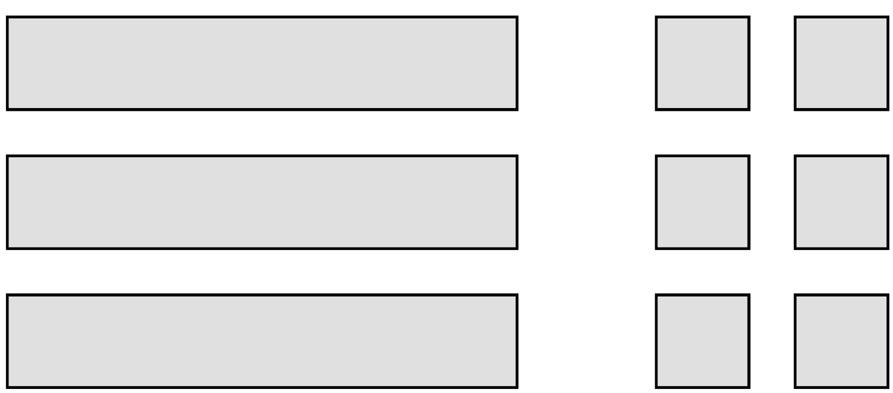
$$(3)\cdot(x + 2)$$

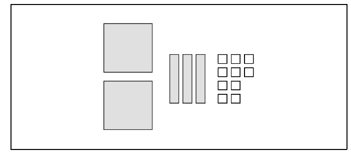
we make a rectangle with dimensions (sides) of **3** and ***x* + 2**, like this:



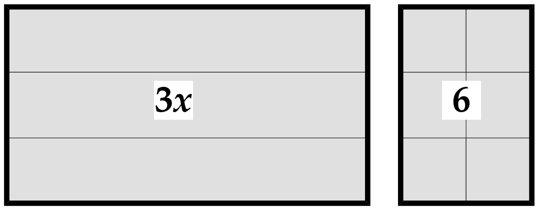
The product, or area, will just be the sum of the pieces we use, which is three bars (**3*x***) and six little squares (**6**):

$$3(x + 2) = 3x + 6$$

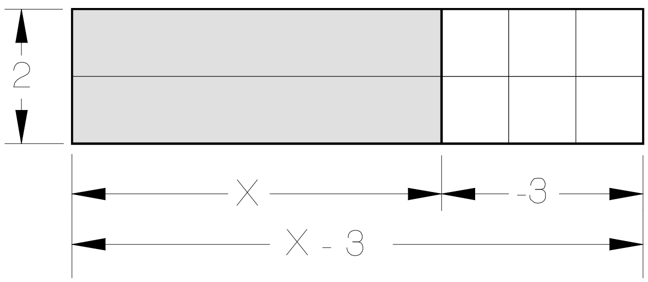




The product, or area, is **$3x + 6$.** We can think of this as being two smaller rectangles added together: one rectangle **3** by x, and the other rectangle **3** by **2**. In this case both rectangles are positive.



If one piece of our product is negative, the product will look like this:



This time one of the smaller rectangles is positive ($2 \cdot x = 2x$) while the other smaller rectangle is negative ($2 \cdot {}^{-}3 = {}^{-}6$). Thus the product is.

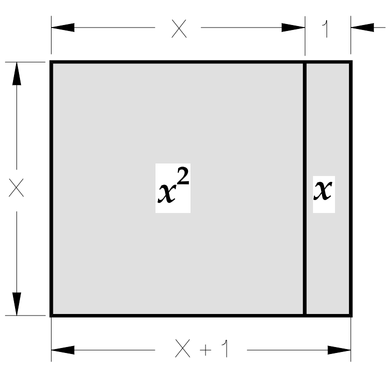
$$2(x - 3) = 2x - 6$$

Using Unknowns in Both Dimensions

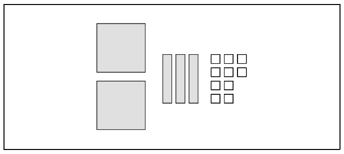
If we wish to find the product

$$x(x + 1)$$

we build a rectangle x wide and $x + 1$ long. This is a rectangle made up of two smaller rectangles. One is x by x or x^2, the other is x by **1** or x**:**



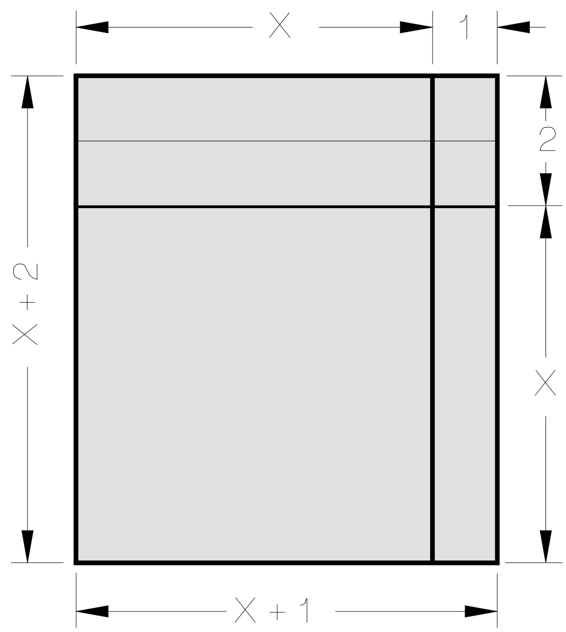
$$(x)(x + 1) = x^2 + x$$



If we wish to find the product

$$(x + 2)(x + 1)$$

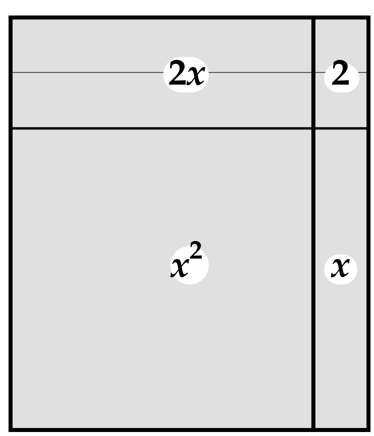
we must build a rectangle having each factor (***x* + 2** and ***x* + 1**) as one dimension of length or width.



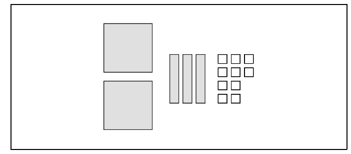
As can be seen in this illustration, the result is a large rectangle which can be subdivided into four smaller rectangular areas. In the upper right are two small chips, a rectangle 1 by 2 units. At the top left are two bars, defining a rectangle 2 by x. At the lower right is one bar, in a rectangle **1** by $\boldsymbol{x}$. Finally, the large square forming the lower left corner is the rectangle with sides $\boldsymbol{x}$ by $\boldsymbol{x}$ and area $\boldsymbol{x^2}$.

The area of the larger rectangle is the sum of these 4 areas:

$$(x + 2)(x + 1) = x^2 + x + 2x + 2$$



This time, two of the terms **are** made of the same size pieces; the x and the $2x$ are both made up of bars, so they can be combined giving



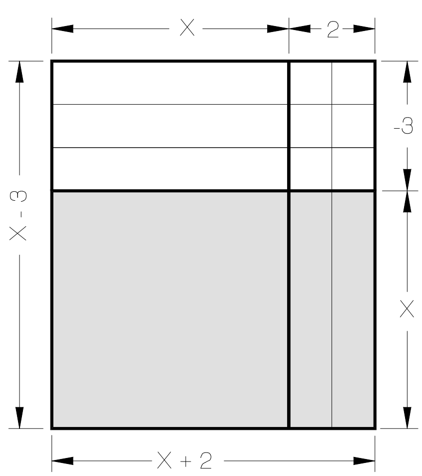
$$(x + 2)(x + 1) = x^2 + 3x + 2$$

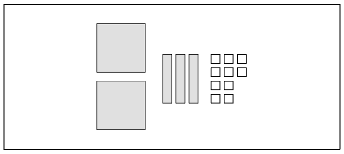
Each of the four smaller rectangles inside the large rectangle represents a piece of the product we are seeking. When using symbols, we can find these four smaller areas by using a technique called the **FOIL method**. This is defined as shown below:

F	**F**irst times **F**irst	$(x + 2)(x + 1)$	$x \cdot x = x^2$
O	**O**utside times **O**utside	$(x + 2)(x + 1)$	$x \cdot 1 = x$
I	**I**nside times **I**nside	$(x + 2)(x + 1)$	$2 \cdot x = 2x$
L	**L**ast times **L**ast	$(x + 2)(x + 1)$	$2 \cdot 1 = 2$

Each piece of the product, the x^2, the $1x$, the $2x$, and the 2, is one of the smaller rectangles in our figure.

If one or both sides of any of these smaller rectangles is negative, then we use our rules for signs to determine the sign of that particular rectangle. For example, let's illustrate the product of $(x - 3)(x + 2)$:



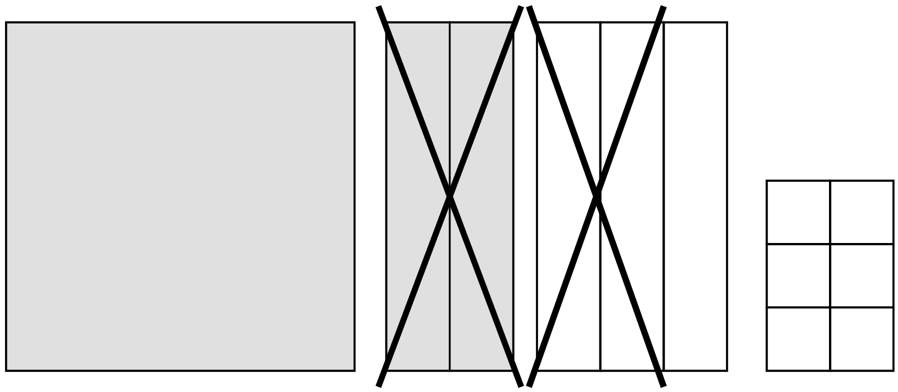


In this example, of the four rectangles within the figure, two are positive and two are negative (white side up).

$$(x-3)(x+2) = x^2 + 2x - 3x - 6$$

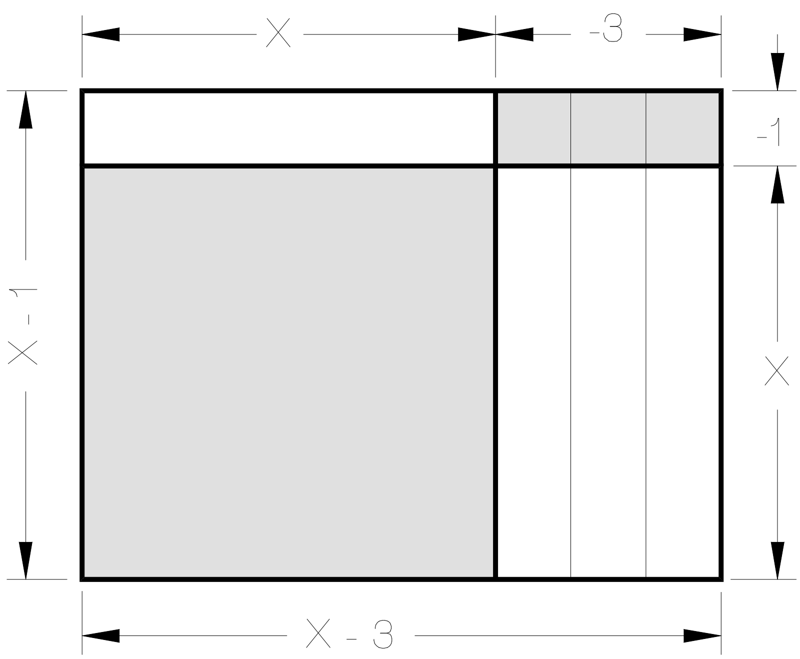
If we combine like terms, the x's (positive bars) will be cancelled out by the negative bars leaving:

$$(x-3)(x+2) = x^2 - x - 6$$

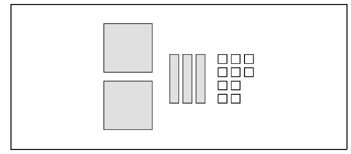


Again, each piece of the rectangle comes from one piece of the product when using the FOIL method. When some parts of our area are positive and other parts are negative, we can think of the product, or area of the figure, as being the difference of the areas, or the area left over when the white is taken away from the colored area.

Here's an example having two negative terms:

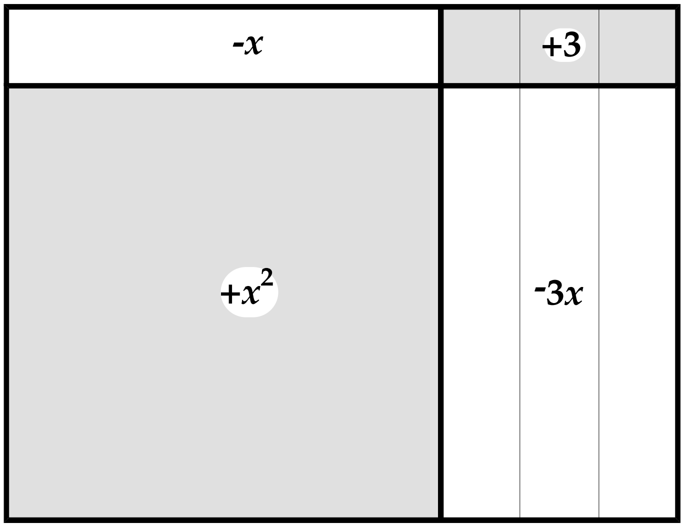


This is



$$(x-1)(x-3)$$

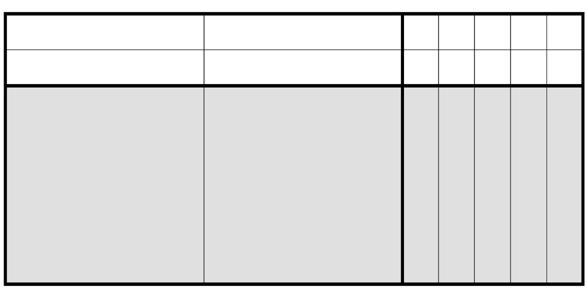
Can you explain why the ***x*-bars** are turned white side up, and the three chips are turned colored side up?



Here is one final example. Find the product

$$(x-2)(2x+5)$$

This requires forming a rectangle of dimensions **(*x* – 2)** by **(2*x* + 5):**



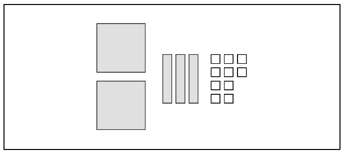
From this we see

$$(x-2)(2x+5) = 2x^2 + 5x - 4x - 10$$

Combining like terms gives

$$(x-2)(2x+5) = 2x^2 + x - 10$$

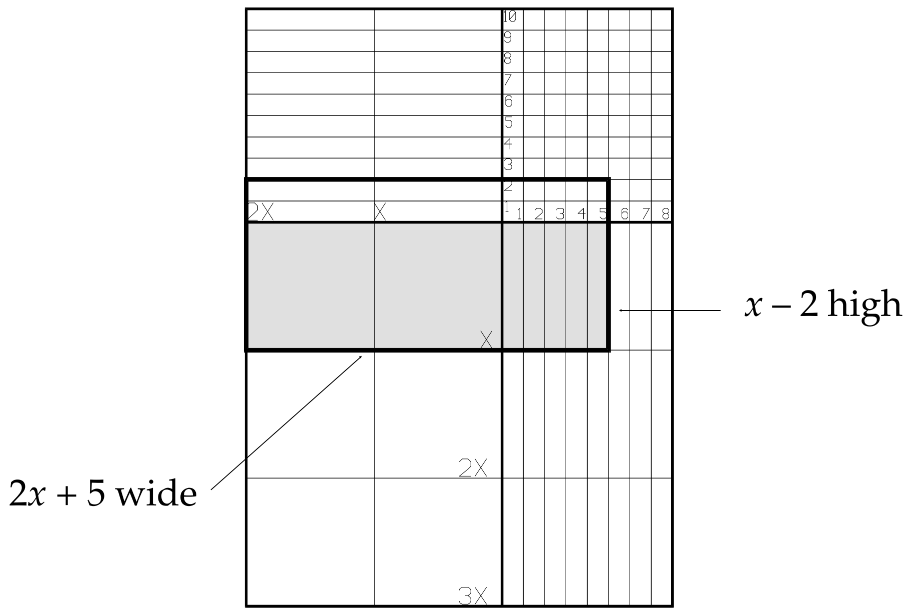
Do you understand where the signs on each term came from?



A plastic grid is included with this book. You can use the grid instead of the chips to plot multiplication of polynomials. Use a water-based marker to outline the rectangles or chips you want to use. You can also mark areas as positive or negative.

The grid is ruled in units of x's and **ones.** The darker lines across the grid (one horizontal and one vertical) are the lines which separate the four smaller rectangles within the larger figure. Remember that each of these smaller rectangles has its own sign and represents one term of the product.

Here is the example above, using the grid:

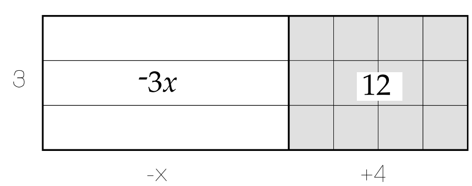


Exercises

Look at the example products and then use your chips to do the following multiplications.

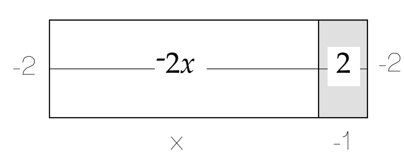
Example: $3(-x + 4)$

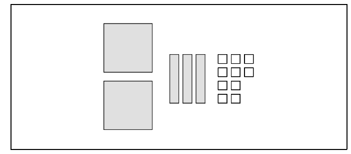
Solution: $^-3x + 12$



Example: $^-2(x - 1)$

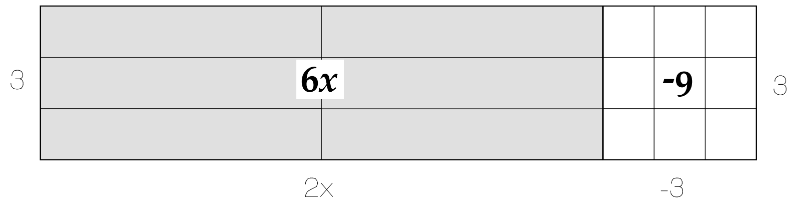
Solution: $^-2x + 2$





Example: $3(2x - 3)$

Solution: $6x - 9$



Multiply:

1. $2(x - 4)$

2. $3(2x + 1)$

3. $3(-x + 1)$

4. $^{-}2(x - 3)$

5. $^{-}2(-x - 1)$

6. $2(3x - 1)$

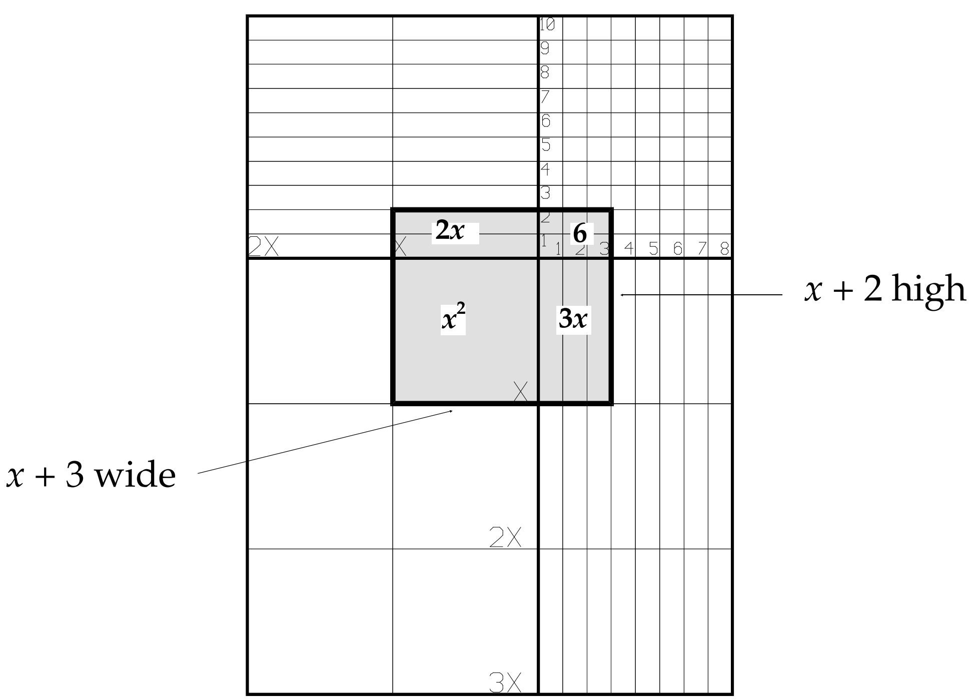
7. $^{-}3(-x + 3)$

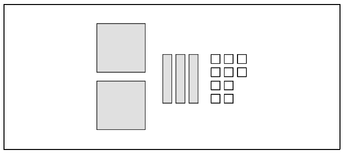
8. $^{-}2(2x - 5)$

Try these problems using chips or the grid:

Example: $(x + 3)(x + 2)$

Solution: $x^2 + 3x + 2x + 6 = x^2 + 5x + 6$





9. $5(2x - 3)$

10. $-3(x - 5)$

11. $2(-2x + 1)$

12. $-5(-2x - 3)$

13. $-5(3x - 2)$

14. $2(5 - 3x)$

15. $-4(3 - x)$

16. $(x + 4)(x + 1)$

17. $(x - 3)(x + 4)$

18. $(x - 1)(x - 5)$

19. $(x + 5)(x - 3)$

20. $x(x - 6)$

21. $(2x + 1)(x - 4)$

22. $-x(3x - 2)$

23. $(2x - 3)(x - 2)$

24. $(x + 3)(x - 5)$

25. $(x - 2)(x - 6)$

26. $(x + 3)(2x - 1)$

27. $(2x - 3)(x + 2)$

28. $-x(3 - 2x)$

29. $(x - 2)(2x + 1)$

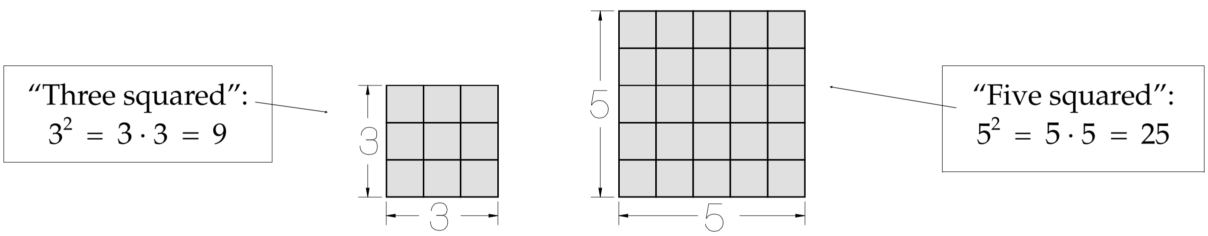
30. $(2x - 1)(2x + 3)$

Section 4
Special Products

Perfect squares

Two types of polynomials are considered special. These special polynomials are called **perfect squares** and **the difference of two perfect squares**.

Any time we make a rectangle where the length and width are the same, we get a square. This is obviously true if the sides of the square are just numbers.



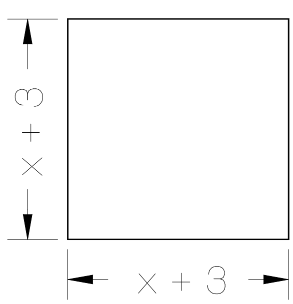
In fact, numbers which can be made into a square in this way are called **perfect square** numbers. The first six perfect square numbers are

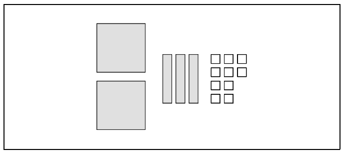
$$1,\ 4,\ 9,\ 16,\ 25,\ 36$$

Can you name the next six perfect square numbers in the series? If you take a number of chips from this list you will be able to arrange them into a perfect square, just as the name suggests.

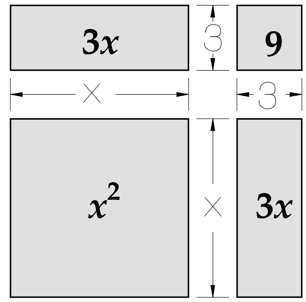
In the same way, if we multiply a polynomial having two terms (a **binomial**) times itself, we get a rectangle which has the same length and width: a *perfect square*.

Using chips, if we multiply the quantity $(x+3)$ times itself, giving $(x+3)^2$ or x plus three, quantity squared, we will be making a rectangle having the same length and width: a square.





Breaking this square into its four smaller areas we find that two of them, the units and the x^2's, are smaller squares.



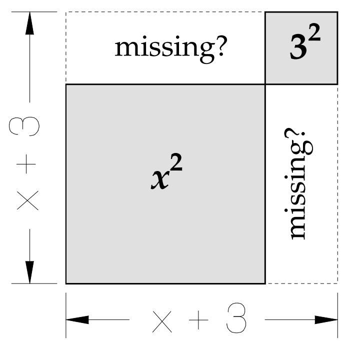
The remaining rectangles, the x's, are both the product of one side of each of these smaller squares ($x \cdot 3$).

Although this example seems obvious when working with chips, it is important to remember when using the symbolic language of algebra, that

$(x + 3)^2$ ***is not*** $x^2 + 3^2$

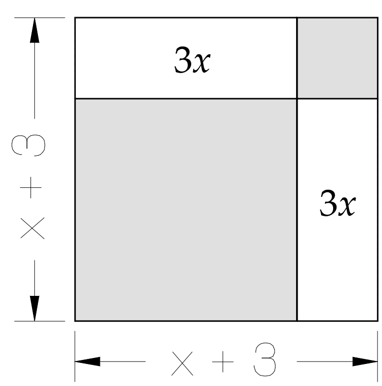
With the chips, we can see that these two expressions cannot be equal:

$(x + 3)^2$ is not just x^2 plus 3^2



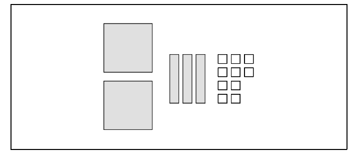
We must include the two rectangles which each have area $3x$. Remember the FOIL method:

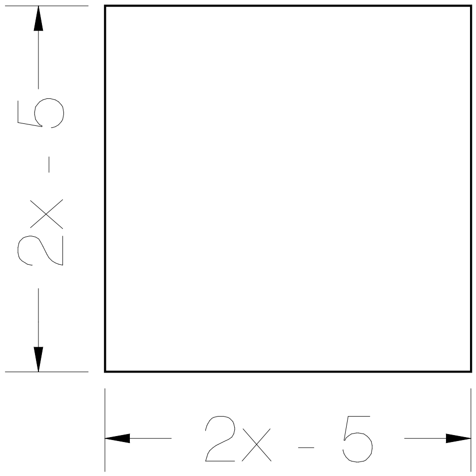
$$
\begin{aligned}
(x+3)^2 &= (x+3)(x+3) \\
&= x^2 + \mathbf{3x} + \mathbf{3x} + 9 \\
&= \mathbf{x^2 + 6x + 9}
\end{aligned}
$$



When we include all four of the areas shown we get the correct result.

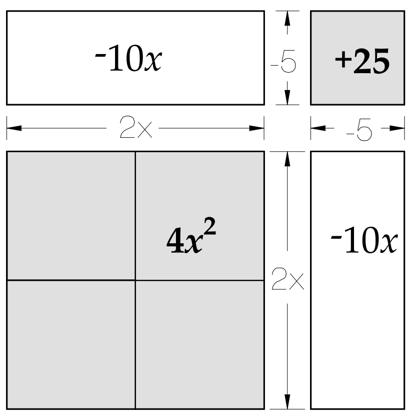
Now consider a second example of multiplying a binomial by itself to form a perfect square: $(2x - 5)^2$, or two x minus 5, quantity squared:





$$(2x - 5)^2 = (2x - 5)(2x - 5)$$

Filling in the four smaller rectangles within this diagram we again find two squares and two rectangles.



The two squares are both positive (colored side up),

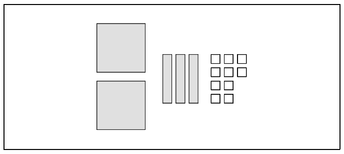
$$(2x)(2x) = 4x^2$$

$$({}^-5)({}^-5) = {}^+25$$

but this time the x-bars are negative (white side up) since

$$({}^-5)(2x) = {}^-10x$$

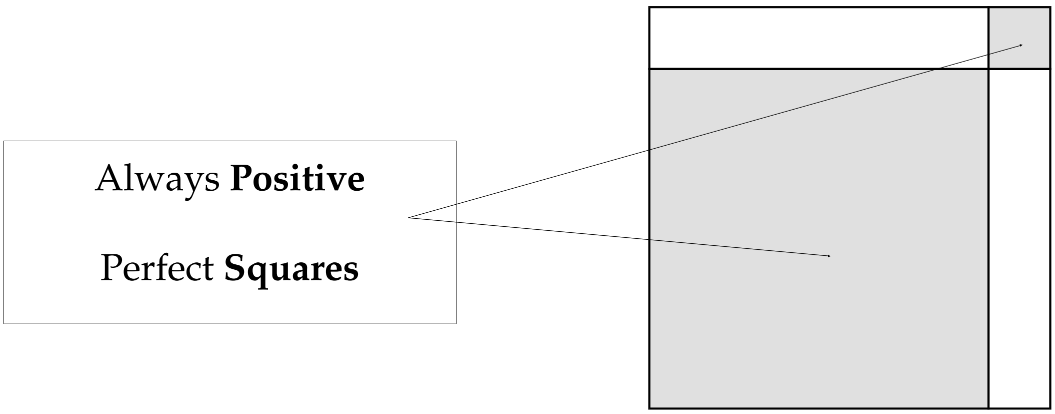
$$(2x)({}^-5) = {}^-10x$$



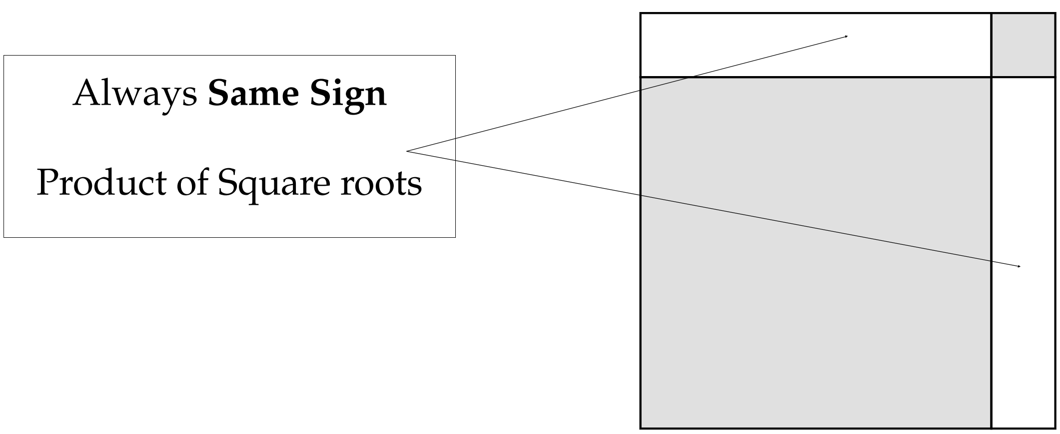
Again, using the FOIL method of symbol multiplication we get all four of the included areas and the correct result:

$$\begin{aligned}(2x-5)^2 &= (2x-5)(2x-5)\\ &= (2x)(2x)+(2x)(-5)+(-5)(2x)+(-5)(-5)\\ &= 4x^2-10x-10x+25\\ &= \mathbf{4x^2-20x+25}\end{aligned}$$

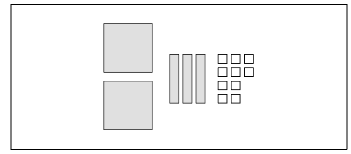
The smaller squares (x^2 pieces and units) within a perfect square are always positive in value (colored side up). This is because we get both of them by multiplying a number times itself, which always gives a positive result.



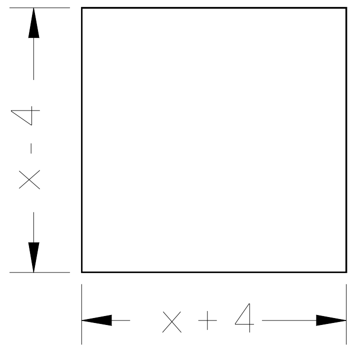
As the two examples demonstrate, the *x*-bars in a perfect square trinomial can sometimes be positive and sometimes be negative,. But in any one perfect square, all of the *x*-bars must be the *same sign*, either all plus or all minus. The number of *x*-bars will always equal the product of the square roots of the units square and the x^2 square, times two (because there are two groups of *x*-bars).



The Difference of Two Perfect Squares

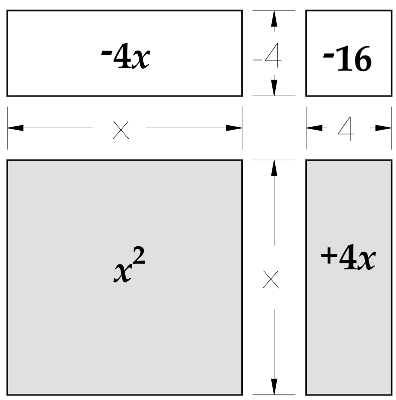


The two binomials $(x + 4)$ and $(x – 4)$ look very similar to each other; their only difference is the sign on the second term. If we multiply these two binomials together we get an interesting result.

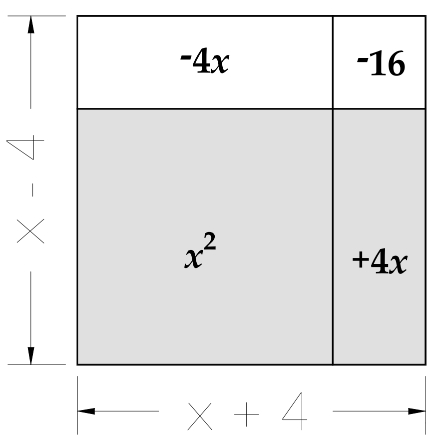


$$(x + 4)(x - 4)$$

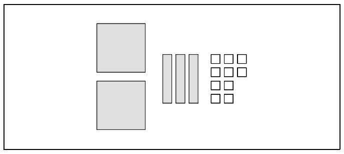
We again have a figure which appears square, but this time the two sides will have some pieces of different colors.



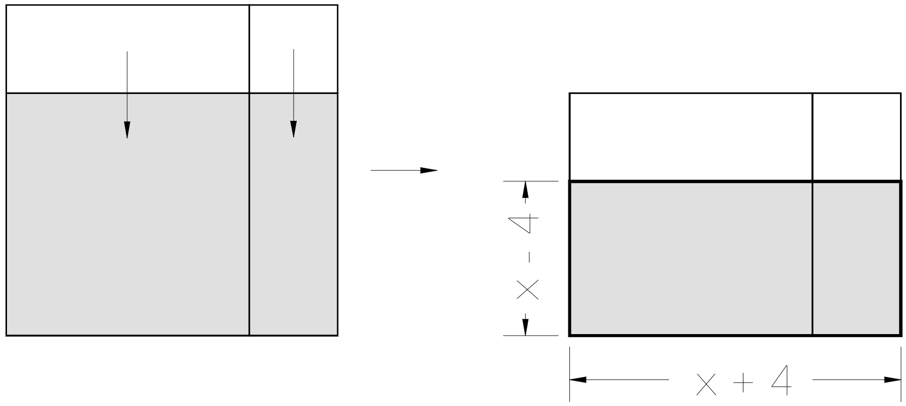
Shown as one rectangle, our example now looks like:



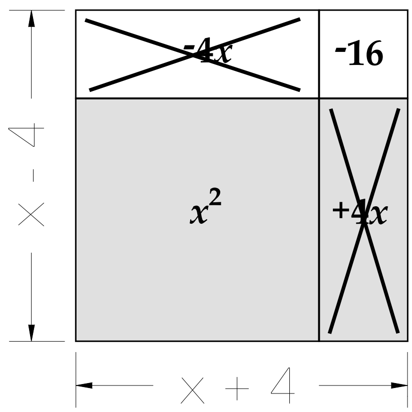
$$(x + 4)(x + 4) \; = \; x^2 - 4x + 4x - 16$$



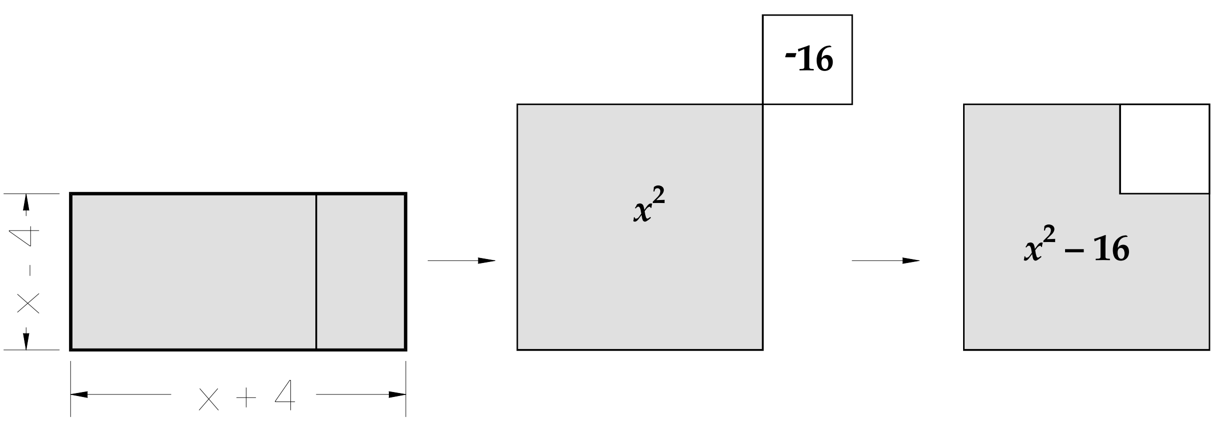
If we overlay the pieces and subtract the negative areas from the positive areas, we see that the resulting area is not really a square, but a rectangle having dimensions $(x + 4)$ and $(x - 4)$.

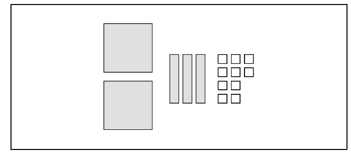


If we let the positive and negative pieces cancel in a different way we get an equivalent and surprising result.



The $^{+}4x$ and the $^{-}4x$ cancel each other out, leaving only x^2 and $^{-}16$. So we see





Now we can see why such products, products of binomials which differ only in the sign on the second term, are called *the difference of two perfect squares*; they are one square subtracted from another. We can use the FOIL method of multiplying symbols to verify this result.

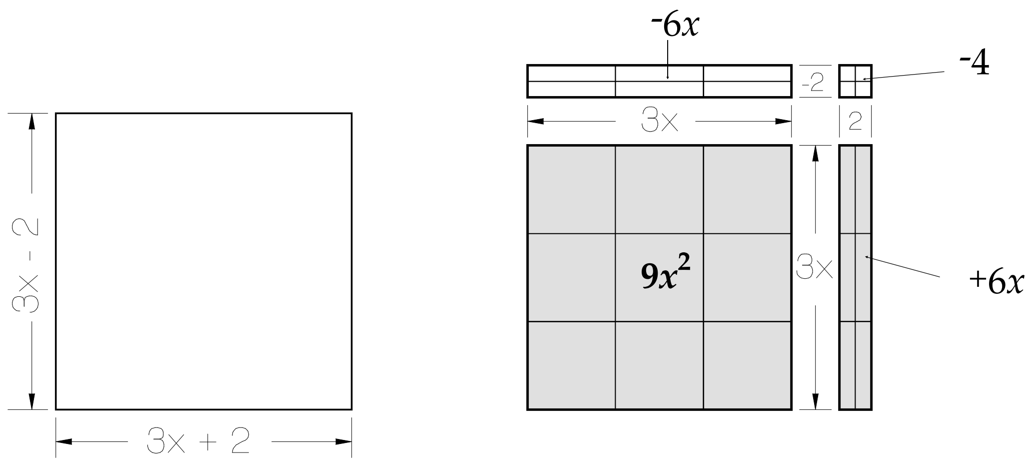
$$\begin{aligned}(x+4)(x-4) &= (x)(x)+(x)(-4)+(4)(x)+(4)(-4)\\ &= x^2-4x+4x-16\\ &= x^2+0x-16\\ &= x^2-16\end{aligned}$$

In the result, each of the two terms is a perfect square and the negative sign means to take the *difference*, or subtract, one perfect square from the other.

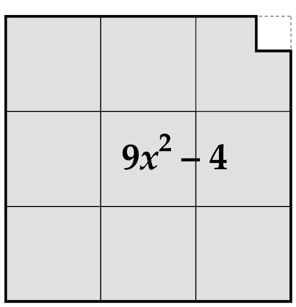
Here is a second example of a product which will generate the difference of two perfect squares:

$$(3x+2)(3x-2)$$

Again the two binomials in the product differ only in the sign on the second term.



We can use both a diagram and the FOIL method to obtain the results of the product.



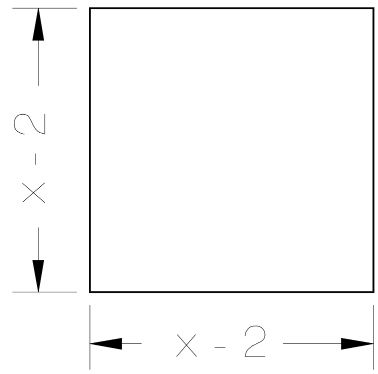
$$\begin{aligned}(3x+2)(3x-2) &= (3x)(3x)+(3x)(-2)+(2)(3x)+(2)(-2)\\ &= 9x^2-6x+6x-4\\ &= 9x^2-4\end{aligned}$$

In both results we are subtracting one perfect square from another; *the difference of two perfect squares*.

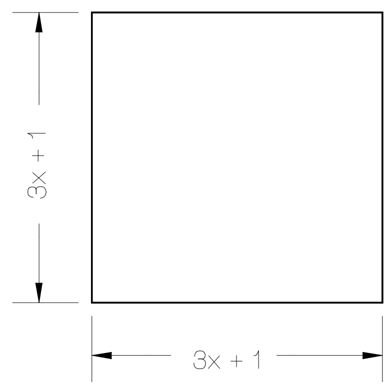
Exercises

Fill in the four smaller rectangles included in these perfect squares and then use the FOIL method to get the same results using algebraic symbols.

1. $(x - 2)^2 =$



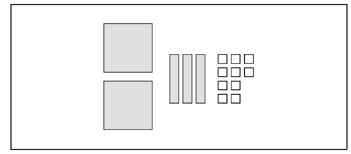
2. $(3x + 1)^2 =$



Multiply out the following perfect squares; verify using a sketch.

3. $(7)^2$
4. $(2x - 7)^2$
5. $(3x)^2$
6. $(3x + 2)^2$
7. $(x + 4)^2$
8. $(2x - 1)^2$
9. $(x - 9)^2$
10. $(5x + 3)^2$

Choose only the products which will generate the difference of two perfect squares, and work out only those products both in a diagram and using the FOIL method to verify your results.



11. $(2x + 5)(2x - 3)$

12. $(x + 2)(x - 2)$

13. $(3x + 4)(3x + 4)$

14. $(3x + 5)(5x - 3)$

15. $(3x + 5)(3x - 5)$

16. $(x - 7)(x + 7)$

17. $(2x - 1)(2x + 3)$

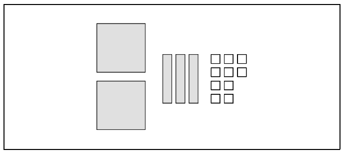
18. $(2x - 1)(x + 1)$

19. $(2x + 1)(2x - 1)$

20. $(3x - 2)(2x + 3)$

21. $(5x - 6)(5x + 6)$

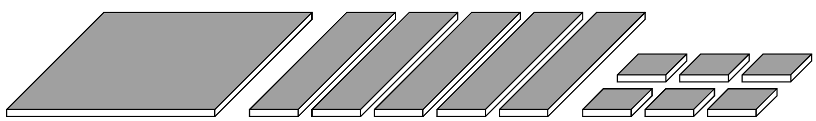
22. $(7x - 1)(7x + 1)$

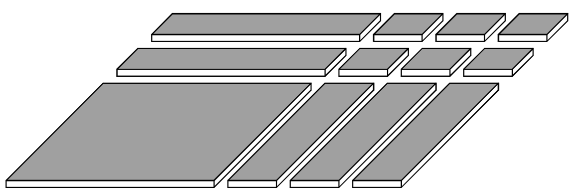


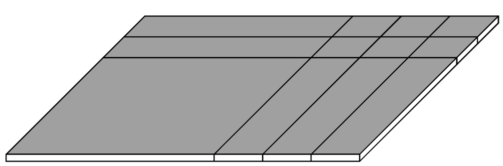
(This page is intentionally blank)

Chapter 10

Factoring Polynomials



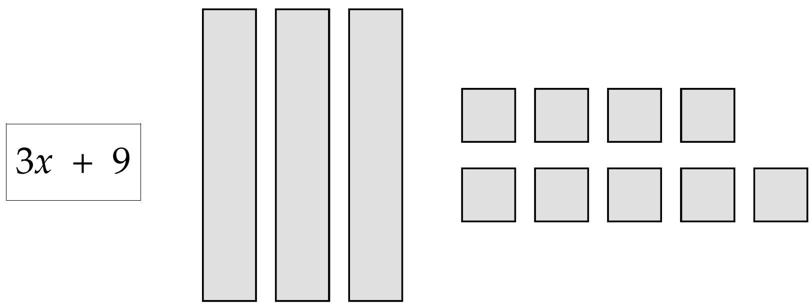




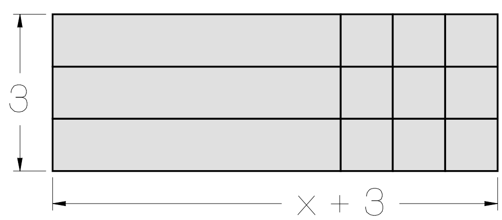
Section 1
Introduction: Rectangles and Factoring

The Meaning of Factoring

Factoring means taking an amount and rewriting the amount as a multiplication problem. Using chips, factoring is the process of taking a group of pieces and arranging them to form a rectangle. The factors are the dimensions (the length and width) of the rectangle. Start with $3x + 9$:

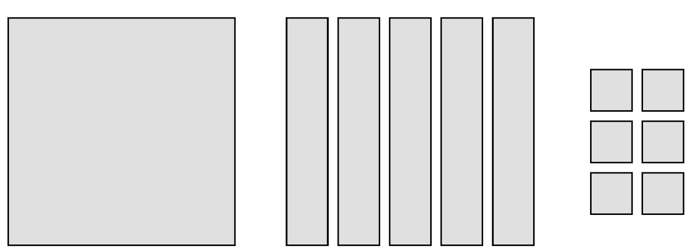


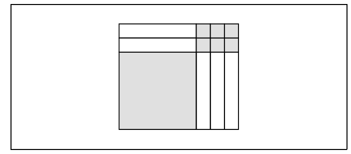
The polynomial $3x + 9$ makes a rectangle that is **3** by $\boldsymbol{x + 3}$:



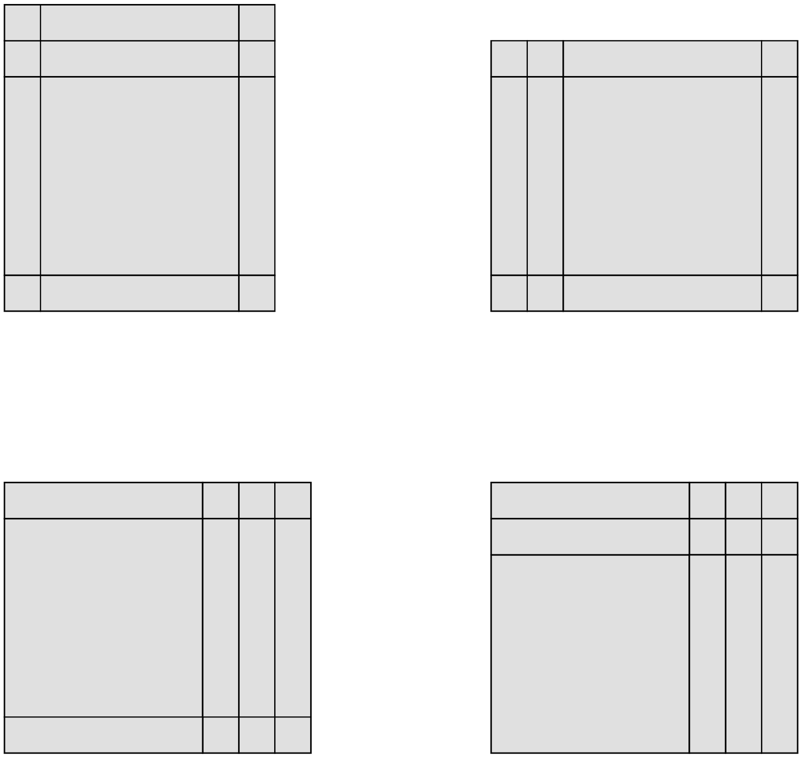
The **factors** of $3x + 9$ are **3** and $\boldsymbol{x + 3}$. This is the same as saying that the product of 3 and $x + 3$ is $3x + 9$. In both forms, *the rectangle means multiplication.*

Sometimes there are several ways to make a rectangle from a group of pieces. Start with the following chips:





From these pieces we could make the following rectangles:

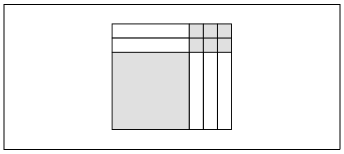


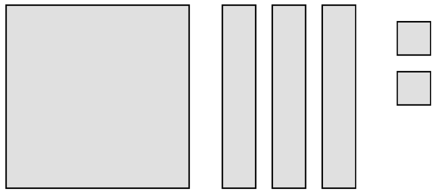
If we check the side lengths of each of these rectangles we find that they all have one direction which is a bar and three unit squares long ($x + 3$), and another which is a bar and two unit squares long ($x + 2$):

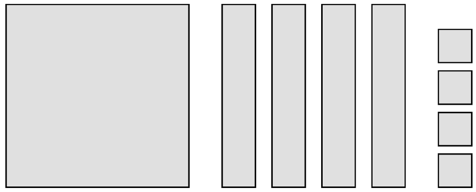
large square		five bars		six units	Factors into a rectangle	1 bar and 3 units	by	1 bar and 2 units
x^2	$+$	$5x$	$+$	6	$=$	$(x + 3)$	$\cdot$	$(x + 2)$

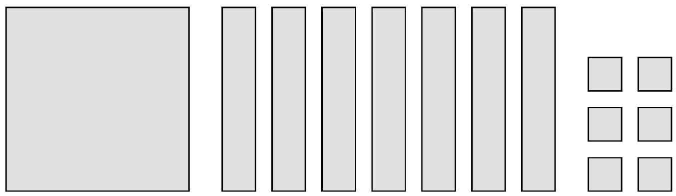
Exercises

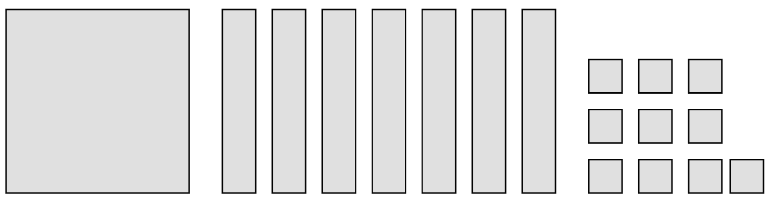
Make rectangles from the following groups of pieces. Remember that there must be no pieces left over and no holes or bumps in the rectangle.



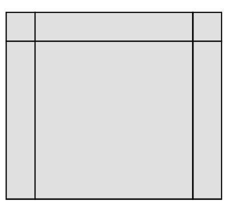
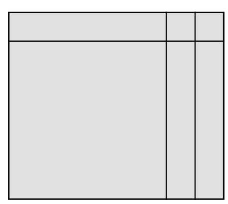
1. 

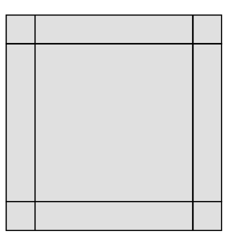
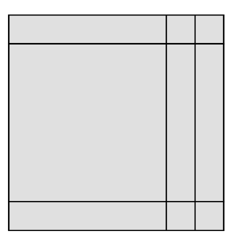
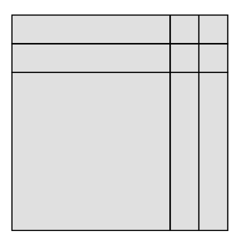
2. 

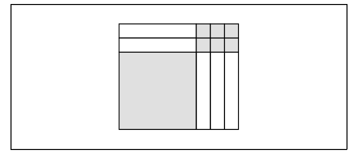
3. 

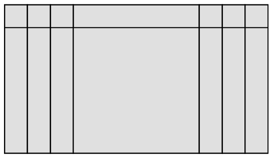
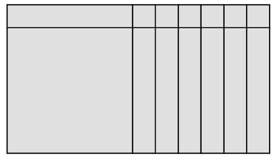
4. 

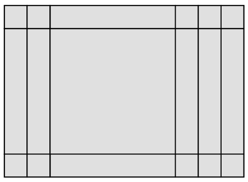
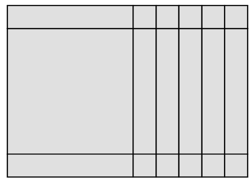
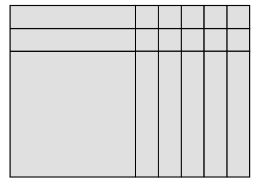
Some possible solutions look like this:

1.  *or* 

2.  *or*  *or* 

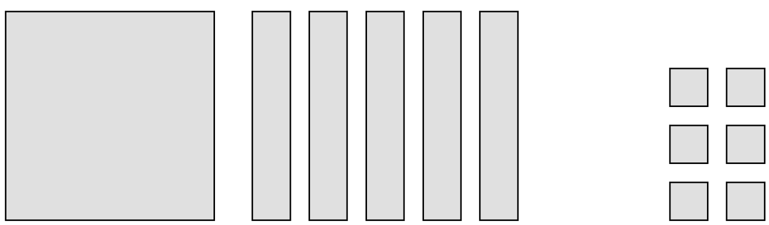


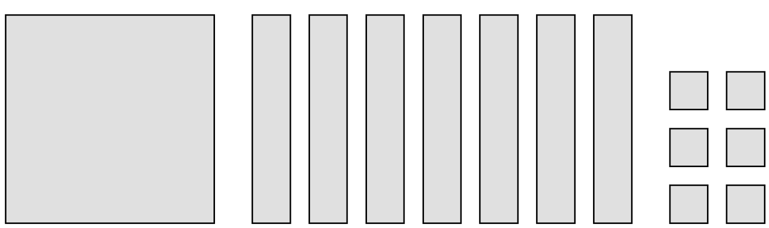
3.  *or* 

4.  *or*  *or* 

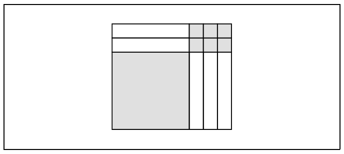
A Clue is in the Units

Look at these two similar examples (both shown before):



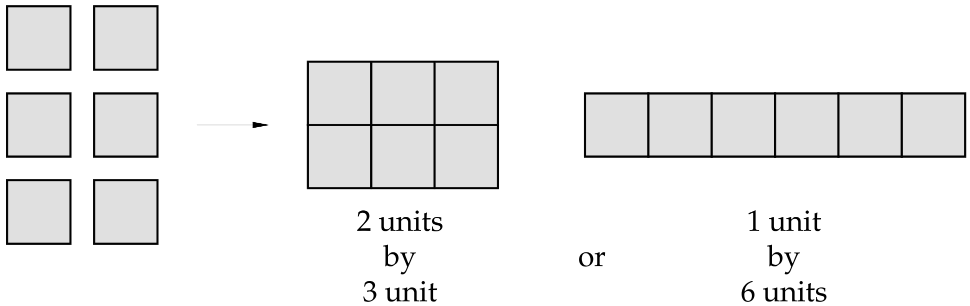


These two groups of pieces differ only in the number of bars. Obviously, since the two groups shown have different numbers of bars, the rectangles they make must have different dimensions.

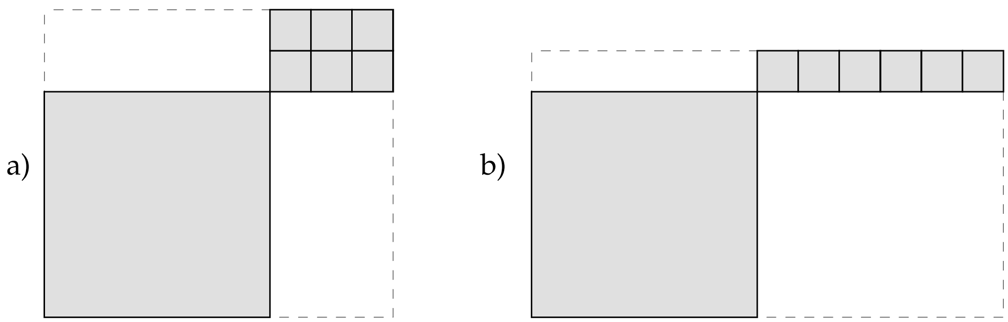


How can you tell before trying different rectangles which ones will work? A clue is in the number of units. Both of the groups shown have six unit squares; let's just look at the units.

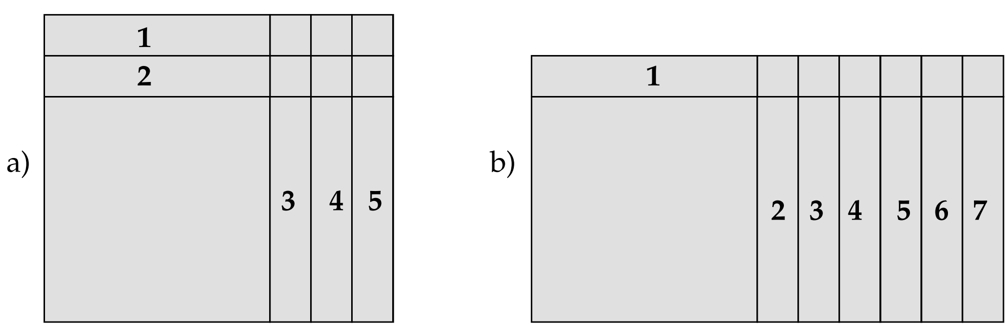
How many ways can you make rectangles using just six units?



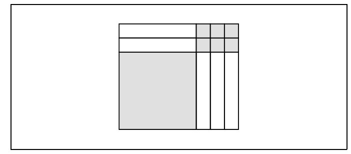
If we think of placing either of these smaller rectangles of units at the corner of the larger rectangle (the total amount), we see two different possible shapes for the larger rectangle.

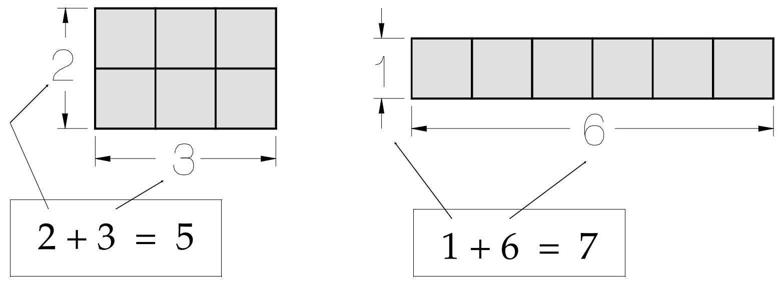


In picture (a) we could fill in the rectangle using two bars on the top and three bars on the side, for a total of five bars. In picture (b) we would need to fill in with one bar on top and six bars on the side, for a total of seven bars.



In each case the number of bars we need to complete the figure depends on the dimensions of the small rectangle of units.





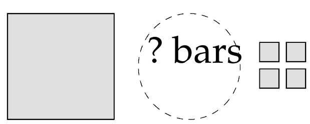
If the units rectangle is (2) by (3) we need 2 + 3 or 5 bars to complete the figure. If the units rectangle is (1) by (6) we need 1 + 6 or 7 bars to complete the figure.

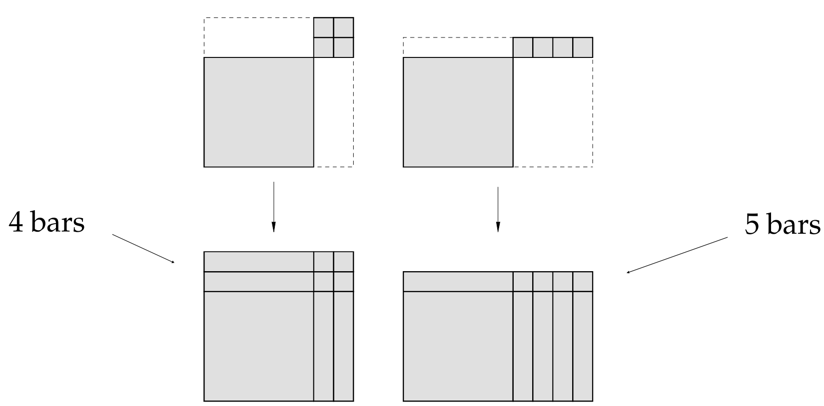
There are only these two ways to make small rectangles using six unit chips. So if we start with one big square and six unit chips we must have either five bars or seven bars in order to make a rectangle which has no holes and no pieces left over. (You can try making rectangles using one big square and six unit chips to see if any are possible with numbers other than five or seven bars.)

Let's Try Predicting

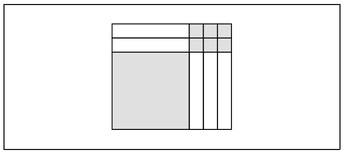
If you are given one big square and any specific number of unit chips, you can learn to predict how many bars you will need to complete each figure.

What if you have one big square and four unit chips? How many ways could you make rectangles and how many bars would you need for each? There are two ways to make a small rectangle using four unit chips:





So we could use either four bars or five bars to make a rectangle.



The product is given by the total number of pieces—large squares, x-bars and units—while the dimensions of the rectangle are the factors:

large square		four bars		four units		1 bar and 2 units	by	1 bar and 2 units
x^2	$+$	$4x$	$+$	4	$=$	$(x + 2)$	$\cdot$	$(x + 2)$

large square		five bars		four units		1 bar and 4 units	by	1 bar and 1 unit
x^2	$+$	$5x$	$+$	4	$=$	$(x + 4)$	$\cdot$	$(x + 1)$

Exercises

Set up the following polynomials with chips and factor:

1. $x^2 + 4x$

2. $x^2 + 5x$

3. $x^2 + 6x + 9$

4. $x^2 + 5x + 4$

5. $x^2 + 8x + 15$

6. $x^2 + 8x + 12$

7. $x^2 + 7x + 12$

8. $x^2 + 9x + 14$

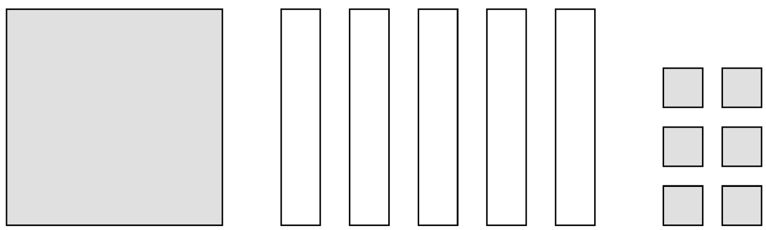
9. $x^2 + 8x + 16$

10. $x^2 + 9x + 20$

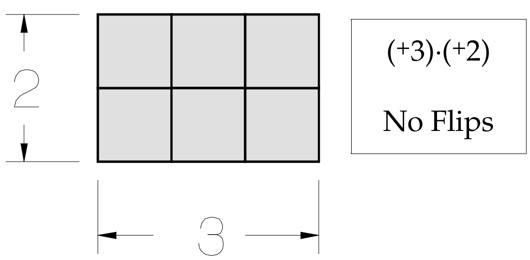
Section 2
Positive Units, Negative Bars

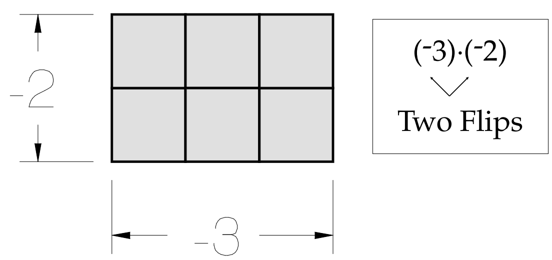
Factoring with Negative Bars

How can we factor polynomials with negative bars?

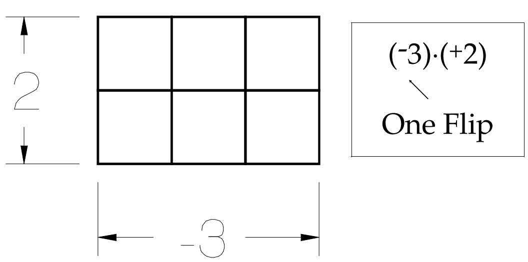


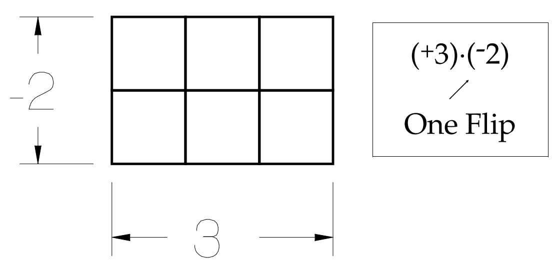
To make a rectangle from pieces having positive units but negative bars we need to remember how to multiply two numbers having signs (see POSITIVE AND NEGATIVE NUMBERS, Section 5 or POLYNOMIALS, Section 1). We can get a positive answer (colored rectangle) from multiplying two positive numbers, or from multiplying two negative numbers.

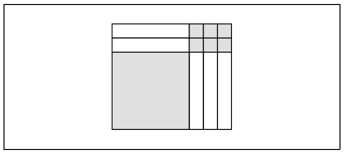




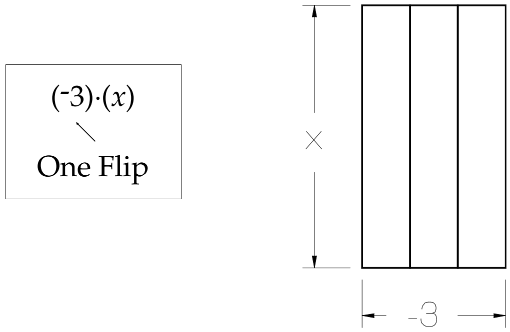
Similarly, we get a negative answer (white rectangle) from multiplying two units having different signs.

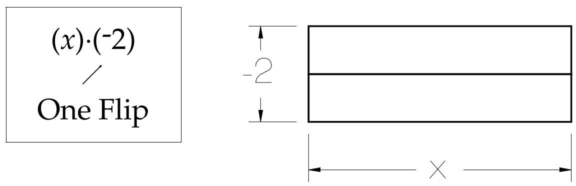




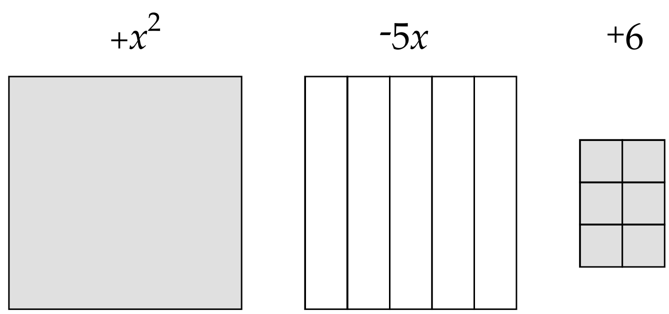


In the same way, we get rectangles made from white (negative) bars whenever one dimension (factor) of the rectangle is positive but the other dimension (factor) is negative.

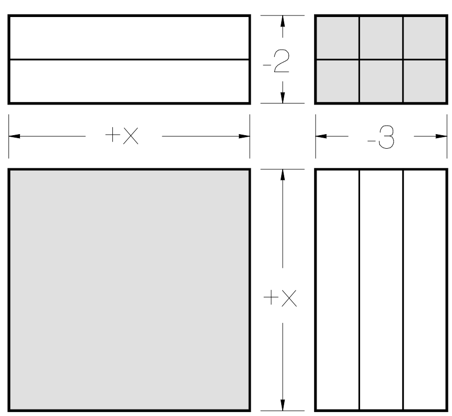




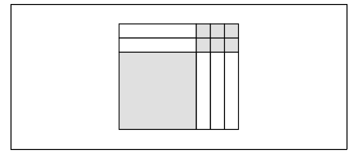
For example, look at the pieces below:

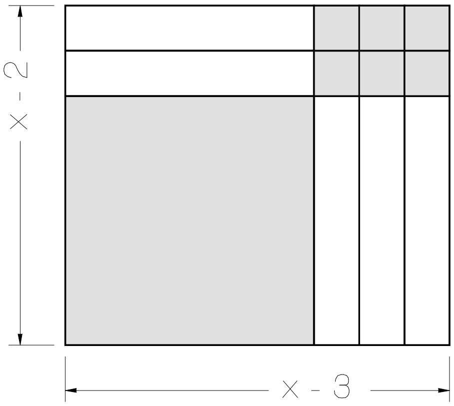


We can make a large rectangle (using four smaller rectangles) in the following way:



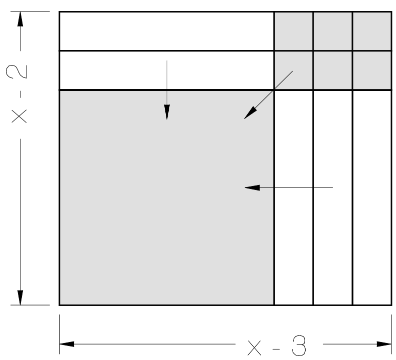
Each of the four small rectangles has its color (sign) determined by the signs of its two dimensions. Then the composite large rectangle looks like this:



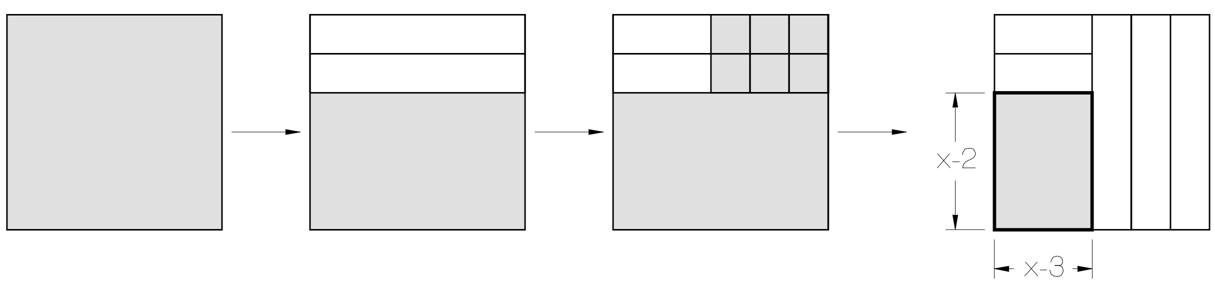


$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

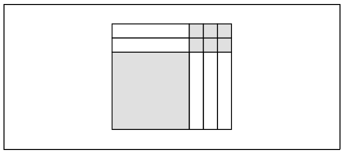
The dimensions of the rectangle are most easily read along the bottom and up the left side. The edges of the large colored square are each $+x$, and the short ends of the white bars are each a negative one ($^-1$). In this figure the white areas can be thought of as canceling out colored areas leaving a rectangle with actual dimensions of $x - 2$ and $x - 3$, as shown below.



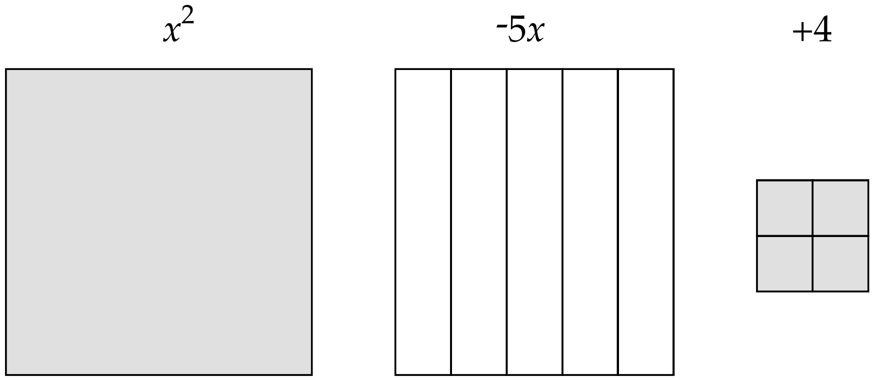
Start with x^2	Put on ^-2x bars. (Cancels)	Put on $^+6$ units. (Adds back)	Put on ^-3x bars (cancels). The result is $(x - 2)$ by $(x - 3)$



As in the above illustration, the x-bars and units subtract from and add to the original x^2 piece. First, place negative x-bars to cancel out some of the area. Then add back area by placing the positive units on top of the negative bars. Finally, cancel out area with the remaining negative bars. The resulting rectangle is two less than x on one side and three less than x on the other.

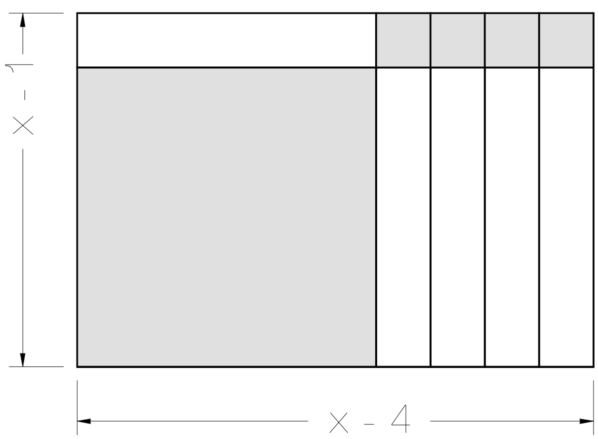


Example: Make a rectangle from the pieces given below:



Solution: The four single chips can only form two possible rectangles—2 by 2 or 1 by 4. Of these two possibilities, only the 1 by 4 corner rectangle would require 5 bars (4+1) which, in this case, are all negative. Looking at the rectangle's dimensions we see that

$$x^2 - 5x + 4 = (x - 4)(x - 1)$$



Exercises

Use chips and factor the following polynomials by making rectangles and noting their dimensions.

1. $x^2 - 4x + 3$

2. $x^2 - 6x + 8$

3. $x^2 - 8x + 12$

4. $x^2 - 7x + 12$

5. $x^2 - 7x + 10$

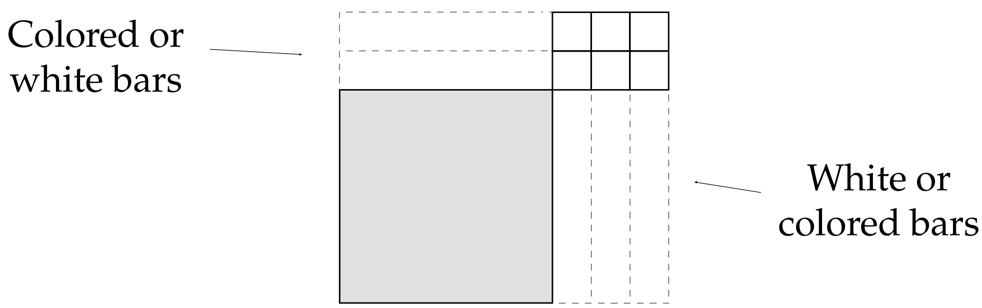
6. $x^2 - 10x + 16$

Section 3
Rectangles Having Negative Units

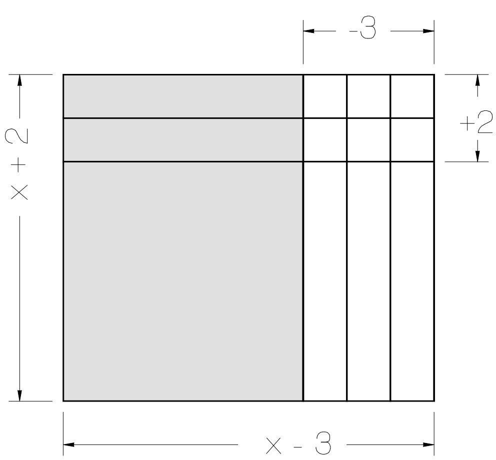
Factoring with Negative Units

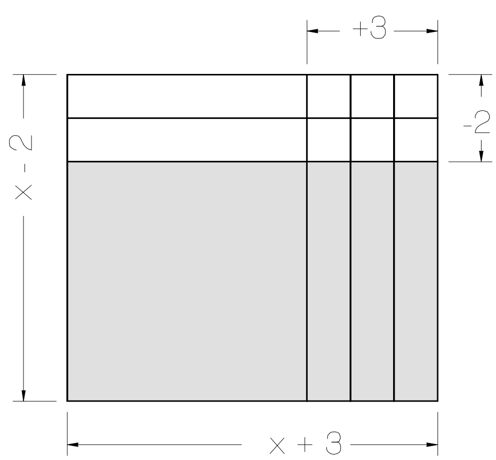
As we just reviewed, if a rectangle has a negative value (white side up), it means that one dimension of the rectangle is positive while the other dimension is negative.

In the case of a polynomial, this means that if the large square (x^2) is colored (positive), and the small units (single chips) are white (negative), then when we make a complete rectangle we will need some colored bars and some white bars. (You may want to review POLYNOMIALS, Section 3.)

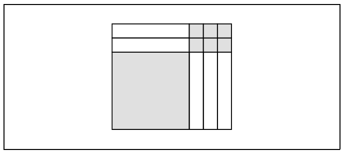


The color and number of the bars will match the positive or negative values of the dimensions of the units rectangle. For example we could imagine two different rectangles having $+x^2$ and $^-6$ units:



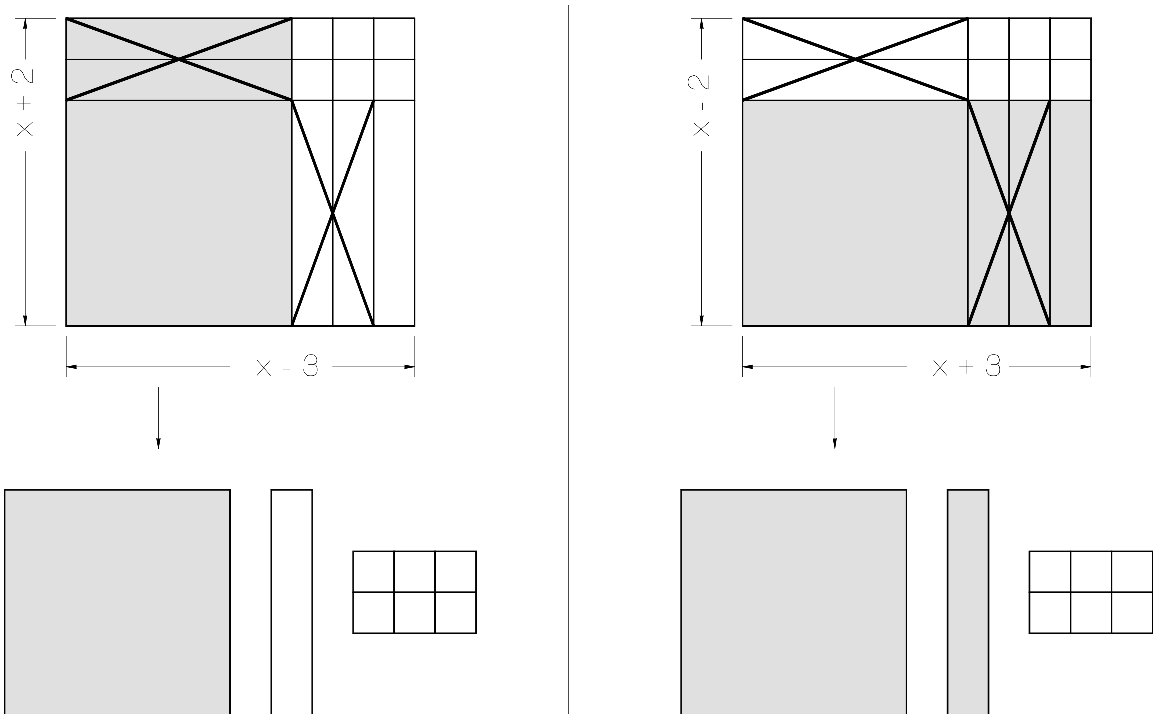


One of these rectangles has two positive bars and three negative bars; the other rectangle has two negative bars and three positive bars.



How Can We Tell Which to Use?

If we begin with a polynomial where some of the bars are positive and some are negative, when we combine like terms (put all the bars together), some of them are going to cancel out.

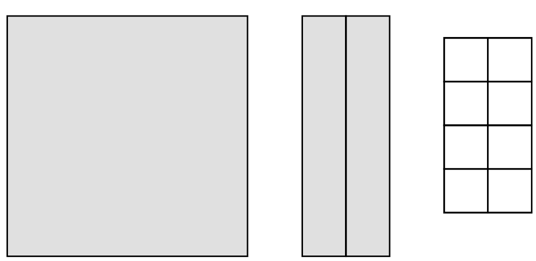


$x^2 - x - 6$ $\qquad$ $x^2 + x - 6$

The sign of the bars which are left over after canceling will match the sign of the larger dimension of the units rectangle.

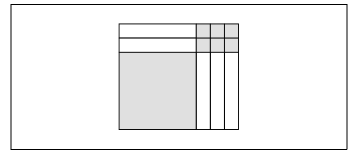
Working Backwards

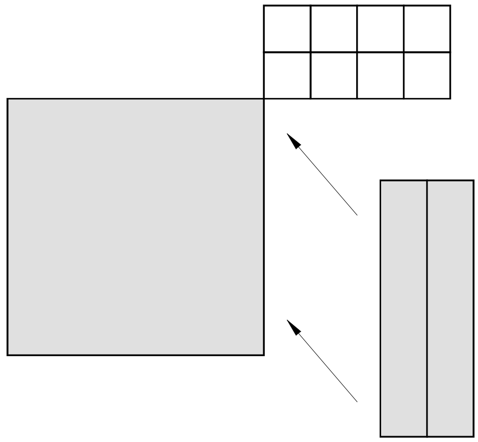
In order to factor a polynomial having negative units, like the following one,



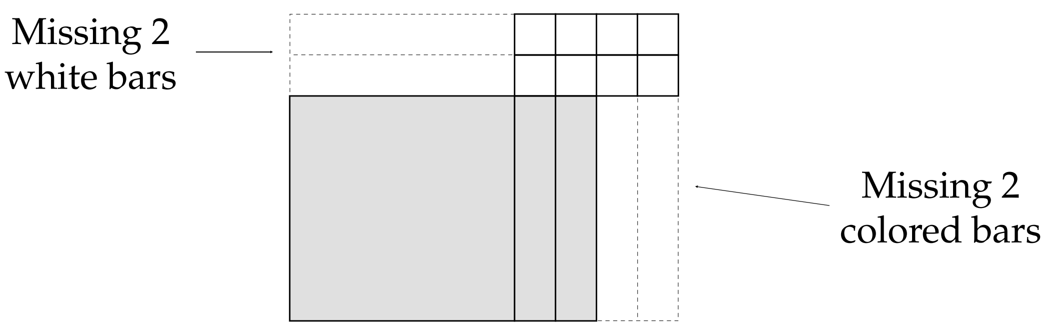
$$x^2 + 2x - 8$$

we begin by putting the unit rectangle at the corner of the big square and putting the bars we have along the *longer side* of the units rectangle.

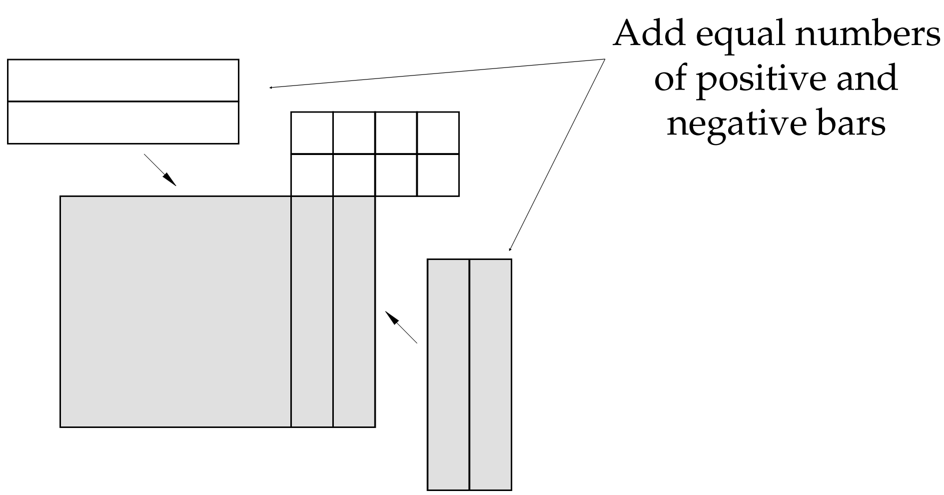


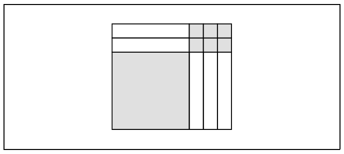


When we look at the result we should see that we are missing *equal numbers* of positive and negative bars.



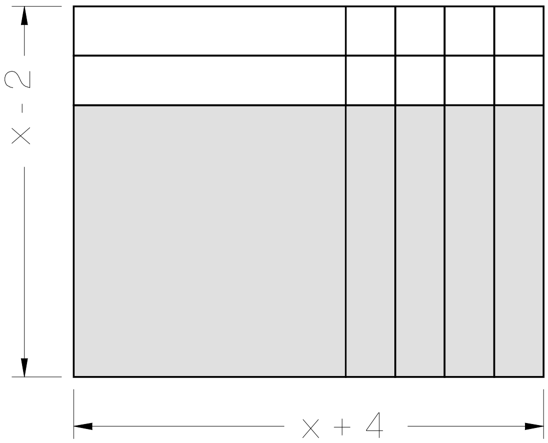
We know that if we add positive and negative bars to the figure in equal numbers we are adding zero, because these pairs of white and colored bars would cancel out.



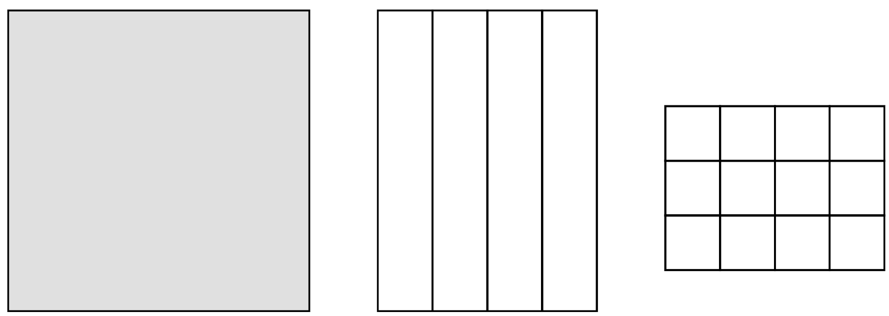


So our final figure looks like this:

$$x^2 + 2x - 8 \; = \; (x-2)(x+4)$$

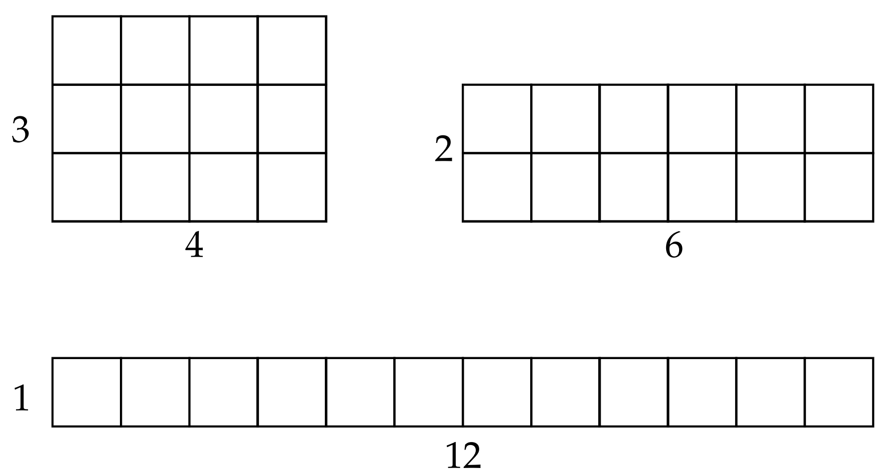


Let's look at another example. Factor $x^2 - 4x - 12$:

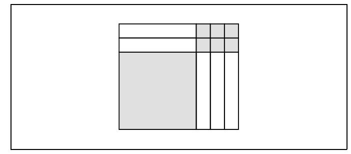


x^2 $\quad - 4x$ $\quad - 12$

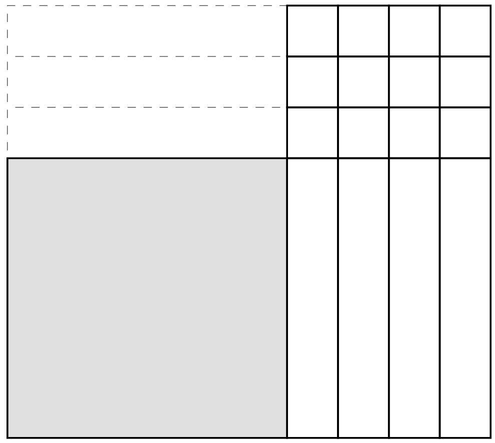
In this case the $^{-}12$ units can be put into three different possible rectangles:



Putting each of these three small rectangles into the larger complete rectangle we have the following options:

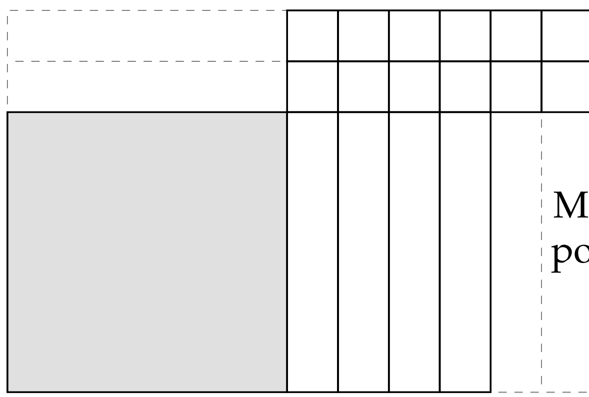


No



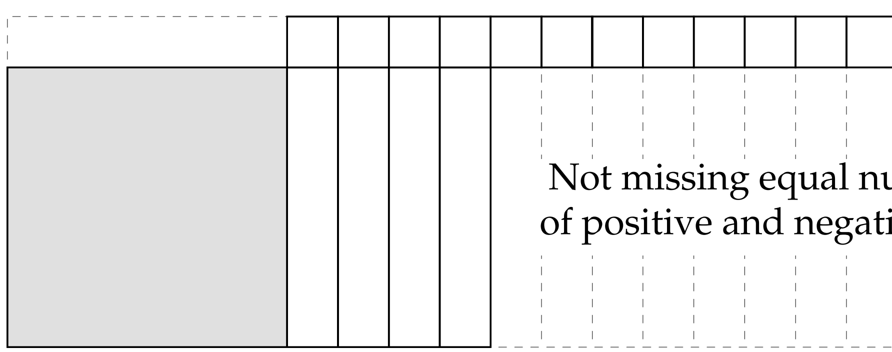
Not missing equal numbers of positive and negative bars

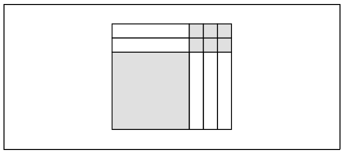
YES



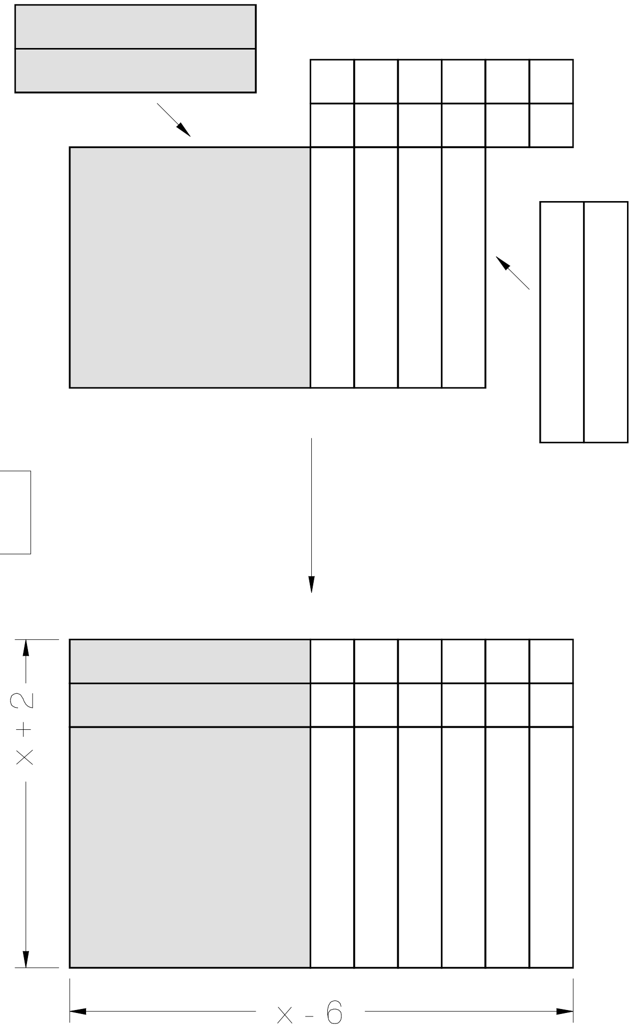
Missing equal numbers of positive and negative bars

No





To complete the figure we must add equal numbers of white and colored bars, so only the middle figure will work. The solution to our example is



$$x^2 - 4x - 12 = (x - 6)(x + 2)$$

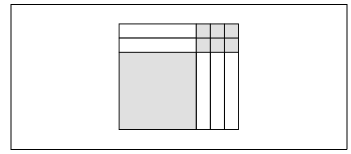
Summary

When factoring a polynomial, remember to take all the pieces and fit them into a large rectangle made up of four smaller rectangles. Each of the smaller rectangles has its sign or color determined by the signs of its two dimensions, and in all, they must match both the signs and the numbers of the pieces you start with.

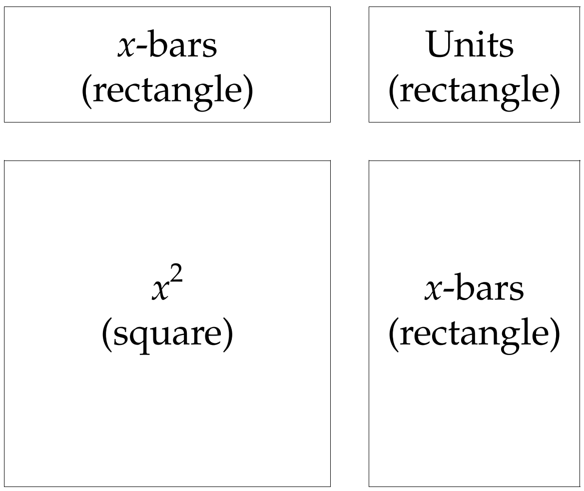
Here are the steps in the factoring process:

- **Consider the possible factors of the units term. (Note the required signs.)**

- **Pick the pair of factors which add together to give the required number of *x*'s.**



- If the units rectangle is positive, then the two factors add, and all of the *x*-bars should just fit along its left and bottom edges.
- If the units rectangle is negative, then equal numbers of positive and negative *x*-bars will be missing when the given *x*-bars are placed along the long side of the units rectangle. In such a case, fill in both the missing positive and negative *x*-bars, remembering that adding equal numbers of positive and negative bars is really adding zero.
- When you have finished this process, the dimensions of the large rectangle you have made are the factors of the polynomial with which you began.



Exercises

Factor the following polynomials:

1. $x^2 + 5x - 6$

2. $x^2 - 2x - 8$

3. $x^2 - 7x - 8$

4. $x^2 - 11x - 12$

5. $x^2 - 5x - 6$

6. $x^2 + x - 12$

7. $x^2 + 8x - 9$

8. $x^2 - 2x - 15$

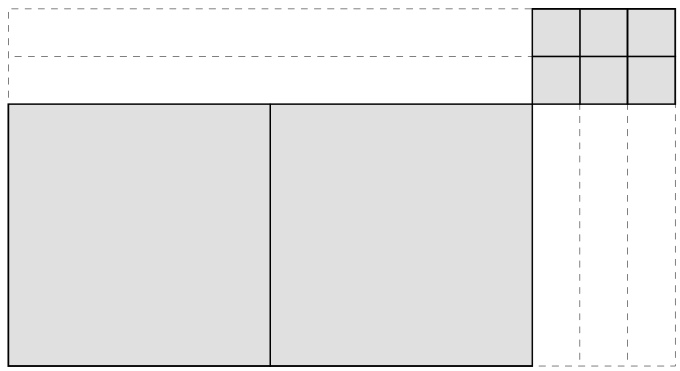
9. $x^2 + 2x - 15$

10. $x^2 - 6x - 16$

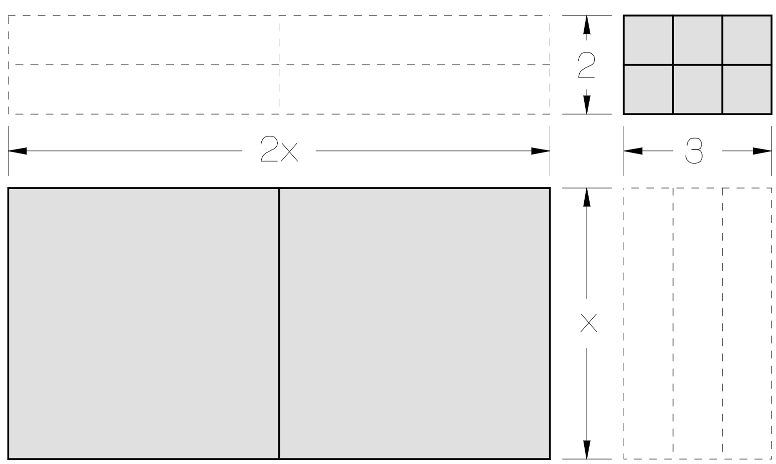
Section 4
Factoring Trinomials with More than One x^2

More than one x^2

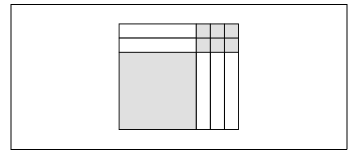
If we make a rectangle out of pieces including two large squares ($2x^2$)



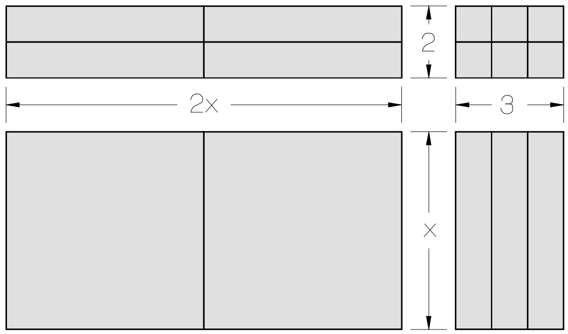
then we can see (from the example shown above) that the number of x-bars needed to complete the figure is more than we would need if we had only one large square.



The top rectangle of x-bars is now twice as long as before because it has to run along the top of two large squares instead of just one. To factor a trinomial having more than one x^2, we make one rectangle out of the large squares, and a second rectangle out of the unit chips, then the dimensions of these two smaller rectangles multiply together to determine the number of x-bars needed to complete the figure.

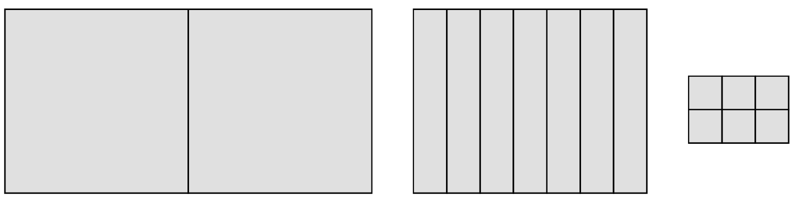


$(2x)(2) = 4x$

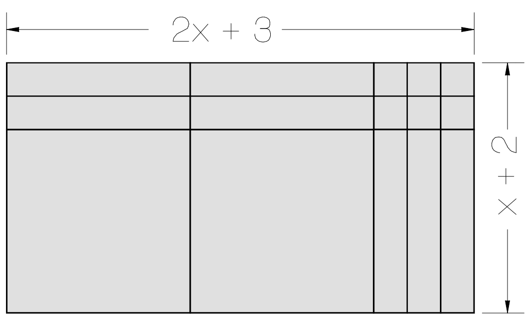


$(x)(3) = 3x$

Working backwards we see the following:

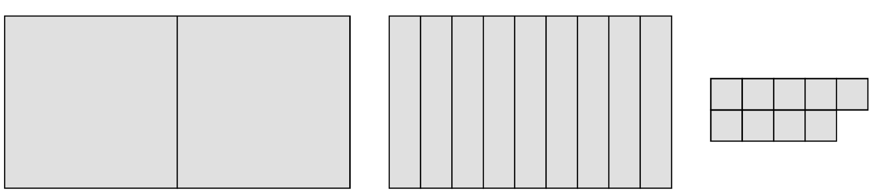


$2x^2$ $7x$ 6

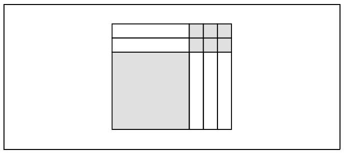


$2x^2 + 7x + 6$
equals
$(2x + 3)(x + 2)$

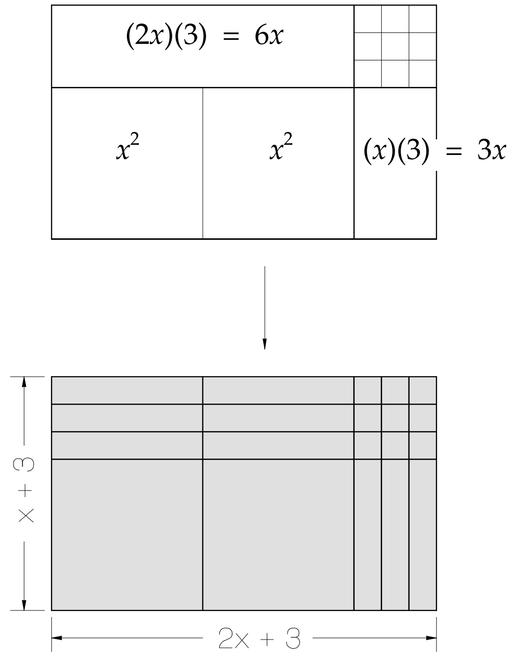
Example: Make a rectangle from the following pieces and use it to determine the factors of the given trinomial.



$2x^2$ $9x$ 9

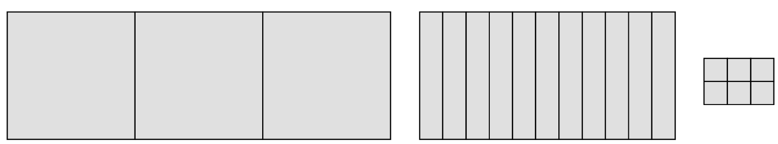


Solution:

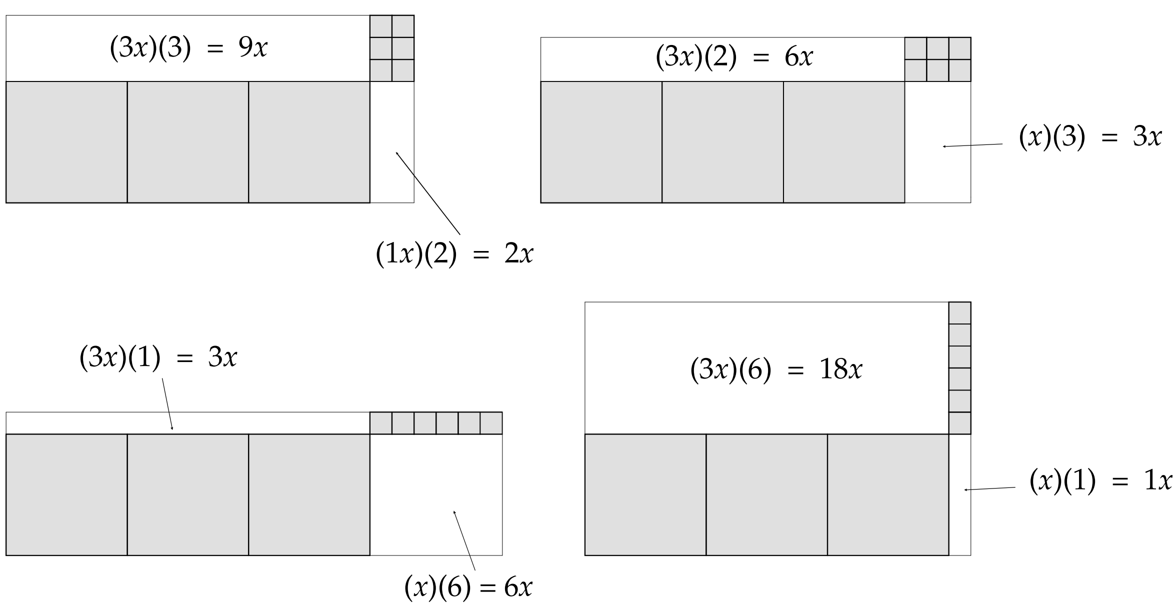


$$2x^2 + 9x + 9 = (2x + 3)(x + 3)$$

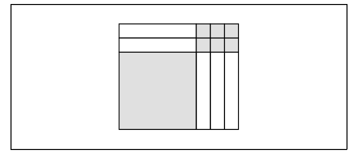
Example 2: Make a rectangle from $3x^2 + 11x + 6$:

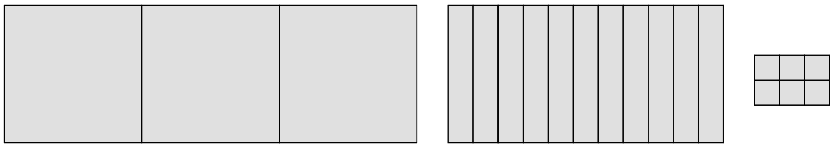


Solution: There are four possible ways to orient rectangles made from the large squares and the unit chips. Each of these will require a particular number of x-bars, as shown below:

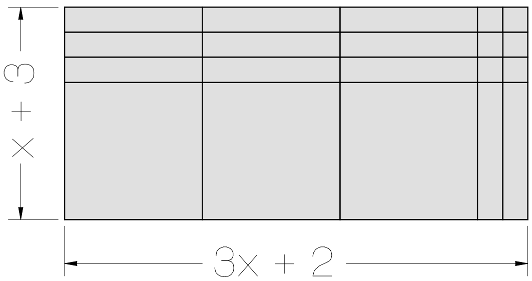


Which of these four possibilities requires 11 x-bars? What are the dimensions of this rectangle?





$$3x^2 + 11x + 6 = (3x + 2)(x + 3)$$



Exercises

Factor these polynomials:

1. $4x^2 + 4x + 1$
2. $3x^2 + 7x + 2$
3. $2x^2 + 7x + 3$
4. $3x^2 + 10x + 3$
5. $2x^2 + 5x + 2$
6. $2x^2 + 3x + 1$
7. $6x^2 + 11x + 3$
8. $6x^2 + 7x + 2$
9. $6x^2 + 11x + 4$
10. $4x^2 + 8x + 3$
11. $12x^2 + 31x + 20$ (Draw a picture instead of using chips)

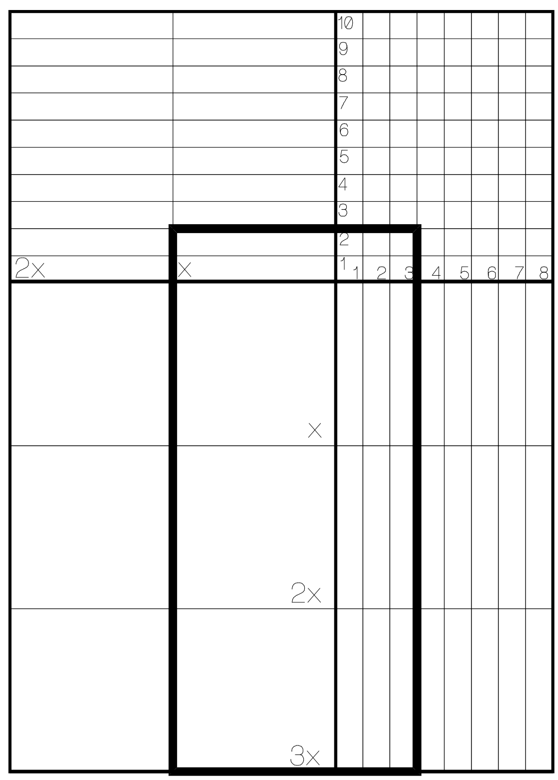
Section **5**
Factoring Using the Grid

The Plastic Grid

The plastic polynomial grid provided with this book can make factoring trinomials much easier than making rectangles out of the chips themselves. You can make a rectangle over your grid which has the proper dimensions for a given factoring problem. The previous example of

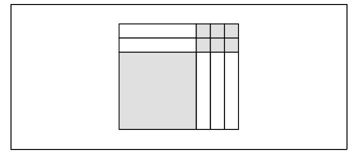
$$3x^2 + 11x + 6 = (3x + 2)(x + 3)$$

would look like this:



Notice that it doesn't matter which direction the rectangle is turned, as long as the correct number of pieces is used. Because the polynomial grid is plastic, it is possible to write on it using *water-soluble* marking pens.(Be sure the marking pens you use have ink which will wash off or you can ruin your plastic grid.) Just outline the areas you want with a heavy line. You can try different arrangements of units and squares in the same way that you move chips around.

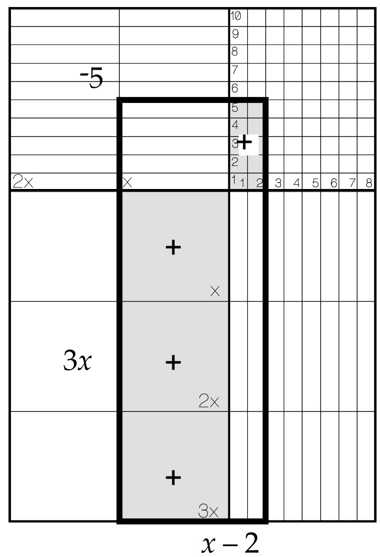
Positive and Negative Areas of the Grid



With the water soluble-marking pen you can mark positive and negative areas on the grid with a plus (+) or a minus (–) sign, and in this way keep them straight. (Of course you will remove the + and – marks after completing each problem). Just as mentioned before, the sign of each portion of the rectangle is determined by the signs of both its dimensions.

For example, let's use the grid to factor the trinomial

$$3x^2 - 11x + 10$$

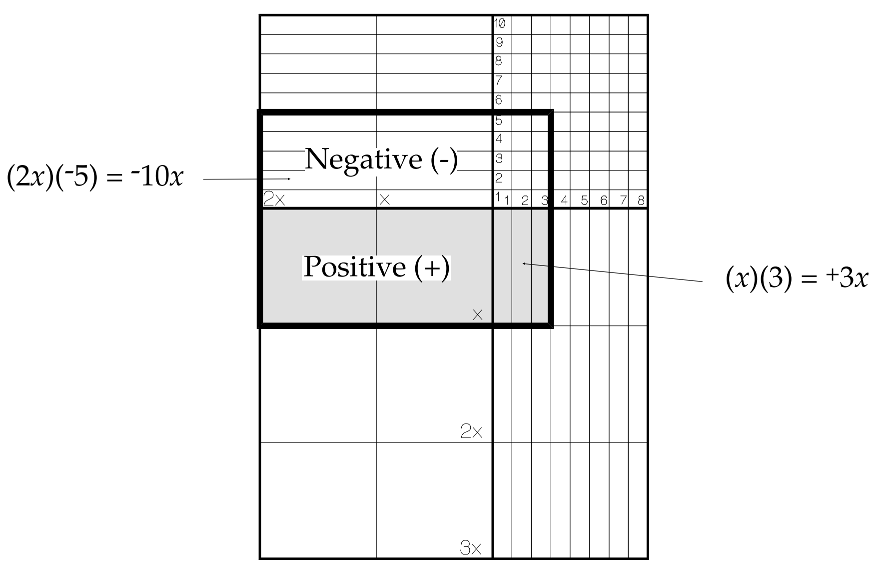


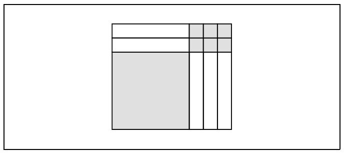
The result is:

$$(3x - 5)(x - 2)$$

Next, use the grid to factor

$$2x^2 - 7x - 15$$



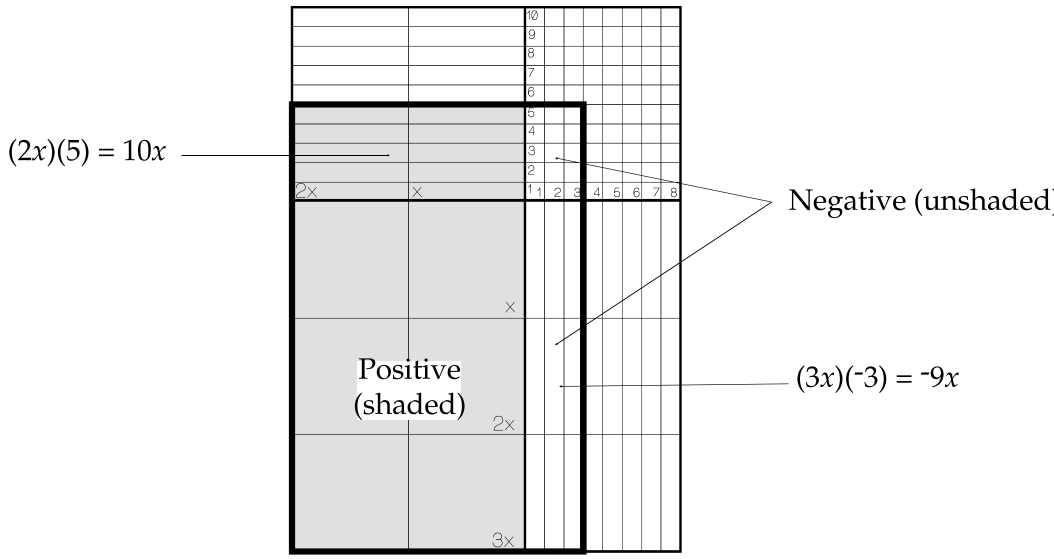


The result is:

$$(2x + 3)(x - 5)$$

Use the grid to factor

$$6x^2 + 1x - 15$$



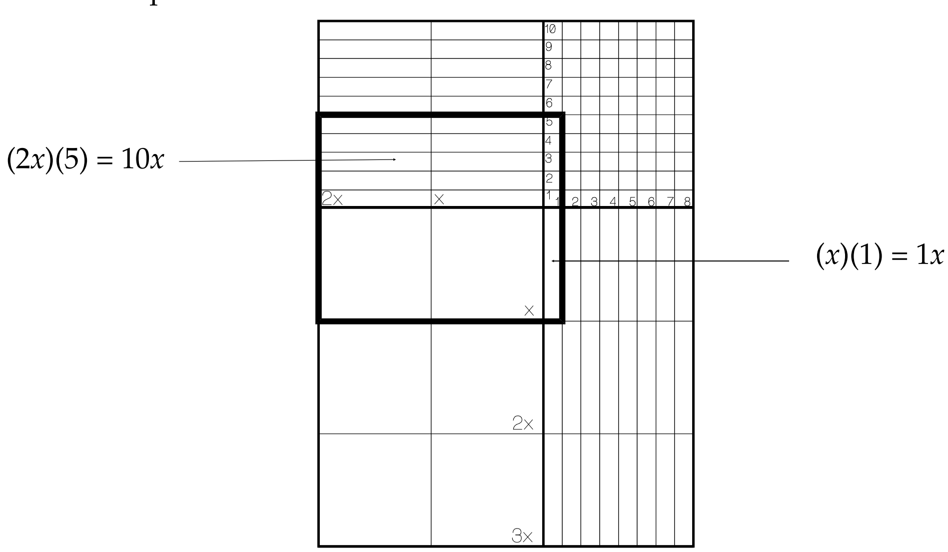
The result is:

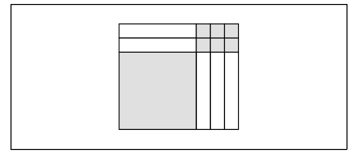
$$(3x + 5)(2x - 3)$$

Exercises

Use your grid to factor the following trinomials:

Example: $2x^2 + 11x + 5$





1. $3x^2 + 8x + 5$

2. $2x^2 + 11x + 12$

3. $3x^2 + 20x + 12$

4. $3x^2 + 10x + 8$

5. $3x^2 + 14x + 8$

6. $3x^2 + 25x + 8$

7. $2x^2 + 13x + 15$

(Remember, start by considering possible rectangles for the large x^2-squares, and for the small unit squares, then figure out which possibility gives the correct number of x-bars.)

Complete the following factoring problems using the plastic grid:

8. $x^2 - x - 6$

9. $x^2 + 4x - 12$

10. $2x^2 + 3x - 5$

11. $2x^2 - 7x + 6$

12. $4x^2 - 4x - 15$

13. $2x^2 + 7x - 15$

14. $6x^2 - x - 15$

15. $6x^2 + 11x - 10$

16. $2x^2 - 13x + 15$

17. $3x^2 - 2x - 5$

18. $2x^2 - x - 6$

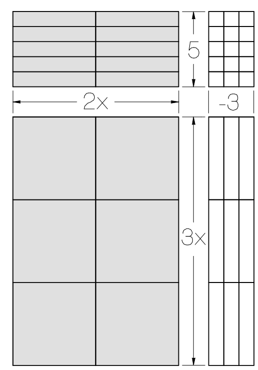
19. $6x^2 + x - 2$

Section 6
A Shortcut Method

A Shortcut for Factoring

Let's look closely at the solution to the last example.

$$6x^2 + 1x - 15 = (3x + 5)(2x - 3)$$

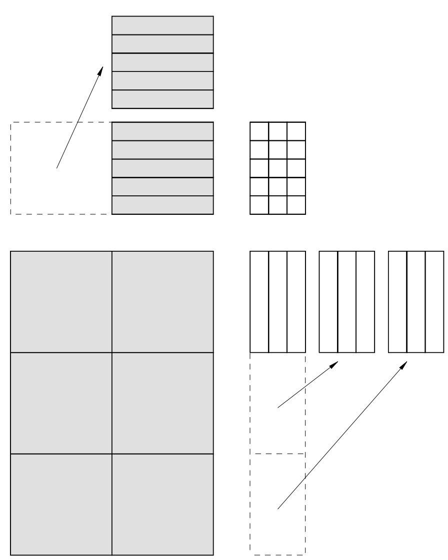


The two rectangles which have the x-bars in this figure have dimensions

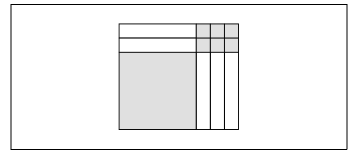
$$(2x)(5) = 10x \quad \text{and} \quad (^-3)(3x) = {}^-9x$$

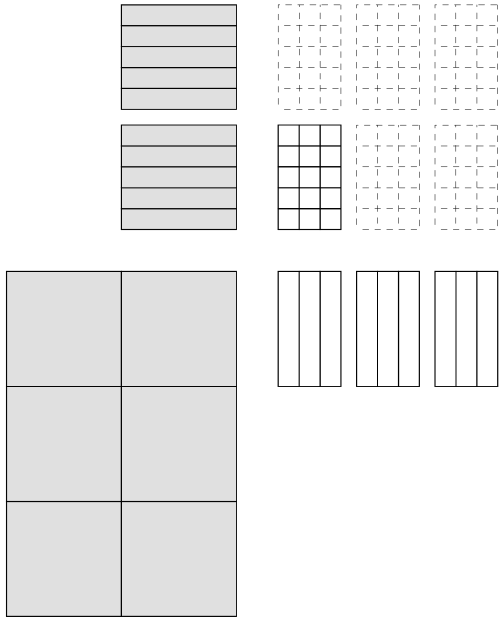
Notice that each of these rectangles of x-bars has one dimension which is a factor of $6x^2$ and another dimension which is a factor of $^-15$.

Mentally move the x-bars to the new positions shown here:



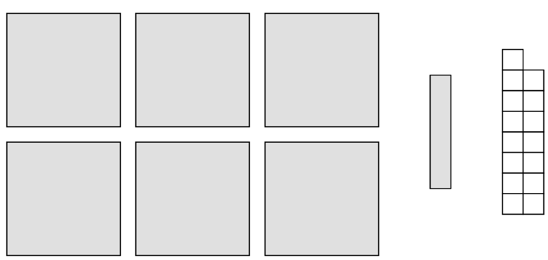
This configuration suggests imagining six rectangles, each having $^{-}15$ chips, as shown in the next diagram.



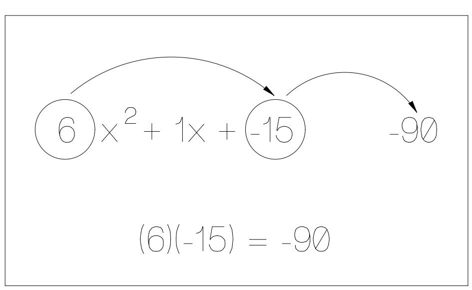


This arrangement will be the key to a shortcut factoring method for polynomials having more than one large square (x^2). *For a more detailed explanation of why this method works, please see Appendix 5.*

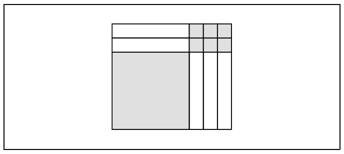
Let's begin with the original polynomial $6x^2 + 1x - 15$ and work through the shortcut factoring method.



First, multiply the 6 times the $^{-}15$. (Note that although we cannot know in advance how the chips are to be arranged, *any* arrangement of $6x^2$ and $^{-}15$ units will give 6 groups of $^{-}15$, or $^{-}90$ imagined unit chips in the corner.



This step corresponds to the picture we "imagined" above (when we started from knowing the solution).



Second, we list all of the ways we could possibly factor $^{-}90$, with the negative sign meaning that one factor will be positive (+) and the other negative (-).

Factors of $^{-}90$	
Factors	**Difference**
$90 \cdot 1$	89
$45 \cdot 2$	43
$30 \cdot 3$	27
$18 \cdot 5$	13
$15 \cdot 6$	9
	1

One factor is negative
One factor is positive.
The difference is positive
or negative

This list shows the dimensions of all the possible rectangles we could make using 90 white chips. But remember that besides multiplying to give $^{-}90$, the factors we are interested in must add together to give us the total number of x-bars we need. The expression

$$6x^2 + 1x - 15$$

has only $^{+}1$ x-bar, so we must find a pair of factors which add together to give a $^{+}1$. This requires that we use the factors

$$(^{+}10) \text{ and } (^{-}9),$$

and tells us that the two rectangles made from x-bars *must* have

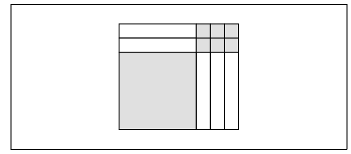
$$^{+}10x \text{ and } ^{-}9x$$

Knowing this we rewrite our original polynomial and replace the term $^{+}1x$ with the two terms $^{+}10x - 9x$, as shown below:

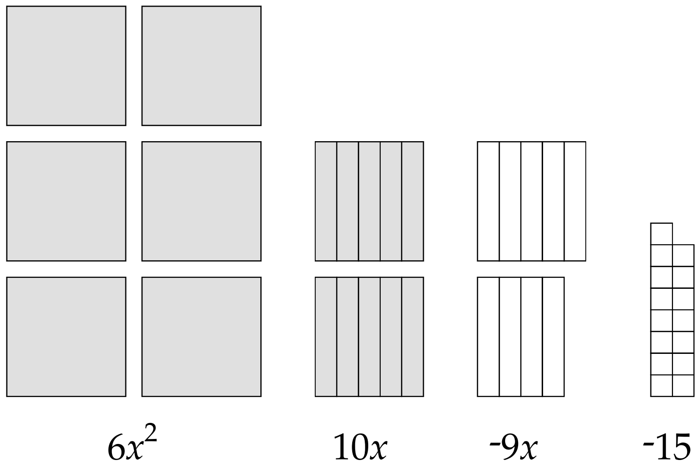
$$6x^2 + 1x - 15$$

$$6x^2 + (10x - 9x) - 15$$

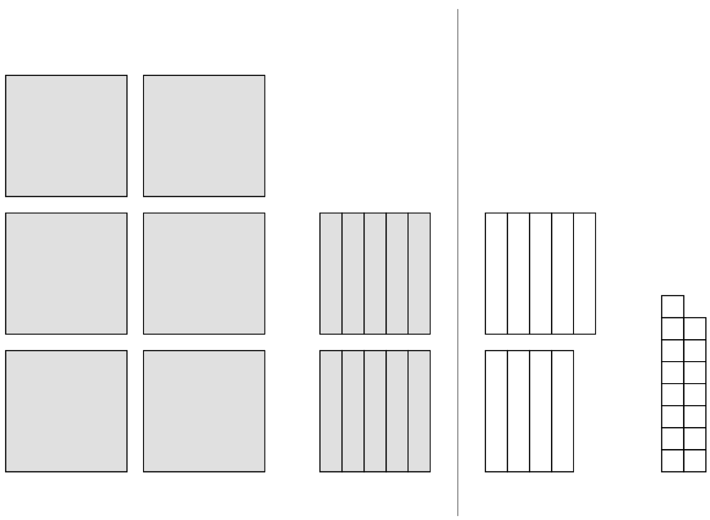
Notice that these four terms correspond to the four parts of the rectangle which we know will be our final factored solution.



Now we have the following pieces to use:



The third step in the process separates these four terms into two groups. Move the first two terms (the $6x^2$ and the ^+10x pieces) to one place, and move the last two terms (the ^-9x and the $^-15$ pieces) to a different place.

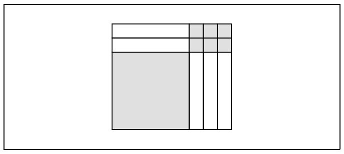


From each of these two groups take the largest common factor.

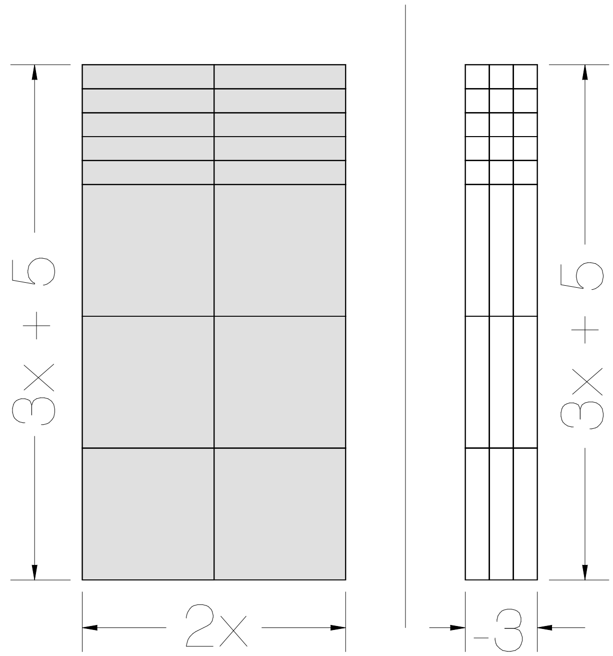
$6x^2 + 10x$	$-9x - 15$
$2x(3x + 5)$	$^-3(3x + 5)$

Largest Common Factors

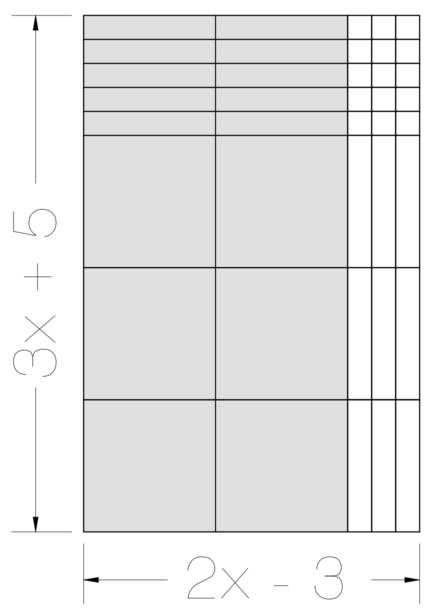
In each case the largest common factor is the width of a rectangle which can be made from the group of pieces, and the parentheses holding two terms is the length of the same rectangle.



This idea is illustrated below:



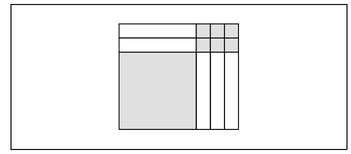
The surprise, which you may have already noticed, is that the rectangles we have made from the two separate groups of pieces have *the same length* ! We can put them side by side—they will form one large rectangle.



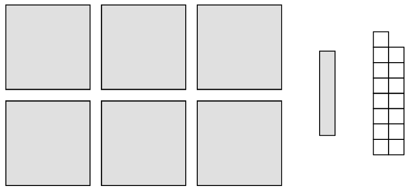
The dimensions of this rectangle are the factors of the original expression.

$$6x^2 + 1x - 15 = (2x - 3)(3x + 5)$$

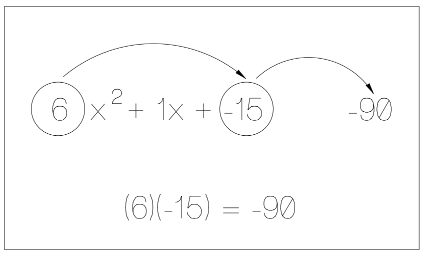
Shortcut Method: Summary



Begin with the original expression: $6x^2 + 1x - 15$:



- **Step 1: Multiply the first coefficient times the last number.**

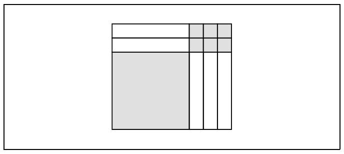


- **Step 2: List all the possible factors of the product.**

Factors of -90	
Factors	**Difference**
$90 \cdot 1$	89
$45 \cdot 2$	43
$30 \cdot 3$	27
$18 \cdot 5$	13
$15 \cdot 6$	9
	1

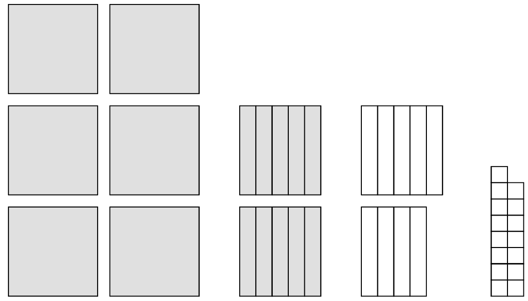
- **Step 3: Select the pair of factors which adds together to give the needed number of *x*'s.**

$$+10x - 9x = +1x$$



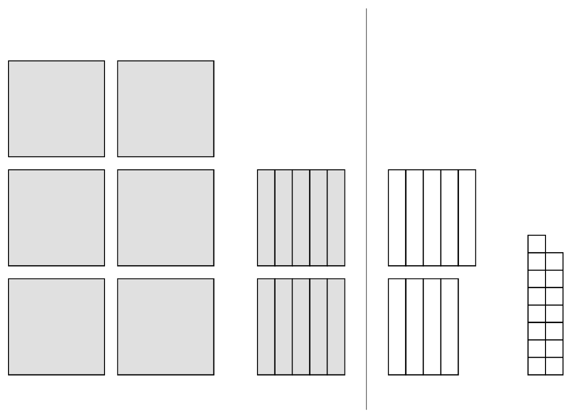
- **Step 4: Rewrite the given expression using four terms instead of three.**

$6x^2 + 10x - 9x - 15$



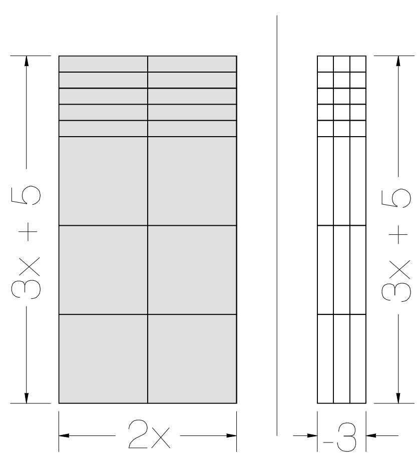
- **Step 5: Separate the first two terms and the last two terms. This makes two groups.**

$(6x^2 + 10x) + ({}^{-}9x - 15)$

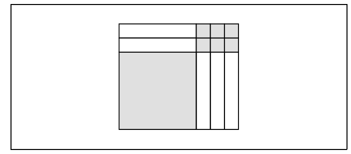


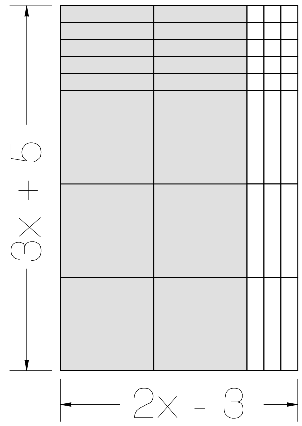
- **Step 6: Take the largest common factor out of each pair of terms. Make two rectangles.**

$2x(3x + 5) + {}^{-}3(3x + 5)$



☐ **Step 7: Put the two pieces together. (The two common factors go together in one new factor.)**





$(2x - 3)(3x + 5)$

Here's what you write down without using pictures:

$$6x^2 + 1x - 15$$

$$6x^2 + (10x - 9x) - 15$$

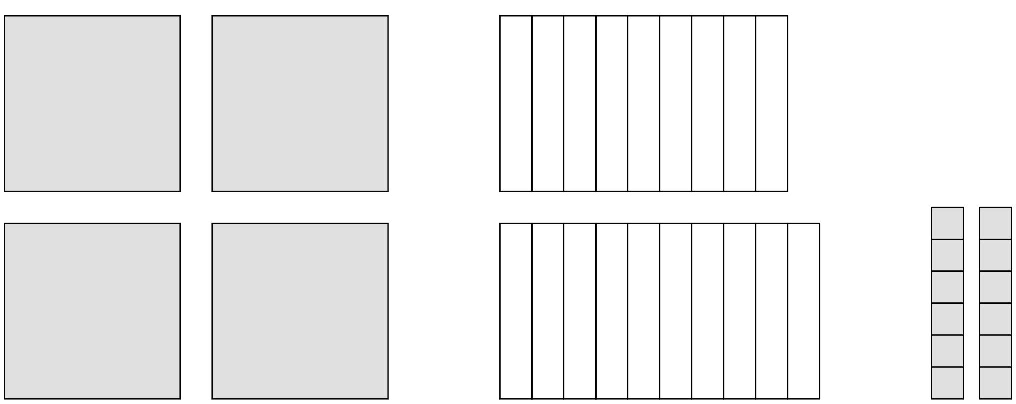
$$(6x^2 + 10x) + (^-9x - 15)$$

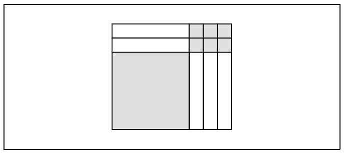
$$2x(3x + 5) + {}^-3(3x + 5)$$

$$(2x - 3)(3x + 5)$$

(6)(-15) = -90	
90	1
45	2
30	3
18	5
15	6
10	9

Let's try one more example. Factor: $4x^2 - 19x + 12$:





Solution:

$$4x^2 - 19x + 12$$

$$4x^2 + ({}^-16x + {}^-3x) + 12$$

$$(4x^2 - 16x) + ({}^-3x + 12)$$

$$4x(x - 4) + {}^-3(x - 4)$$

$$(4x - 3)(x - 4)$$

(4)(12) = 48

48	1
24	2
16	3
12	4
8	6

Notes: Since our product is positive 48, the two factors will add. Since we need two factors that add to be $^-19$, we use $^-16$ and $^-3$. Also, when there is a negative sign on the third term of the four terms, *always* use this negative as part of the common factor. If you do not do this, there will be no shared factor to join the two products together in the last step.

Exercises

Use the shortcut method to factor the following polynomials:

1. $2x^2 - 7x - 15$

2. $2x^2 - 3x - 5$

3. $2x^2 + 3x - 5$

4. $2x^2 - 7x + 6$

5. $4x^2 - 4x - 15$

6. $2x^2 + 7x - 15$

7. $6x^2 - x - 15$

8. $6x^2 + 11x - 10$

9. $2x^2 - 13x + 15$

10. $12x^2 + 25x + 12$

11. $20x^2 - 26x - 6$

12. $15x^2 + 8x + 1$

13. $25x^2 + 30x + 9$

14. $12x^2 - 7x - 12$

15. $3x^2 + 2x - 5$

16. $4x^2 + 8x + 3$

17. $2x^2 + x - 6$

Section 7
Recognizing Special Products

Introduction

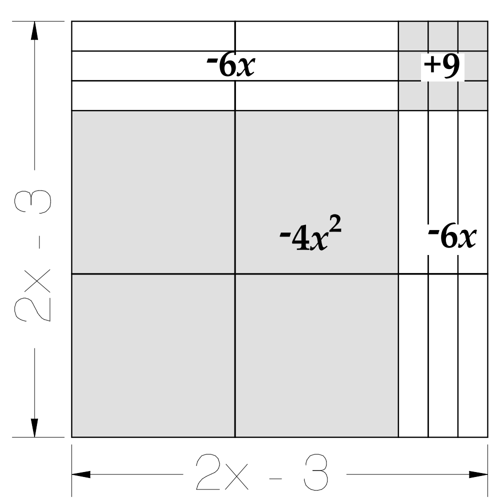
The factoring methods discussed so far in this chapter will work for any quadratic expression *which can be factored*. Many quadratic expressions cannot be factored, and they will be discussed briefly in this section. It may be useful to learn to recognize some special types of quadratic expressions so that factoring them will be even easier. The special expressions we are talking about are **perfect squares** and the **difference of two perfect squares**, both of which were discussed at the end of the previous chapter.

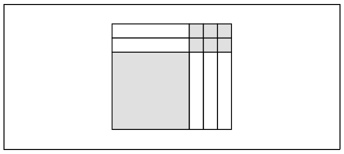
Recognizing Perfect Squares

As you will recall from our earlier discussion, perfect square trinomials have some very specific characteristics which make them relatively easy to recognize. An example of a perfect square can be generated by multiplying a binomial times itself, such as

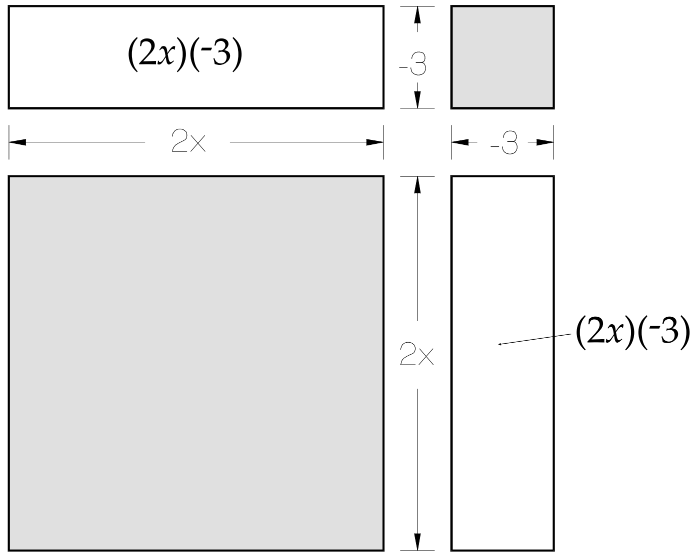
$$\begin{aligned}(2x-3)^2 &= (2x-3)(2x-3)\\ &= 4x^2-6x-6x+9\\ &= 4x^2-12x+9\end{aligned}$$

We can illustrate this product with the following diagram.





From the diagram we can see that both the x^2 term and the units term are themselves positive perfect squares. (Do you recognize the perfect square numbers?) Also we see that there are two equal groups of negative x-bars, each group being the product of the square roots of the squares.



The fact that the x-bars are all negative tells us that both dimensions of one of our squares (x^2 pieces or units) must be negative. (We generally put the negative signs on the units square, giving dimensions of $(2x - 3)$, but both dimensions could also be written $(^-2x + 3)$ and the result would still be correct.)

From this we see that perfect square trinomials *always* have the following characteristics:

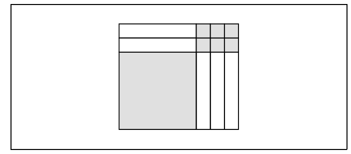
- **The x^2 term and the units term are always positive perfect squares. Look for numbers associated with each of these terms which are perfect square numbers.**

- **The x term may be either positive or negative, but its value is always twice the product of the square roots of the other two terms.**

If you look for these characteristics when factoring you will recognize a perfect square trinomial.

$4x^2 - 12x + 9$	Perfect squares
$(2x)^2 \quad (3)^2$	Square Roots
$2 \cdot (2x)(3)$	Twice the product of the square roots
$12x$	

Once a perfect square trinomial is recognized, factoring it is very easy. The terms in each of the binomial factors are the square roots of the x^2 term and the units term, separated by the sign of the x term.



$$4x^2 - 12x + 9$$

$$(2x)^2 \qquad (3)^2$$

$$(2x - 3)^2$$

$$4x^2 - 12x + 9 \; = \; (2x - 3)^2$$

Let's look at another example. Factor:

$$9x^2 + 6x + 1$$

Is this a perfect square?

$$9x^2 + 6x + 1$$

Perfect squares

$$(3x)^2 \qquad (1)^2$$

Square Roots

$$2 \cdot (3x)(1)$$

$$6x$$

Twice the product of the square roots

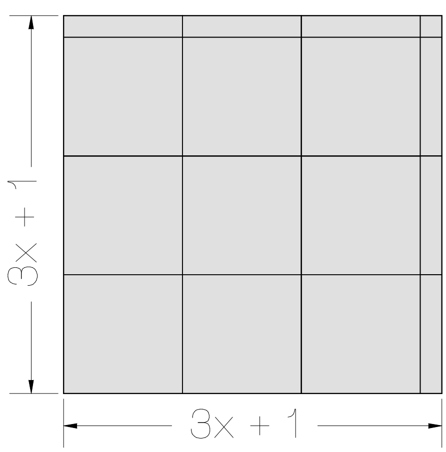
Yes, this is a perfect square. What are its factors?

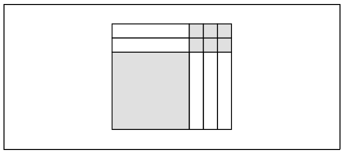
$$9x^2 + 6x + 1$$

$$(3x)^2 \qquad (1)^2$$

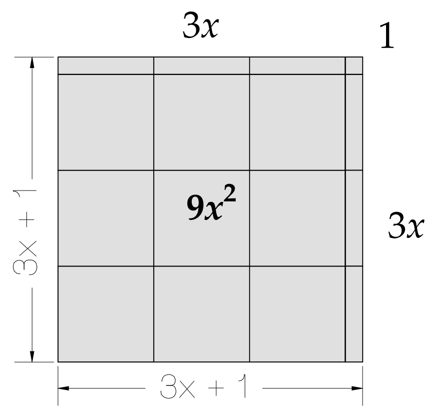
$$(3x + 1)^2$$

$$9x^2 + 6x + 1 \; = \; (3x + 1)^2$$





To check your work draw a diagram and/or multiply out your answer using the FOIL method to verify that the product equals the given trinomial.



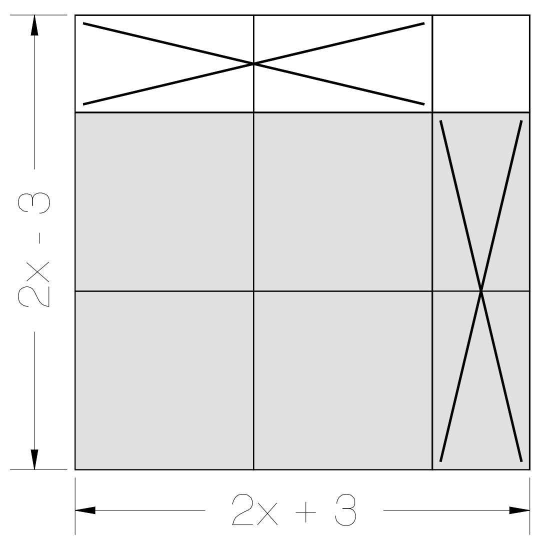
$$(3x+1)^2 = (3x+1)(3x+1)$$
$$= 9x^2 + 3x + 3x + 1$$
$$= 9x^2 + 6x + 1$$

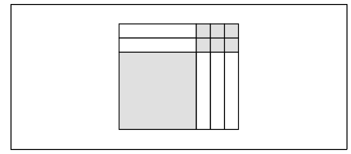
Recognizing the Difference of Two Perfect Squares

The difference of two perfect squares is the result of multiplying two binomials which are the same except for the signs on their second terms.

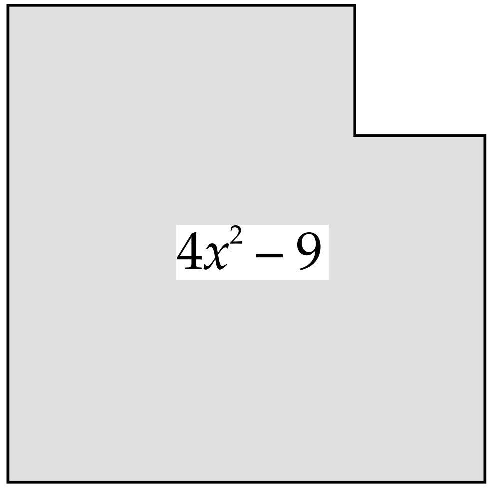
Different Signs

$$(2x+3)(2x-3)$$
$$= 4x^2 - 6x + 6x - 9$$
$$= 4x^2 - 9$$





Our result is one square (the units) taken away from another square (the x^2's), with all the x-bars canceling out.



From this we see that the difference of two perfect squares should be easy to recognize when factoring. This is due to several specific characteristics:

- **The x^2 term is a positive perfect square.**
- **The units term is a negative perfect square.**
- **The x term is missing altogether.**

There are other expressions which look a little like the difference of two squares, but if you look carefully you can always tell them apart.

For example:

$$4x^2 - 9x$$

or

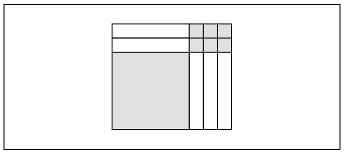
$$16x - 25$$

or

$$x - 25$$

Both of these expressions have two terms separated by a minus sign, and the number associated with each term is a perfect square number. Still these examples are **not** the difference of two perfect squares, because each expression has an x term, and since x is a bar, not a square, the x term cannot be a perfect square. (The top example can still be factored, however, by taking out the common factor of x.)

When you are asked to factor an expression having only two terms separated by a minus sign, look to see if one term is x^2 pieces and the other is units, with no x term; and then see if both the x^2 and the units terms are perfect squares. If they are, the expression is the difference of two perfect squares, and the factorization will be quite easy.



$$4x^2 - 9$$

$$16x^2 - 49$$

$$x^2 - 1$$

$$25x^2 - 1$$

These are all the difference of two perfect squares

Once you have identified an expression as the difference of two perfect squares, factoring is a breeze.

$$4x^2 - 9$$ Two perfect squares

$$(2x)^2 \quad (3)^2$$ Square Roots

$$(2x + 3)(2x - 3)$$ Use different signs

Different signs

Further factoring examples:

$$16x^2 - 49$$

$$(4x)^2 \quad (7)^2$$

$$(4x + 7)(4x - 7)$$

$$x^2 - 1$$

$$(x)^2 \quad (1)^2$$

$$(x + 1)(x - 1)$$

As with perfect squares, recognizing when you have an expression with a difference of two perfect squares is more than half of the work involved in factoring the expression. If you don't recognize an expression right away as a special type, but you see that it has no x term, place a zero-x in as the middle term, and then look again:

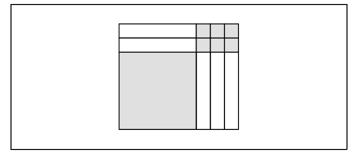
$$25x^2 - 1$$

$$25x^2 + 0x - 1$$

Ask yourself, "How can I get zero for the middle term?". The answer is to multiply two binomials which are the same except that they have opposite signs on the second term. The result must be

$$(5x + 1)(5x - 1)$$

Exercises



Identify which of the following are perfect square trinomials. Label each example as YES or NO. Factor only the perfect square trinomials.

1. $x^2 + 6x + 9$

2. $x^2 + 5x + 6$

3. $2x^2 + 3x - 9$

4. $4x^2 + 20x + 25$

5. $9x^2 + 6x - 1$

6. $4x^2 - 4x + 1$

7. $6x^2 + 11x + 5$

8. $x^2 + 8x - 9$

9. $3x^2 - 5x + 2$

10. $16x^2 - 24x + 9$

11. $4x^2 + 21x - 25$

12. $4x^2 - 28x + 4$

Label the following expressions either **PS** for perfect squares, **DTPS** for the difference of two perfect squares, or **neither**. Factor those labeled PS or DTPS. Do not attempt to factor the examples that are not PS or DTPS

13. $4x^2 - 1$

14. $x^2 + 1$

15. $x^2 + 6x + 9$

16. $x^2 - 9$

17. $4x^2 - 6x$

18. $9x^2 + 12x - 4$

19. $4x^2 - 12x + 9$

20. $9x - 1$

21. $16x^2 + 8x + 1$

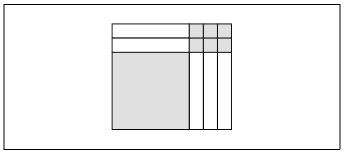
22. $25x^2 - 4$

23. $x^2 - 5x + 6$

24. $x^2 - 10x + 25$

25. $4x^2 + 9$

26. $4x^2 - 25$



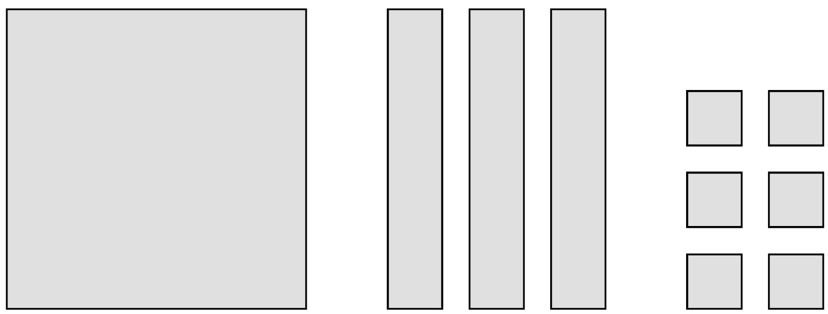
27. $x - 4$

28. $x^2 + 6x - 16$

Section 8
Expressions Which Cannot Be Factored

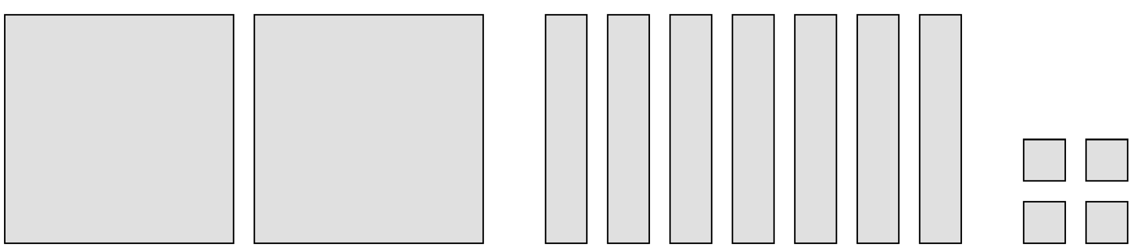
Introduction

Using the chips, factoring means to form a rectangle from the given pieces, with no missing pieces and no pieces left over. *For many groups of pieces, making such a rectangle is not possible*. For example, try making a rectangle out of these pieces:



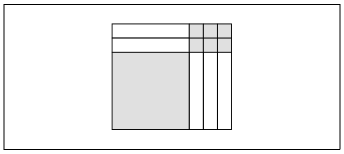
$$x^2 + 3x + 6$$

Or these



$$2x^2 + 7x + 4$$

Actually there are many more expressions which *cannot* be factored than those that *can* be factored. So if you are faced with a tough factoring problem, try all the approaches you have learned, but realize that *not factorable* is a possible answer.



Remember: Look for Common Factors First

Perhaps the most often forgotten step in factoring is to *always* look for common factors first. Removing a common factor will always simplify an expression and will sometimes turn an apparently impossible problem into an easy problem.

For example, factor:

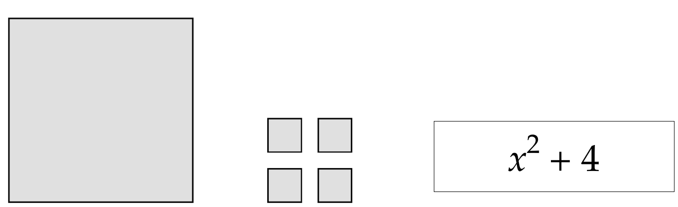
$18x^2 - 8$	
$2(9x^2 - 4)$	Common Factor
$2(3x + 2)(3x - 2)$	Difference of Squares

$3x^2 - 24x + 48$	
$3(x^2 - 8x + 16)$	Common Factor
$3(x - 4)^2$	Perfect Square

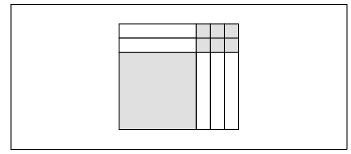
$3x^3 + 15x^2 + 18x$	
$3x(x^2 + 5x + 6)$	Common Factor
$3x(x + 2)(x + 3)$	Factor

The Sum of Two Squares

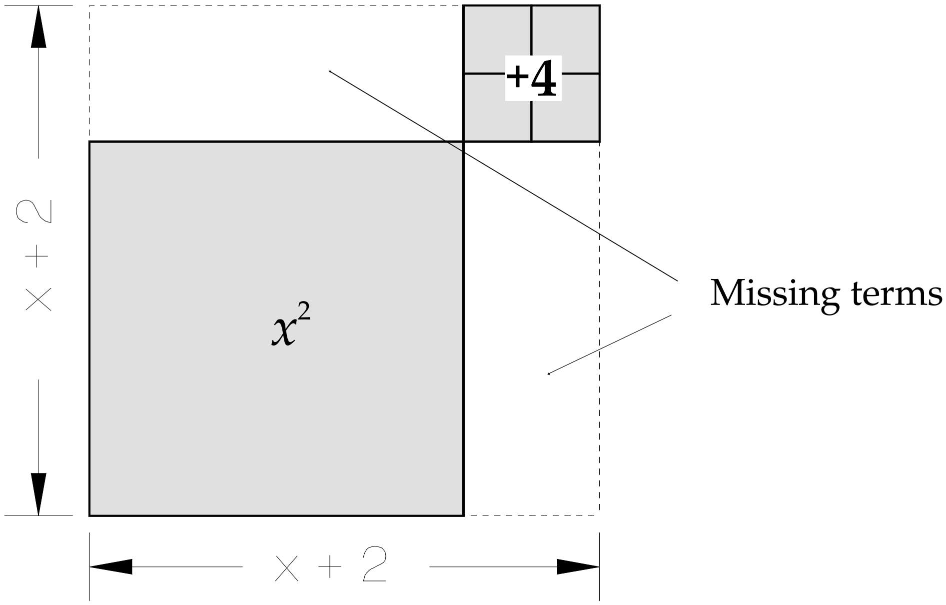
Perhaps the type of expression most often mis-factored is the sum of two squares.



Using chips it may be obvious that no rectangle can be made from the pieces given. But students often try to suggest the following:



$$x^2 + 4 = (x + 2)(x + 2) \qquad \textbf{(Not True !!)}$$



Although the above suggestion may *seem* reasonable, the picture illustrates that there are terms missing which are needed to make a perfect square. If the units square (the $^+4$) were negative, then the two missing terms would have had opposite signs and would have canceled out. But if the units square is positive, the missing terms must both have the same sign, and therefore they can't cancel.

This is why we *can't factor* the *sum* of two squares, but we *can* factor the *difference* of two squares.

Exercises

Factor completely *if possible*.

1. $3x^2 + 15x + 18$

2. $4x^2 + 9$

3. $2x^2 - 18$

4. $3x^2 + 18x + 27$

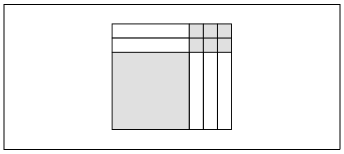
5. $x^2 - 3x + 5$

6. $x^2 + 4x - 5$

7. $3x^2 + 2x - 5$

8. $2x^2 + 5x + 6$

9. $4x^2 - 24x + 9$



10. $2x^2 + 16x + 32$

11. $5x^2 - 20$

12. $4x^2 - 9x$

13. $3x^2 + 12$

14. $x^3 + 2x^2 + x$

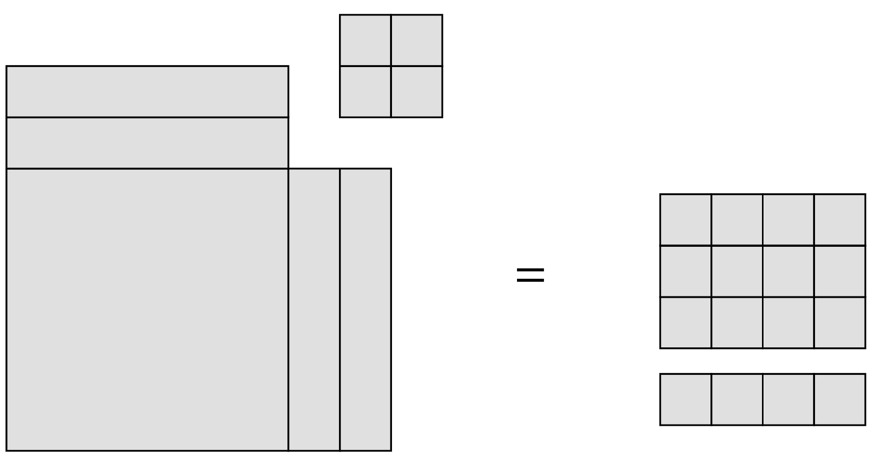
15. $x^2 + 6x + 5$

16. $x^2 + 5x - 6$

17. $x^2 + 7x - 6$

18. $18x^2 - 8x$

Chapter 11
Quadratic Equations



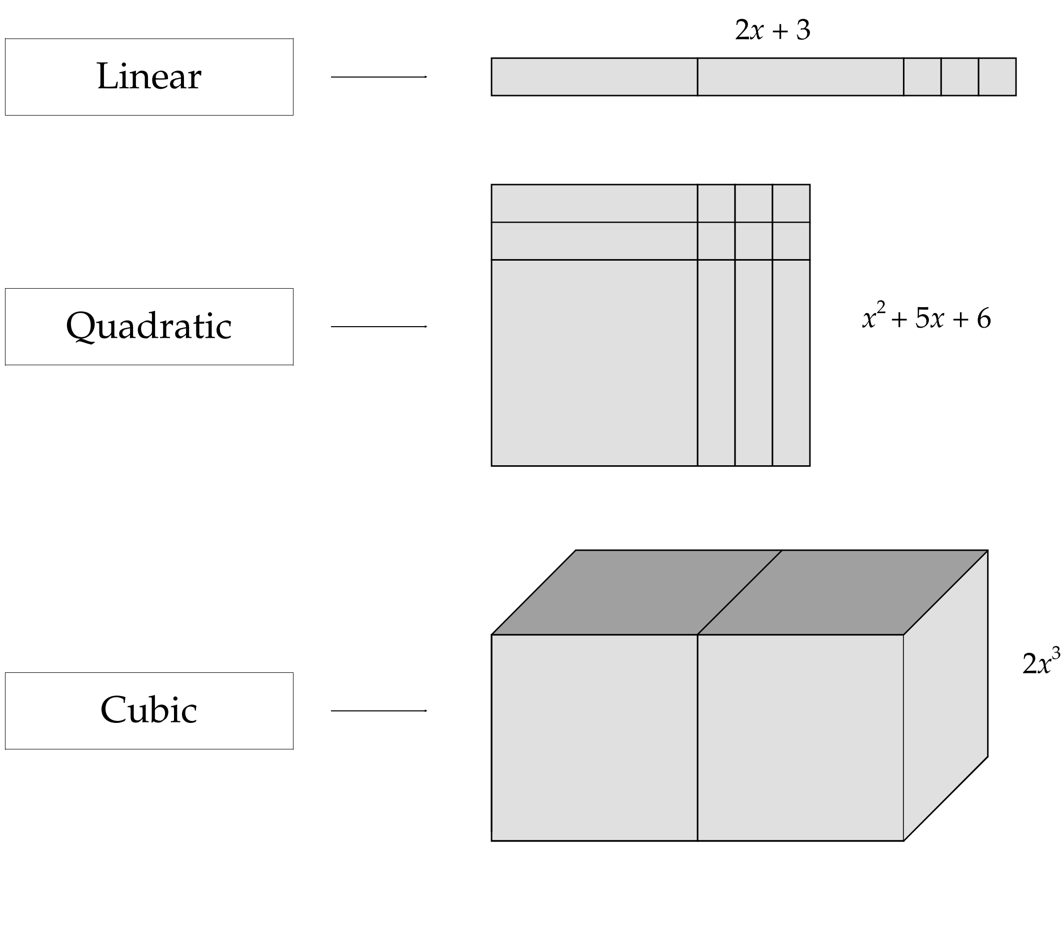
Section 1
Introduction

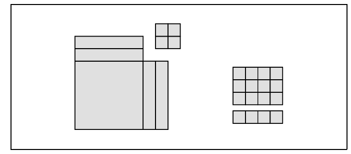
Polynomials and Quadratic Expressions

In the last two chapters we have worked with polynomials. We have learned about terms, combining polynomials, and factoring. The terms in any polynomial expression can include units (small squares), x's (bars), x^2 pieces (large squares), x^3 pieces (cubes, which we don't have in our kit), and higher powers of x (like x^4, x^5, …).

When a polynomial expression contains x^2 pieces (large squares), but does *not* contain any higher powers of x, we call the expression a **quadratic expression**. Expressions having combinations of squares, bars and chips are called *quadratic expressions* is because they can be represented using flat *four*-sided figures—squares or rectangles. The prefix "Quadri" means having *four* parts, in this case four sides.

In a similar way, expressions having only x's and units are called **linear**, because they can be represented using *lines*; and expressions having x^3's as their highest term (biggest piece) are called **cubic**, because they can be represented using *cubes*.





Just as a quadratic expression is an expression having x^2-squares as its biggest piece (highest term), so **quadratic equations** are *equations* having x^2-squares as their biggest piece, along with some combination of x-bars and unit-chips. Some examples of quadratic equations are

$$x^2 + 2 = 3x + 6$$
$$x^2 - 6x + 8 = 0$$
$$2x^2 - 7 = x^2 - 7x + 1$$

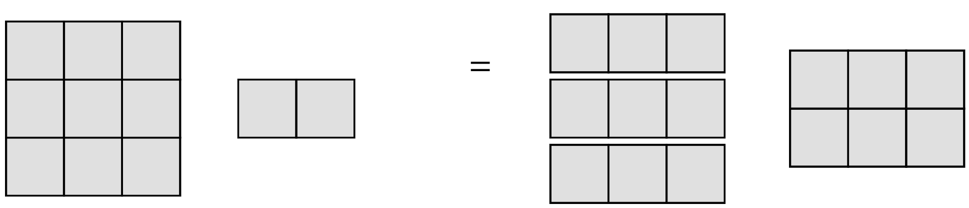
These are all equations having x^2 as the highest term. As with all equations, each statement says that the quantities on the left and the right of the = (equal) sign have the same value. However these statements of equality will only actually be true for *certain values* of the unknown (values of x). To be sure you understand this idea you should try guessing a value for x which will make one of the given equations into a true statement. For example, look at the first equation:

$$x^2 + 2 = 3x + 6$$

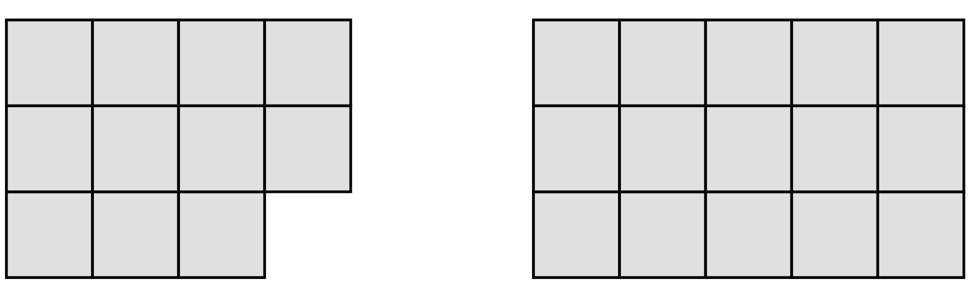
If we guessed that x might be 3, we would get

$$(3)^2 + 2 = 3(3) + 6$$
$$9 + 2 = 9 + 6$$
$$11 = 15 \quad \text{(Not true)}$$

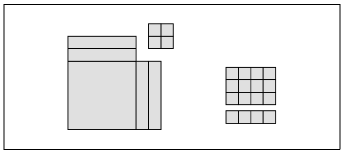
$$(3)^2 \quad + \quad 2 \quad = \quad 3(3) \quad + \quad 6$$



$$11 \neq 15$$



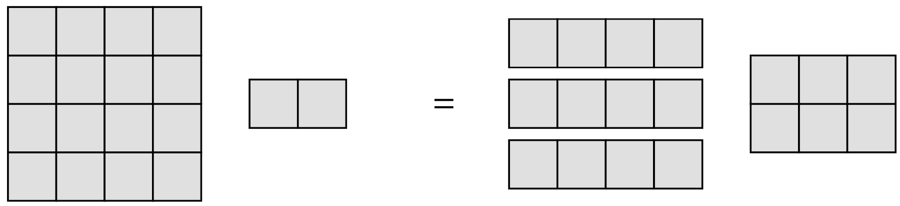
This is obviously not true because there are 11 units one the left and 15 units on the right side.



However, next we might choose to try $x = 4$; this would give us

$$\begin{aligned} (4)^2 + 2 &= 3(4) + 6 \\ 16 + 2 &= 12 + 6 \\ 18 &= 18 \end{aligned}$$

$$(4)^2 \quad + \quad 2 \quad = \quad 3(4) \quad + \quad 6$$



$$18 = 18$$

This *is* true. Since $x = 4$ makes this equation true, we say that one **solution** for the equation is

$$x = 4$$

The purpose of this chapter is to learn how to find the correct solutions (values of the unknown) to quadratic equations *without guessing !* It's like a puzzle, where the real values required for the unknown are hidden within every quadratic equation, and your job is to solve the puzzle and find the true solution—the value of x. The puzzle-solving process isn't very hard; it is based upon ideas that you already know from previous chapters.

Exercises

Decide if the following items are expressions or equations. Then decide if they are linear, quadratic, or cubic:

1. $1{,}000{,}000x + 17$

2. $x^2 = 25$

3. $x^2 + 3x + 2$

4. $23 + 32x + 2x^2 + 3x^3 = 27$

Section 2
The Zero Product Rule

Multiplying and Zero

There is a very simple fact which is used with astonishing power for solving quadratic equations. This fact, which you already understand, is that *when two numbers are multiplied together the answer is never zero unless one of the numbers being multiplied is zero.* Let us illustrate this. Look at the following products of two numbers:

$$(3)(2) = 6$$

$$\frac{1}{2} \cdot 4 = 2$$

$$(^-3)(^+5) = {}^-15$$

$$\left(-\frac{2}{3}\right) \cdot \left(-\frac{3}{5}\right) = {}^+\frac{2}{5}$$

$$(0)(7) = 0$$

If we multiply any two positive numbers, negative numbers, whole numbers, or fractions, we always get a positive or negative whole number or fraction for an answer, unless we multiply by zero. If we multiply by zero we always get zero for an answer; and *any time we get zero as the answer to a multiplication, then one of the multipliers must be zero.* If you know that:

$$(x)(3) = 0$$

then x must be zero (since 3 can't be zero). If you know that

$$(x)(x - 4) = 0$$

then either one or the other of the two parentheses must be zero. Either

$$x = 0 \quad \text{or} \quad x - 4 = 0$$

In such a situation, there could be two possible values for x:

$$\boxed{x = 0} \qquad \textit{or} \qquad \begin{array}{c} x - 4 = 0 \\ \downarrow \\ \boxed{x = 4} \end{array}$$

Perhaps you have already recognized how this relates to quadratic equations. If we have a quadratic equation like

$$x^2 - 4x = 0$$

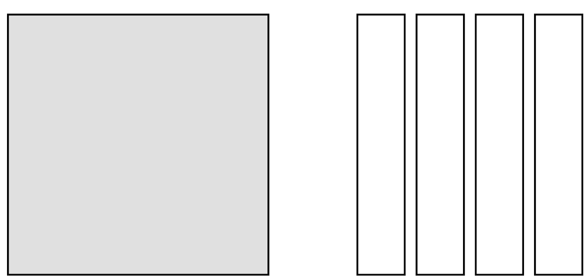
rather than guessing what values of x will make this a true statement, we can get the correct solutions for x very quickly if we can *factor* the expression on the left side of the equation:

$$\begin{aligned} x^2 - 4x &= 0 \\ (x)(x-4) &= 0 \end{aligned}$$

So either:

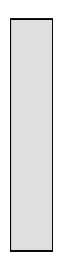
$x = 0$		$x - 4 = 0$
$\boldsymbol{x = 0}$	*or*	$\boldsymbol{x = 4}$

Using chips, this equation would look like:



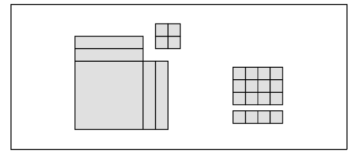
$$= 0$$

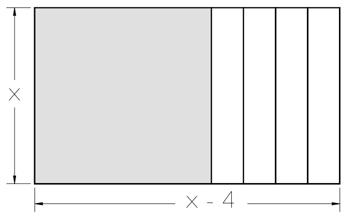
Obviously, if the areas of the pieces on the left combine to make zero, then the positive area (the square), must be canceled out by the negative area (the bars). The question is, what number must x (the length of the bar and the side of square) be for these areas to really cancel?



$$= ?$$

We find out by factoring, or making a rectangle.

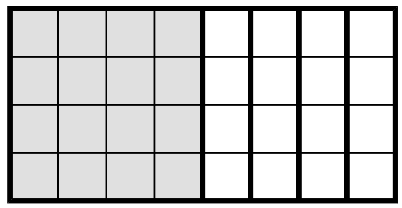




$= 0$

The area of this rectangle will only be zero if either the height is zero or the width is zero. The height will be zero when $x = 0$. The width will be zero when $x = 4$. Substituting these values into our picture we have:

$= 0$ *or*



$= 0$

Both are true, and no other value for x will work. (Try some.) Substituting these values numerically into the original equation gives:

$$(0)^2 - 4(0) = 0$$

$$\mathbf{0 = 0}$$

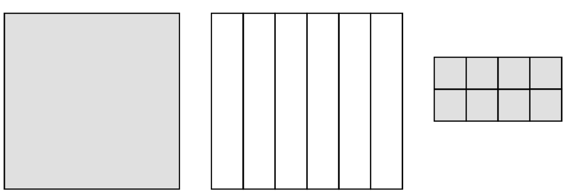
and

$$(4)^2 - 4(4) = 0$$

$$\mathbf{0 = 0}$$

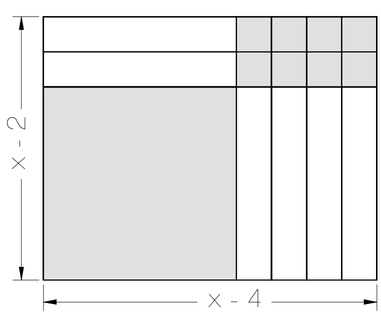
Here is a second example:

$$x^2 - 6x + 8 = 0$$

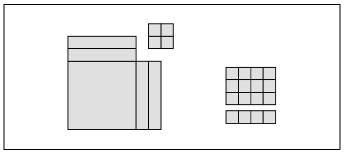


$= 0$

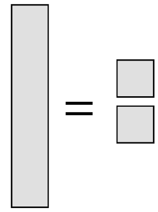
If the total area is zero, then the white pieces must cancel out the colored pieces. What size must x be to make this happen? We factor (make a rectangle):



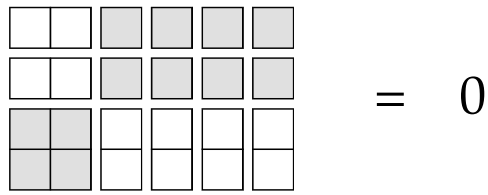
$= 0$



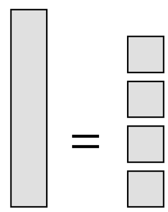
Now we can see that the height would be zero if $x = 2$:



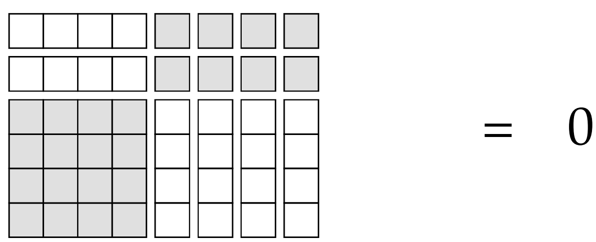
This would give:



Similarly the length would be zero if $x - 4 = 0$, which means that $x = 4$:



The picture would now be:

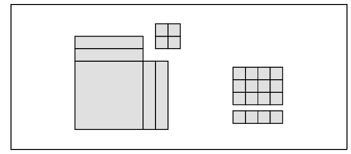


which is also true. Here are the steps with algebra symbols alone:

The equation	$x^2 - 6x + 8 = 0$
Factors into	$(x-2)(x-4) = 0$
Using the zero product rule:	$x - 2 = 0$ *or* $x - 4 = 0$
Gives two solutions:	$x = 2$ *or* $x = 4$

Numerical substitution of these results into the original equation gives:

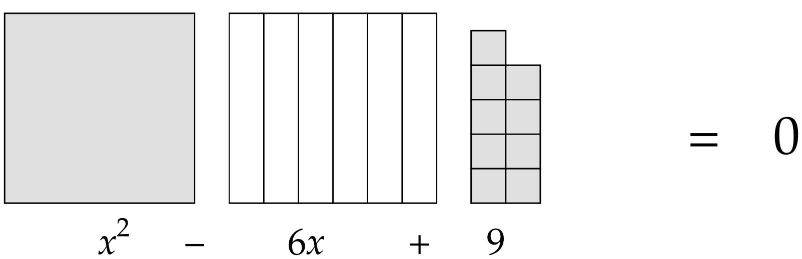
$(2)^2 - 6(2) + 8 = 0$	$(4)^2 - 6(4) + 8 = 0$
$4 - 12 + 8 = 0$	$16 - 24 + 8 = 0$
$0 = 0$	$0 = 0$



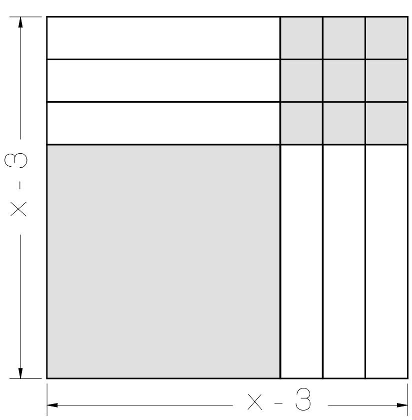
Quadratic equations have at most two solutions for the value of the unknown. When the pieces are factored into a rectangle, either the height or the width of the rectangle can be zero. We get *two* answers because our figures (rectangles) have just *two* dimensions (height and width). The highest term of a quadratic equation is x^2, where the exponent 2 literally means *two dimensions*, giving two possible solutions. (Similarly, linear equations, where the highest term is x, have at most one solution, as seen in the equations chapter. Cubic equations, which have an x^3 term, have up to 3 solutions.)

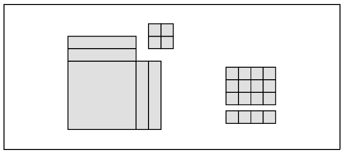
It is possible that the two solutions to a quadratic equation will be the same. For example, find the solutions for

$$x^2 - 6x + 9 = 0$$



Factoring, we find that we have a perfect square:

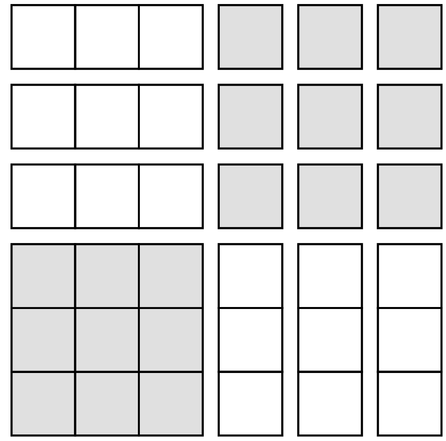




Either the height or the width can be zero, but since the height and the width are equal, our two resulting solutions are both the same:

$$x - 3 = 0 \qquad x - 3 = 0$$

$$\boxed{x = 3} \qquad \boxed{x = 3}$$

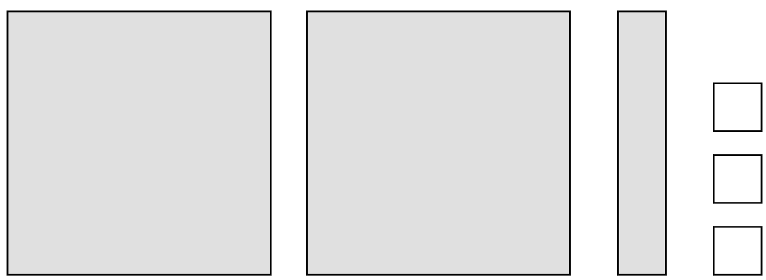


Numerical substitution gives

$$(3)^2 - 6(3) + 9 = 0$$
$$9 - 18 + 9$$
$$0 = 0$$

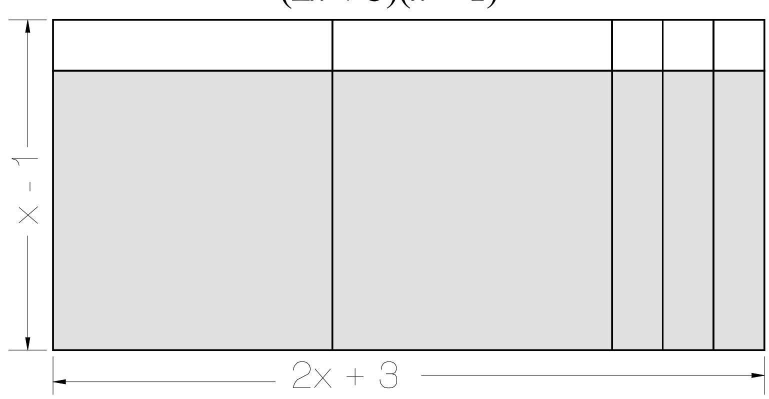
Here is an example with more than one x^2:

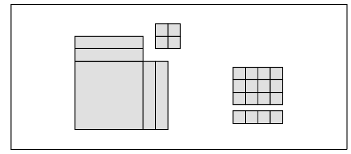
$$2x^2 + x - 3 = 0$$



This factors (with the addition of ^+2x and ^-2x) into

$$(2x + 3)(x - 1)$$





In this case the zero product rule gives us:

$$2x + 3 = 0$$
$$2x = -3$$
$$\boldsymbol{x = -\frac{3}{2}}$$

or

$$x - 1 = 0$$
$$\boldsymbol{x = 1}$$

We can check these results by substituting into the original equation:

$2\left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right) - 3 = 0$	$2(1) + (1) - 3 = 0$
$2\left(\frac{9}{4}\right)^2 - \left(\frac{3}{2}\right) - 3 = 0$	$2 + 1 - 3 = 0$
$\frac{9}{2} - \frac{3}{2} - 3 = 0$	$0 = 0$
$0 = 0$	

Exercises

Solve these quadratic equations:

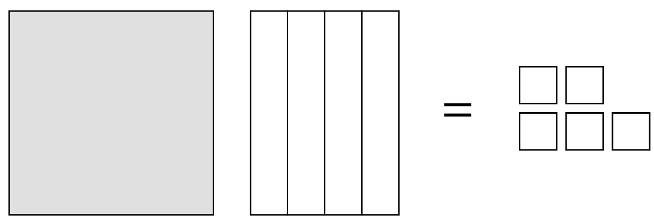
1. $x^2 - 6x + 8 = 0$
2. $x^2 - 8x + 16 = 0$
3. $x^2 - 8x + 12 = 0$
4. $x^2 - 7x + 12 = 0$
5. $2x^2 - 9x + 9 = 0$
6. $3x^2 - 8x + 5 = 0$
7. $3x^2 - 16x + 5 = 0$
8. $2x^2 - 11x + 12 = 0$
9. $3x^2 - 13x + 12 = 0$
10. $x^2 - 10x + 21 = 0$

Section 3
Standard Form

Changing the Form of the Equation

If we are going to use the zero product rule to help solve a quadratic equation, then the first thing we must do is to be sure that one side of the equation is zero. This will allow us to factor the other side of the equation, and then to set each of the factors equal to zero. For example, if we have an equation which starts out as

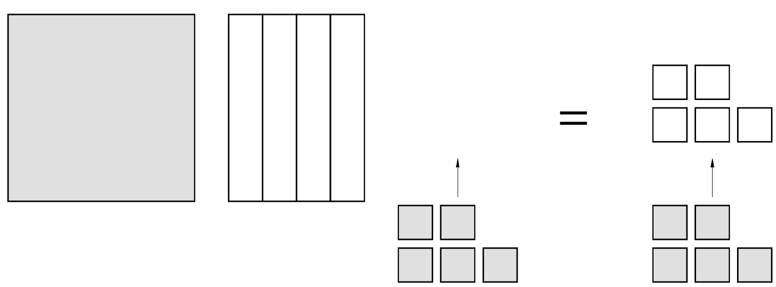
$$x^2 - 4x = {}^{-}5$$



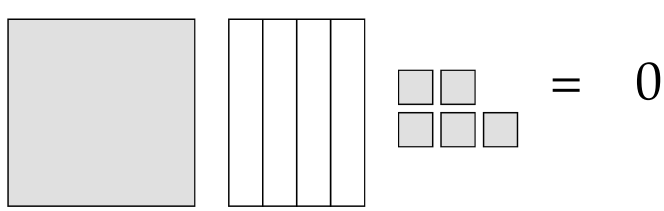
we *must* get one side of the equation to be zero. We can do this easily by adding 5 to both sides of the equation:

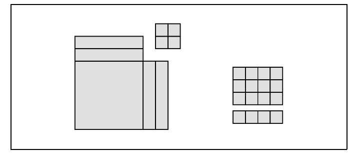
$$x^2 - 4x = -5$$

$$+5 \qquad +5$$



$$\boxed{x^2 - 4x + 5 = 0}$$





Now we are ready to factor the left side of the equation and to set the factors equal to zero, as demonstrated in the last section.

When we write a quadratic equation showing the x^2-term first, followed by the x-term and the units-term, all equaling zero, we say that the quadratic equation is in **standard form**. Standard form is shown as

$$Ax^2 + Bx + C \ = \ 0$$

where A, B and C represent numbers, and x represents the unknown. (A and B are called **coefficients** of x^2 and x respectively; *coefficient* means the number multiplying an unknown.) So if we have an equation like

$$2x^2 + 7x + 5 \ = \ 0$$

we would say $A \ = \ 2, \ B \ = \ 7,$ and $C \ = \ 5.$

In the equation:

$$x^2 - 4x + 5 \ = \ 0$$

$$A = 1, \ B = {}^{-}4, \ \text{and } C = 5.$$

we can see that the coefficient may be implied (1 in $1x^2$) or negative ($^{-}4$ in $-4x$). B is $^{-}4$ because we can write $-4x$ as $+({}^{-}4)(x)$.

It is important to note that in standard form all of the terms of a quadratic equation are in a particular order, with the highest term (x^2) first on the left, the x-term in the middle, and the units-term last, followed by the equal sign (=) and the zero (0).

This order of terms is called **descending order** because the size of the pieces (the power of x) starts with the largest term (highest power) first on the left, and then decreases (descends) as we move to the right. It is standard practice to arrange all expressions in descending order. Keeping this order consistent makes it easier to recognize and combine like terms; it also makes it easier to factor.

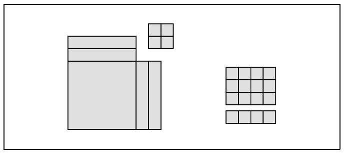
If a quadratic equation starts out written in some form other than standard form, we must first rearrange the terms until we have the standard form before we can proceed to the solution (find the value of x). So equations like

$$x^2 + 2 \ = \ 3x + 6$$

and

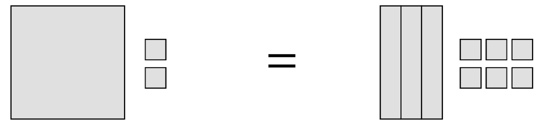
$$2x^2 - 7 \ = \ x^2 - 7x + 1$$

need to be rearranged and put into standard form before they are factored and solved. As mentioned earlier, this is done by adding to both sides until



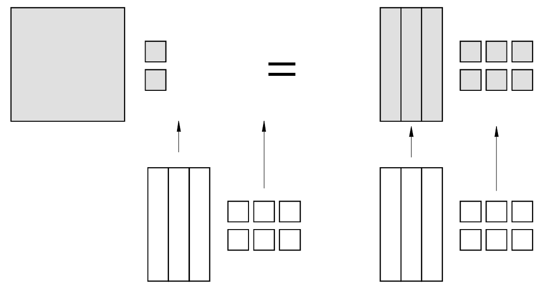
one side (usually the right side) equals zero, while combining like terms and arranging the other side in descending order. For the above examples the process looks like this:

Beginning $x^2 + 2 = 3x + 6$

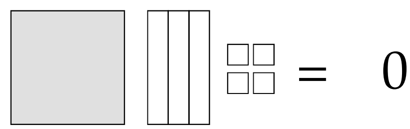


Add Opposites

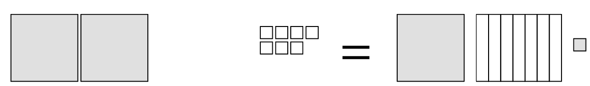
$$\begin{array}{rcl} x^2 + 2 & = & 3x + 6 \\ -3x - 6 & & -3x - 6 \end{array}$$



Standard Form $x^2 - 3x - 4 = 0$

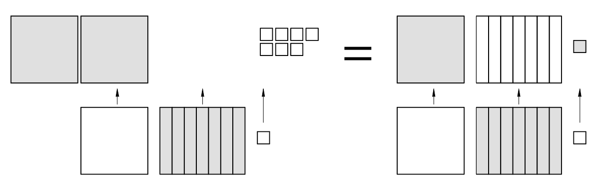


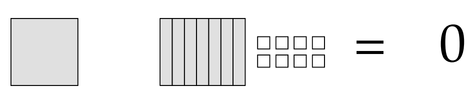
Beginning $2x^2 - 7 = x^2 - 7x + 1$



Add Opposites

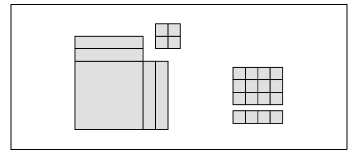
$$\begin{array}{rcl} 2x^2 - 7 & = & x^2 - 7x + 1 \\ -x^2 + 7x - 1 & & -x^2 + 7x - 1 \end{array}$$



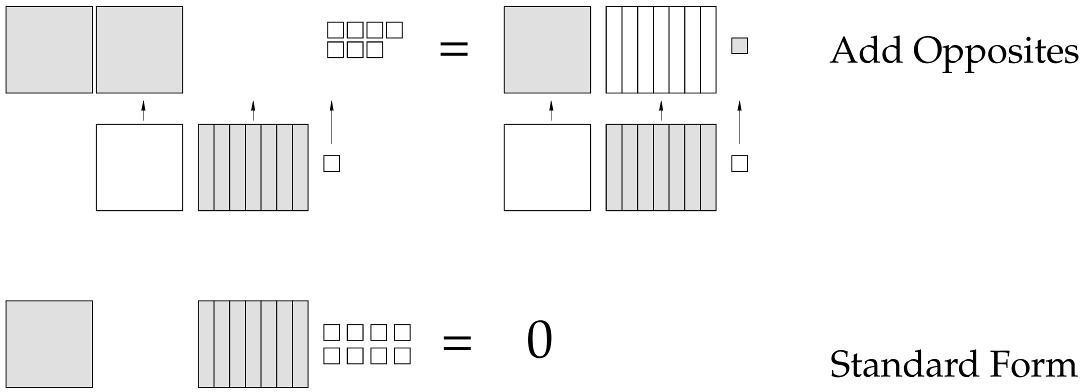


Standard Form $x^2 + 7x - 8 = 0$

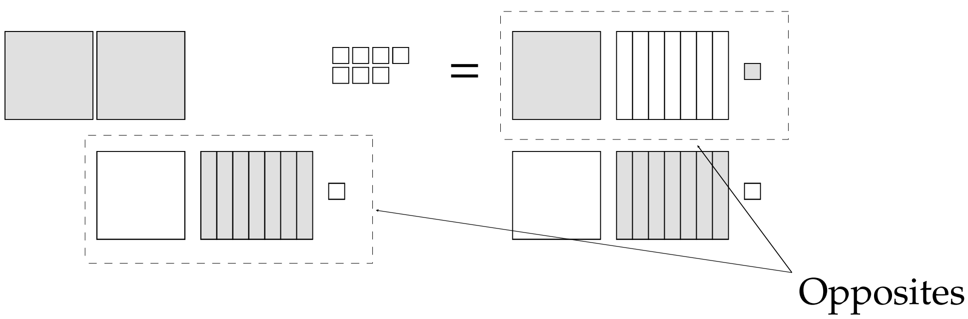
The Flip-Chip Short Cut



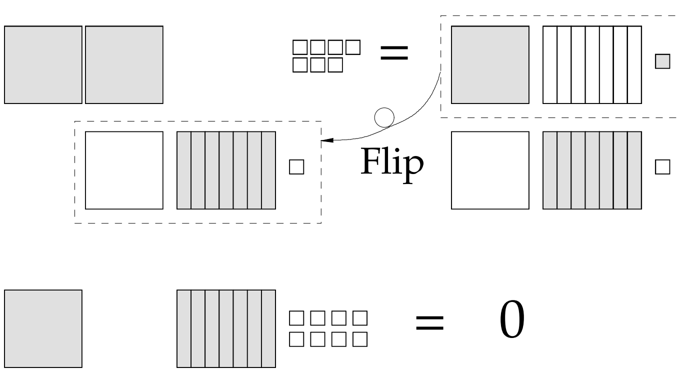
When rearranging an equation into standard form we get rid of all of the pieces (terms) on one side of the equal sign (=) by adding their opposites to both sides:

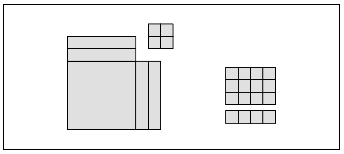


In this process of adding opposites, the equal sign is like a dividing line between the two sides of a balance, and adding the same amount to both sides gets one side to equal zero while maintaining the balance. Notice that when one side gets canceled out, the opposite of each of its terms appears on the other side (across the =).



So as the pieces disappear from one side of the equation, the same pieces appear in flipped over form (with opposite signs) across the equal sign, on the other side of the equation. But this is just the same as taking the pieces we are canceling and, while moving them across the equal sign, flipping them over.





You can move pieces (terms) across the equal sign as long as you flip them over (change their sign) when they cross over from one side of the equal (=) to the other. This is equivalent to adding the opposite of the term to both sides.

Like all short cuts, you must be very careful with this one when you use it. In particular, be sure not to flip terms without reason. Pieces flip (change sign) only when

- **They are multiplied by a negative.**
- **They move across the equal sign.**

Exercises

Arrange these quadratic equations in standard form. Do not solve.

1. $6 = 5x - x^2$
2. $3x^2 - 2x = 5x + 2x^2 - 12$
3. $x^2 + 12 = 11x - x^2$
4. $2x^2 - 6x = x^2 - 8$
5. $3 + 3x^2 = 10 - 5x + x^2$
6. $2 + x^2 - 2x = 20 + 5x$
7. $8x - 1 = -3x^2 - 6$
8. $3x^2 - 6x + 2 = 2x^2 + 4x - 19$
9. $x^2 - 5 = {}^{-}2x^2 + 16x$
10. $2x^2 + 5x = 14x - 9$
11. $x(2x - 3) = 5 - x$
12. $x^2 - 5 = 3(2x + 1)$
13. $x + 8 = 7 - 2x^2$
14. $(x + 2)(x - 5) = 3 - 2x^2$
15. $2x(x - 2) = 17$

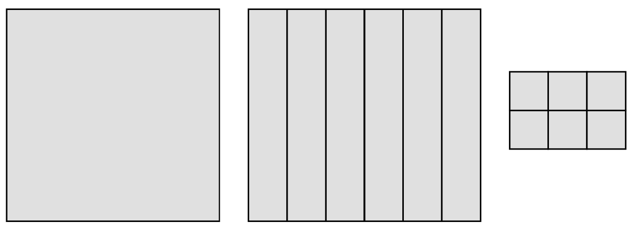
Section 4
Factoring Quadratic Equations

Quadratic Equations Having Positive *x*-bars

In section 2 of this chapter we gave several examples of quadratic equations which had negative *x*-bars when they were written in standard form. Now we will look at other types of quadratic equations.

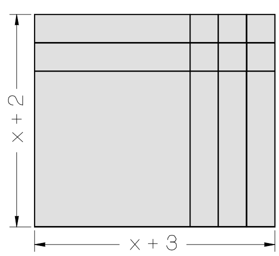
For example, let's factor and solve this quadratic equation:

$$x^2 \quad + \quad 5x \quad + \quad 6 \quad = \quad 0$$



$= \quad 0$

All the chips are colored side up, so how can we get zero for their sum? We know the factors are $(x + 2)$ and $(x + 3)$:

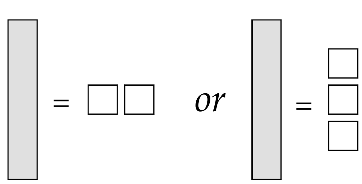


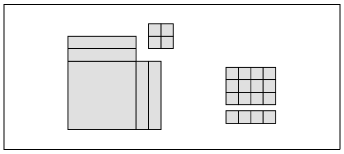
$= \quad 0$

This means the height and width will be zero if

$$\begin{array}{rcl} x + 2 &=& 0 \\ {}^-2 && {}^-2 \end{array} \quad \text{or} \quad \begin{array}{rcl} x + 3 &=& 0 \\ {}^-3 && {}^-3 \end{array}$$

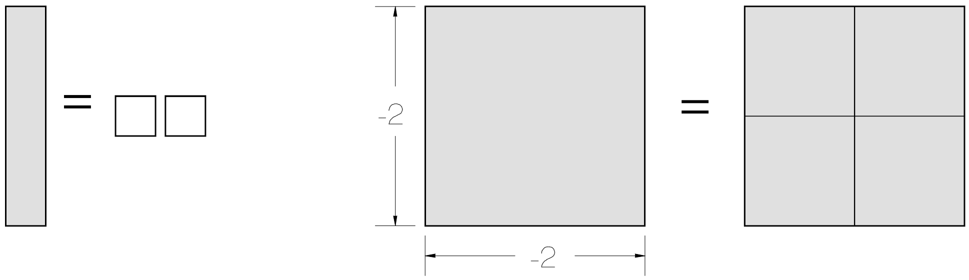
$$\mathbf{x = {}^-2} \qquad\qquad \mathbf{x = {}^-3}$$



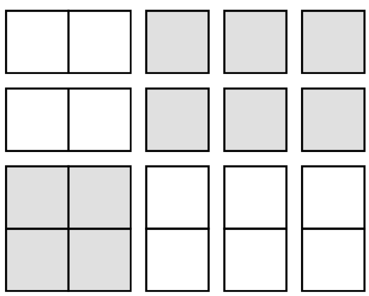


When the x-bar is replaced by any number of negative chips, the x^2 square will still be positive. This is because *both* dimensions of the square will be negative (two flips):

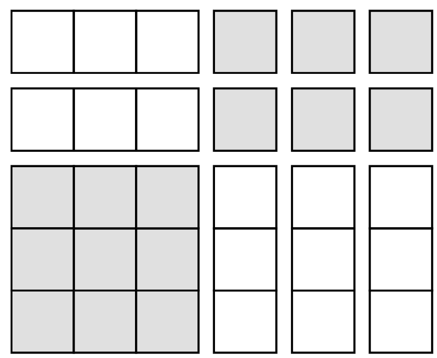
$$(^-2)\cdot(^-2) = {}^+4$$



So we can replace the x-bar with $^-2$ and the x^2 with $^+4$, or we can let the x-bar be $^-3$ and the x^2 be $^+9$, giving:



$x = {}^-2$



$x = {}^-3$

These are both equal to zero.

Substituting these results numerically into the original equation gives:

$$(-2)^2 + 5(-2) + 6 = 0 \qquad (-3)^2 + 5(-3) + 6 = 0$$

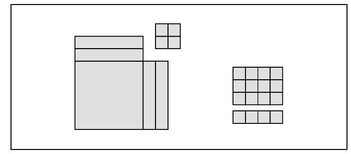
$$4 - 10 + 6 = 0 \qquad 9 - 15 + 6 = 0$$

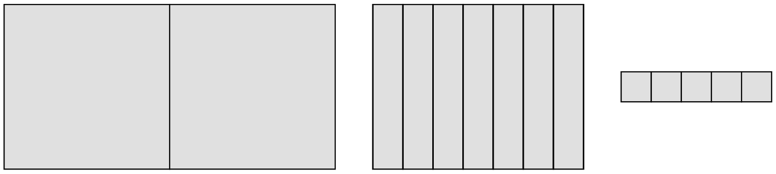
$$0 = 0 \qquad 0 = 0$$

Both answers are correct: $x = {}^-2$ and $x = {}^-3$.

A second example is

$$2x^2 + 7x + 5 = 0$$



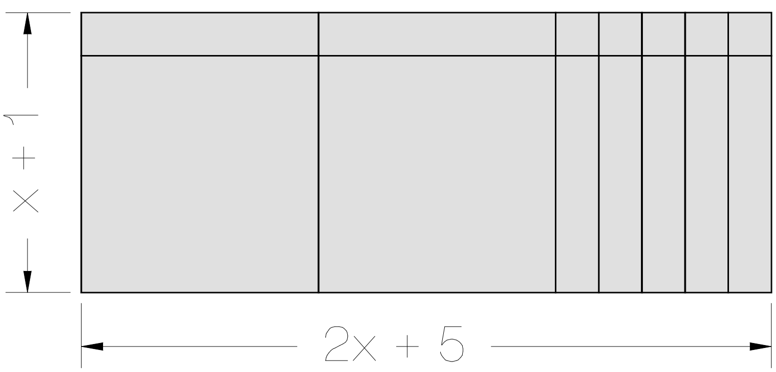


$= 0$

$$2x^2 \quad + \quad 7x \quad + \quad 5 \quad = \quad 0$$

We can factor this, with the result

$$(x + 1)(2x + 5) = 0$$



$= 0$

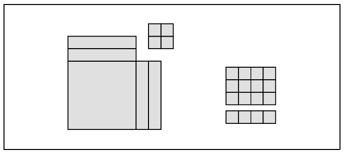
This time:

Either $\quad x + 1 = 0 \quad$ *or* $\quad 2x + 5 = 0$

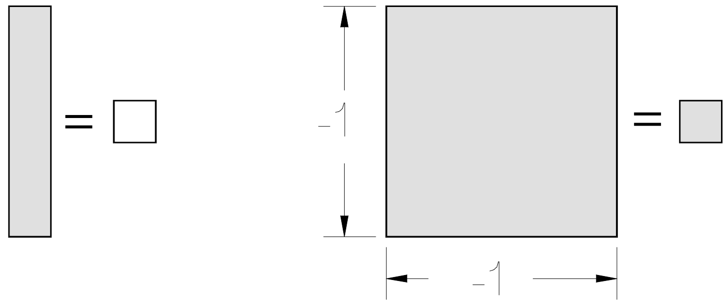
$$\mathbf{x = -1}$$

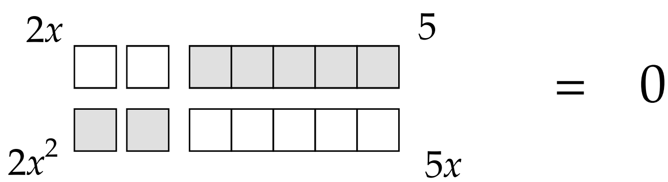
$$2x = -5$$

$$\mathbf{x = -\frac{5}{2}}$$

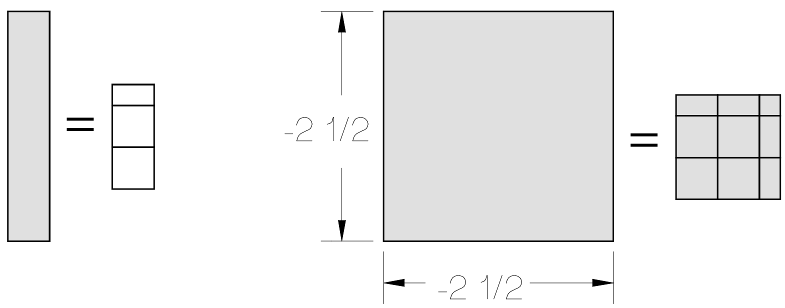


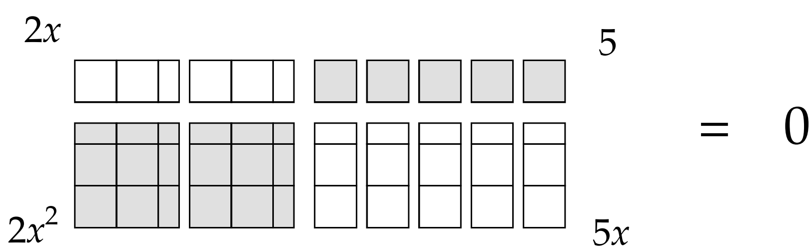
We can picture these results in the following way:



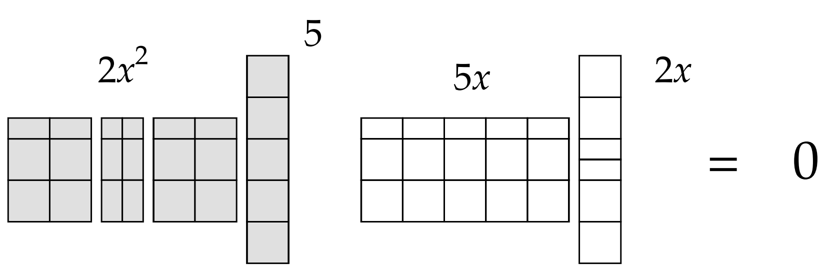


The fractional result can also be demonstrated, but it requires care. Remember that each x is - $5/2$:

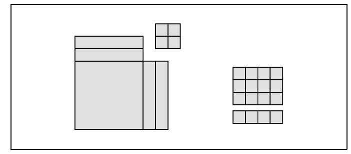




Rearranging shows the positives match the negatives, so the total is zero:



Numerically this result can be checked by substitution:



$$2\left(-2\frac{1}{2}\right)^2 + 7\left(-2\frac{1}{2}\right) + 5 = 0$$

$$2\left(\frac{25}{4}\right) + 7\left(-\frac{5}{2}\right) + 5 = 0$$

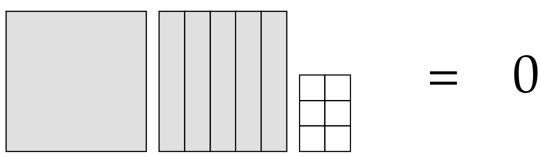
$$12\frac{1}{2} - 17\frac{1}{2} + 5 = 0$$

$$0 = 0$$

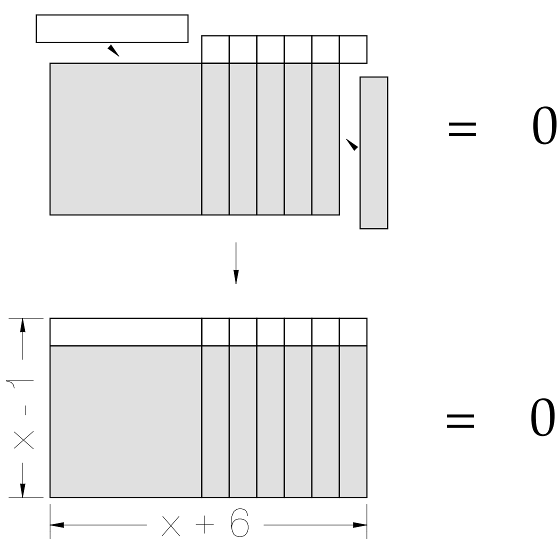
Quadratic Equations Having Negative Units

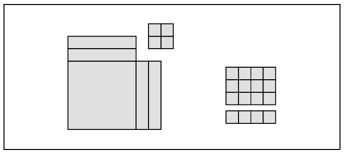
When a quadratic equation in standard form has negative units, we handle it in a way similar to that shown above. For example, consider the equation:

$$x^2 + 5x - 6 = 0$$



To solve this equation we factor by adding one positive and one negative x-bar, as shown.



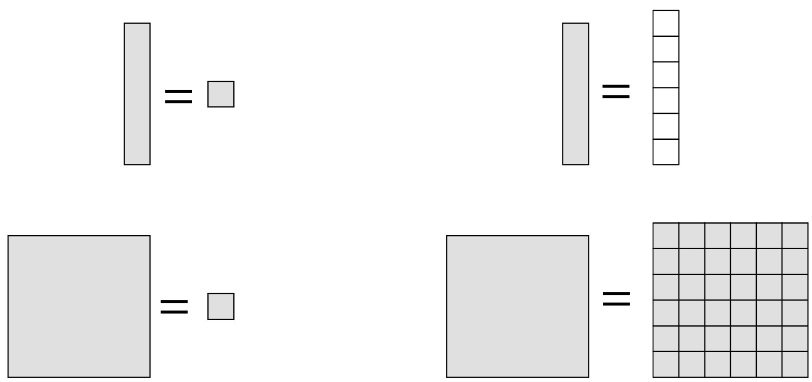


This gives

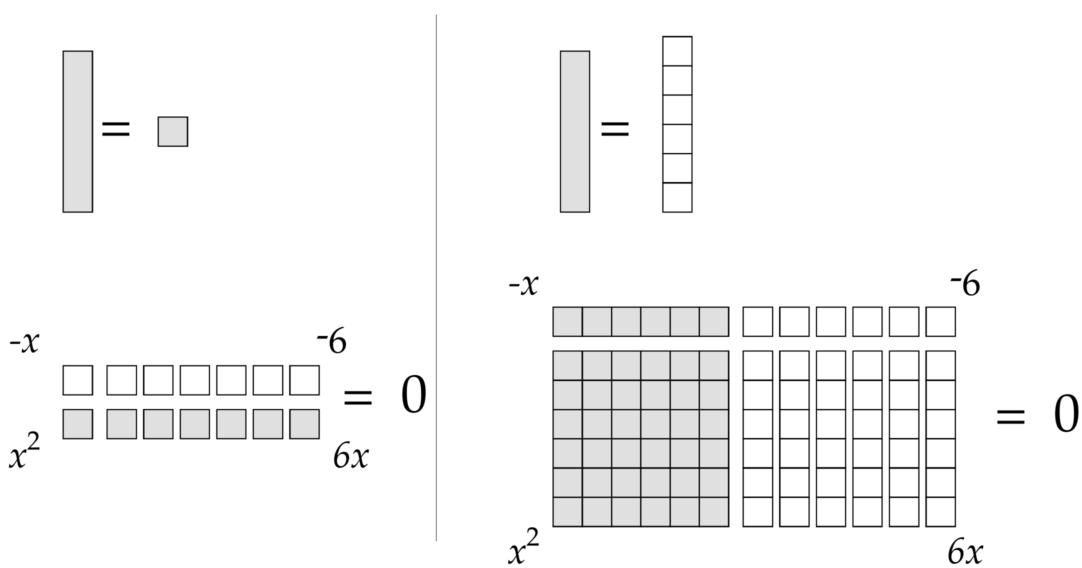
$$(x - 1)(x + 6) = 0.$$

As before, the quadratic expression on the left will equal zero if either the height or the width of the rectangle is zero, which requires that

Either	$x - 1 = 0$	*or*	$x + 6 = 0$
	$+1 \quad +1$		$-6 \quad -6$
	$\mathbf{x = 1}$		$\mathbf{x = -6}$

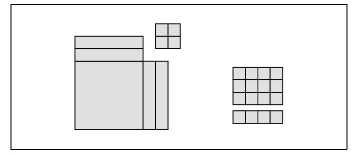


Substituting these values into our diagram we get



If you count carefully, you will see that both of the solutions do give zero for their results.

Numerically we can check our solutions by substituting the values for x into the original equation:



$x^2 + 5x - 6 = 0$	$x^2 + 5x - 6 = 0$
$x = 1$	$x = -6$
$(1)^2 + 5(1) - 6$	$(-6)^2 + 5(-6) - 6$
$1 + 5 - 6$	$36 - 30 - 6$
0	0

Both solutions make the original equation true. (Will any other solutions work? If you think they will work, try checking your suggested solution using chips and using numerical substitution. Work carefully.)

So now you know how to solve any quadratic equation which is factorable when put into standard form. A summary of the steps required is:

- **Put the quadratic equation into standard form:**

  $$Ax^2 + Bx + C = 0$$

- **Factor the quadratic expression on the left side of the equation.**
- **Use the zero product rule to set each of the factors equal to zero, giving two linear equations.**
- **Solve each of these linear equations for the unknown, giving two possible solutions.**
- **Check these two solutions by substituting each value for the unknown in the original equation .**

Exercises

Solve each of these quadratic equations, giving both solutions.

Check your work.

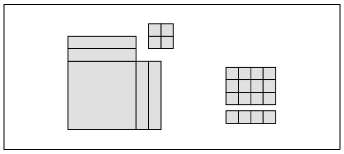
1. $x^2 - 5x + 6 = 0$

2. $2x^2 + 5x - 7 = 0$

3. $x^2 - 6x + 8 = 0$

4. $3x^2 + 8x + 5 = 0$

5. $x^2 = 2x + 8$



6. $2x^2 + 9 = -2x^2 + 12x$

7. $3x^2 + 6x = 2x^2 - 3x - 8$

8. $x^2 - 4 = -x^2 - 7x - 10$

9. $x^2 + 2x - 15 = 0$

10. $2x^2 - 3x - 2 = 0$

11. $x^2 + 3x = 5 - x$

12. $6x(x + 2) = x + 10$

13. $x^2 + 15x = 3(x - 9)$

14. $2x^2 - 6x = 5(x - 3)$

15. $x^2 - 1 = 16(x - 4)$

Section 5
Completing the Square

Quadratic Equations that Won't Factor

Now that you can solve any quadratic equation which can be factored, you might not wish to know that there are more quadratic expressions which *cannot* be factored than there are expressions which *can* be factored.

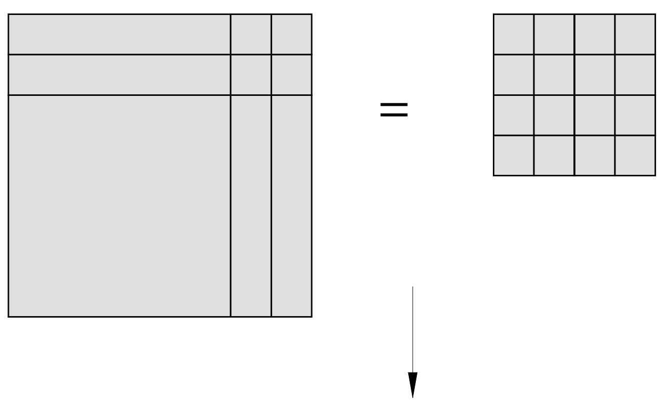
But do not fear; in this section and the next we are going to learn a method which will allow us to solve nearly any quadratic equation, whether we can factor it or not. To do this we will use another obvious and simple idea:

- **If two squares have equal area, then their sides are also equal. (The length of the side is the square root of the area.)**

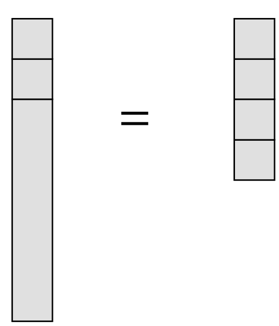
This method is called **completing the square.**

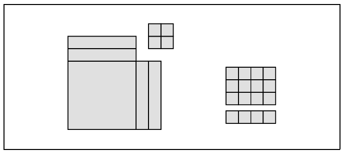
Translated into an example using chips, this means:

If 2 squares are equal in area,



Then their sides are equal in length.





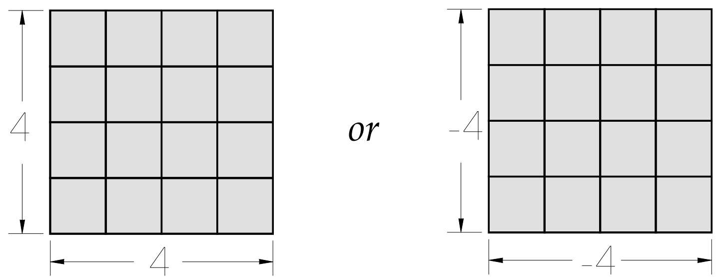
In symbols, if

$$(x+2)^2 = (4)^2$$

then

$$x+2 = 4$$

This idea shouldn't seem too hard. There is one extra twist: if the square on the right has 16 positive units in it, there is the possibility that its sides can be either $^{+}4$ or $^{-}4$, since either possibility will give the same number of positive units in the square.



So we include both possibilities by saying that if

$$(x+2)^2 = (4)^2$$

then

$$x+2 = {}^{+}4 \text{ or } {}^{-}4$$

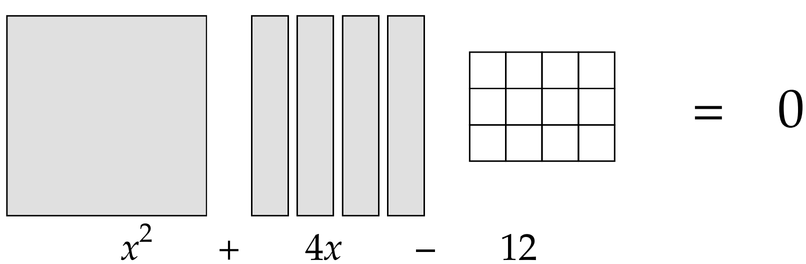
The two options $^{+}4$ or $^{-}4$ are usually written in a shorthand form as $\pm$ **4**:

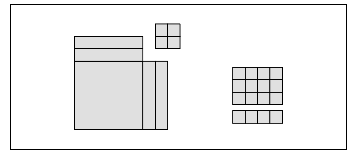
$$x+2 = \pm 4$$

where the symbol $\pm$ is read **plus or minus**; it means that there are two answers, both of which are equally valid.

Now that we see how the situation works with the equal squares, let's use this idea to solve a quadratic equation. We'll begin with this equation:

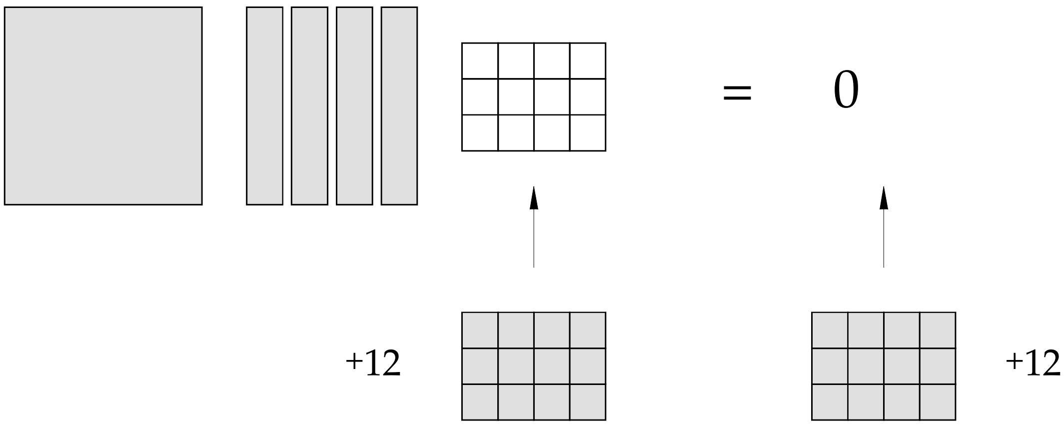
$$x^2 + 4x - 12 = 0$$



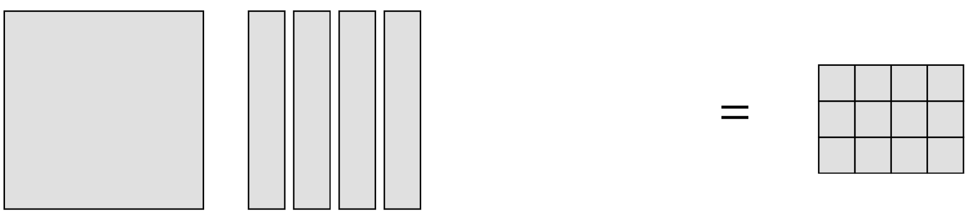


Even though this equation is factorable, let's pretend that we can't see the factors, which means that we can't use any of the methods we have learned so far to solve for the unknown (x). We are going to follow a different approach; we are going to rearrange things instead of factoring.

First we are going to get all of the units away from the x^2 and x terms on the left side of the equation by adding 12 units to each side.

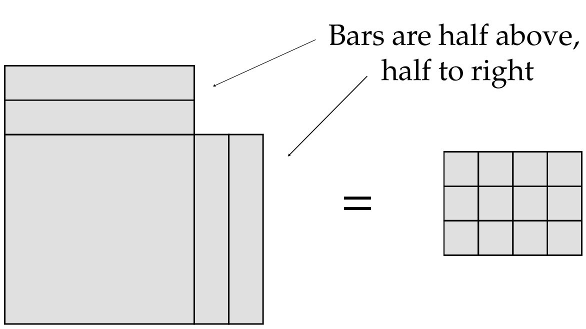


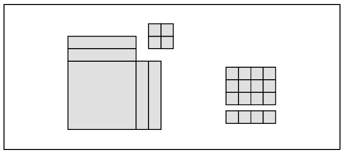
This gives



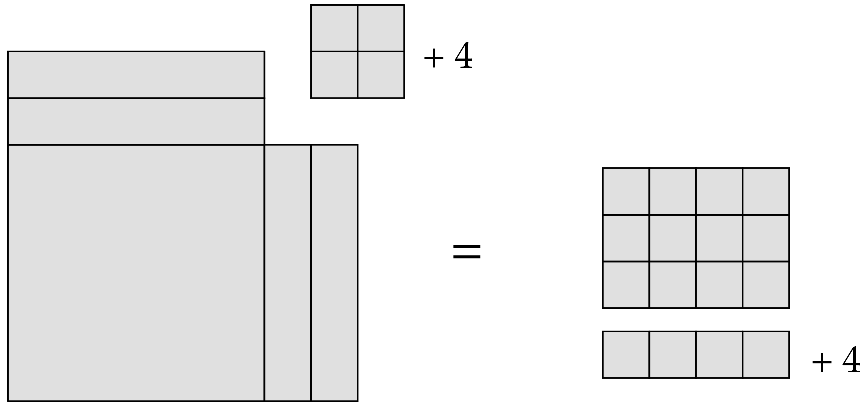
$$x^2 + 4x = 12$$

Next we arrange the x^2 and the x-bars on the left side of the equation into a form which is as close to a square as we can get without having any units. To do this we put half of the bars above the square, and half of the bars beside it on the right:

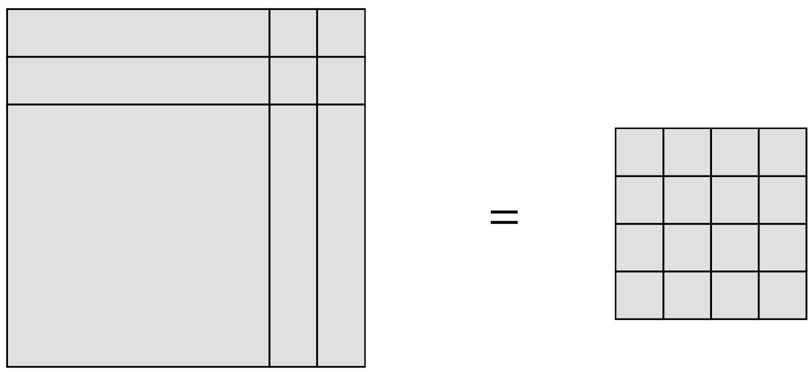




Finally we add enough unit chips to each side to **complete the square** on the left; we can see that four (4) chips will be required.

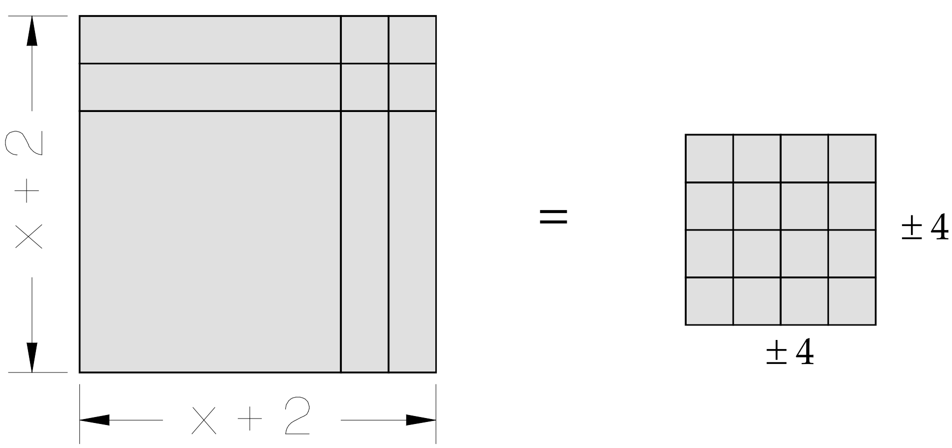


This gives us a picture we have seen before.

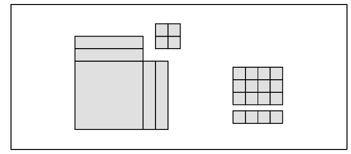


$$x^2 \quad + \quad 4x \quad + 4 \quad = \quad 16$$

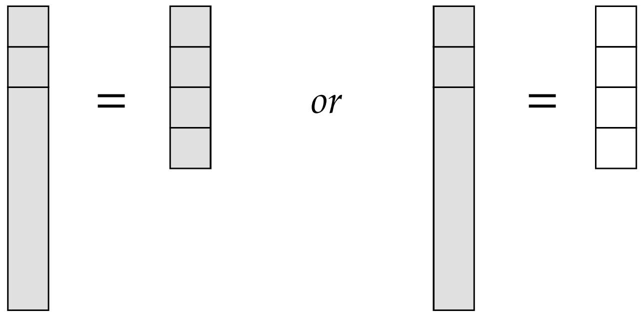
We have two squares that we know are equal, so their sides (square roots) must also be equal.



$$x \quad + \quad 2 \quad = \quad \pm 4$$

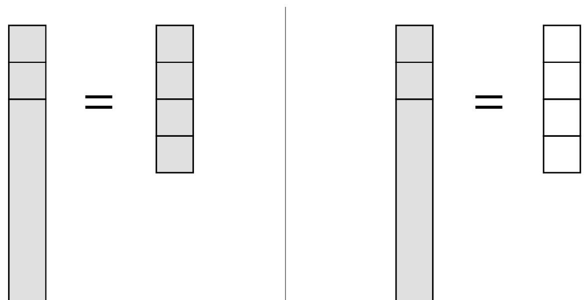


Now we see that if

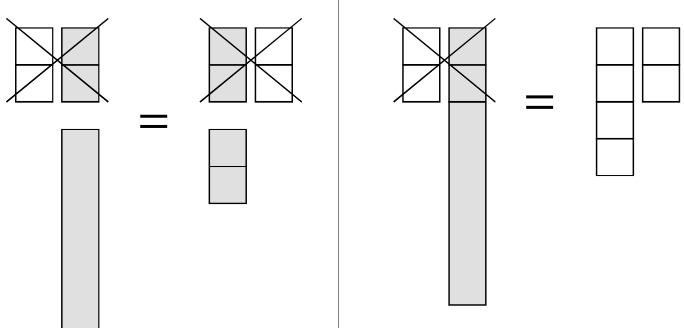


$x + 2 = {}^{+}4$ *or* $x + 2 = {}^{-}4$

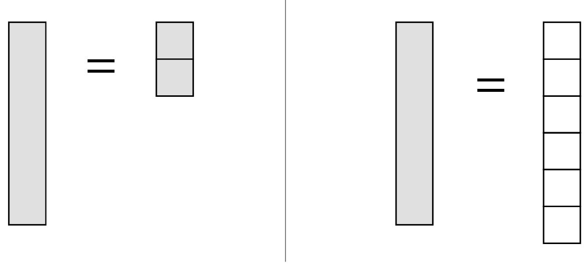
then, adding ${}^{-}2$ to both sides gives us:



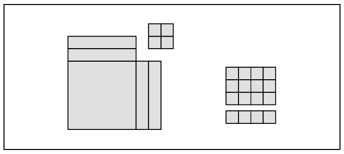
$x + 2 = {}^{+}4$ *or* $x + 2 = {}^{-}4$



$x + 2 - 2 = 4 - 2$ *or* $x + 2 - 2 = {}^{-}4 - 2$

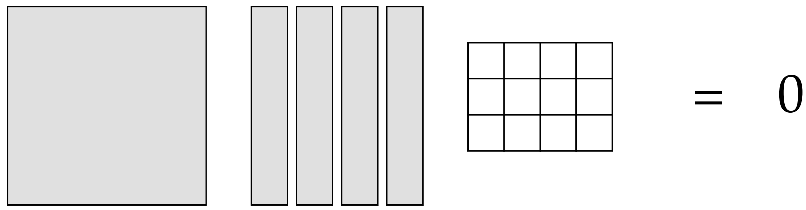


$x = 2$ *or* $x = {}^{-}6$



Here we have the two possible solutions for our original quadratic equation. To check them we can substitute these solutions into the original equation, either using chips or using numbers.

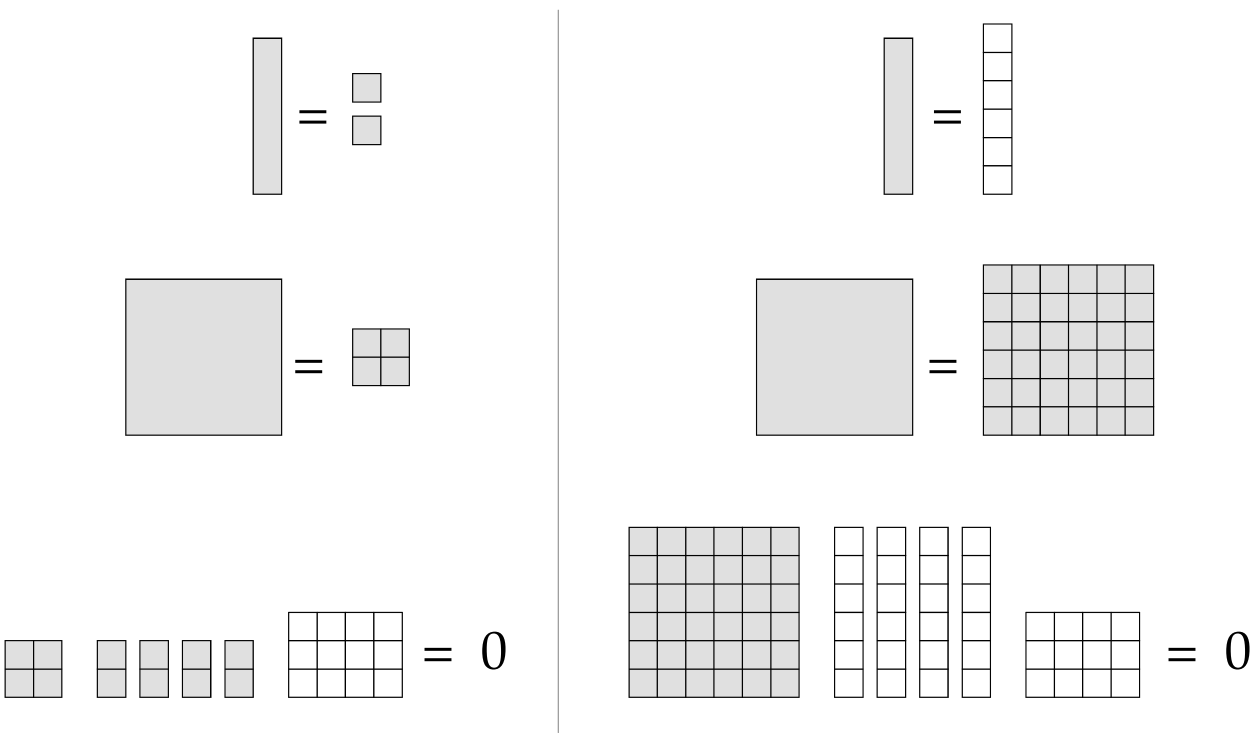
We started with:



$$x^2 \quad + \quad 4x \quad - \quad 12 \quad = \quad 0$$

Now we know that either

$x = 2$ *or* $x = {}^{-}6$

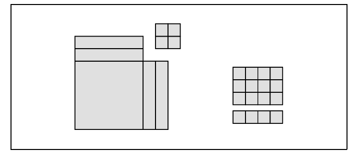


$$(2)^2 + 4(2) - 12 = 0 \qquad\qquad (-6)^2 + 4(-6) - 12 = 0$$

$$4 + 8 - 12 \qquad\qquad 36 - 24 - 12$$

$$0 \qquad\qquad 0$$

Both solutions check. (If we had done this problem by factoring would we have gotten the same result? Try it and see.)



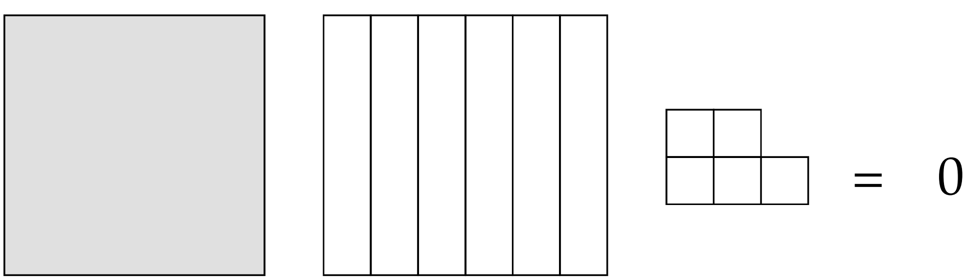
Summary of Steps

The method of *completing the square* which we used to solve this example has several specific steps. So far these steps are:

- **Arrange the quadratic equation in standard form.**
- **Move the units (chips) across the equal sign by adding their opposite to both sides.**
- **Now that the x^2-square and the x-bars are isolated, use them together to make a figure as close to a square as possible. Put half of the bars above the square and half of the bars beside the square. Because you have no units, there will be a square hole in the corner.**
- **Add enough units to both sides of the equation to *complete the square* begun by the x^2 and the x-bars. Across the equal sign, also make the units into a square, or as close to a square as possible.**
- **Take the square root of both squares by noting the length of their sides. Remember that the side length of the units square can be either + or – its square root.**
- **Set the square roots of the expressions equal to each other.**
- **Isolate the unknown (x) on the left side of the equation by adding the necessary positive or negative units to each side of the equation. This gives the two solutions for x.**

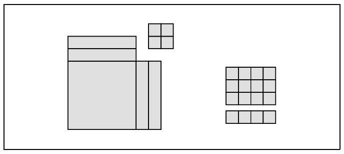
Another Example

Let's work through a second example. (This time we will choose one that



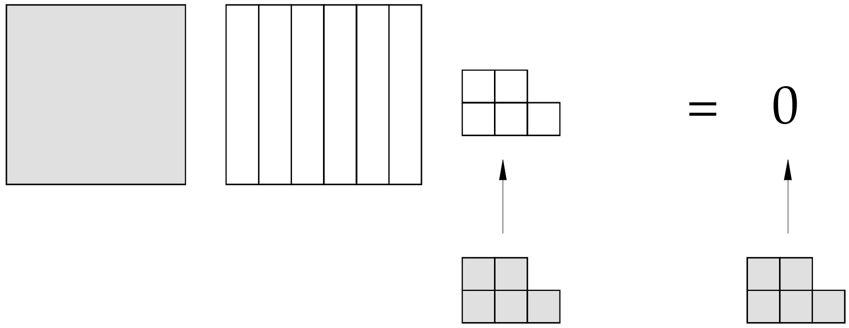
$$x^2 \quad - \quad 6x \quad - \quad 5 \quad = \quad 0$$

really can't be factored.)

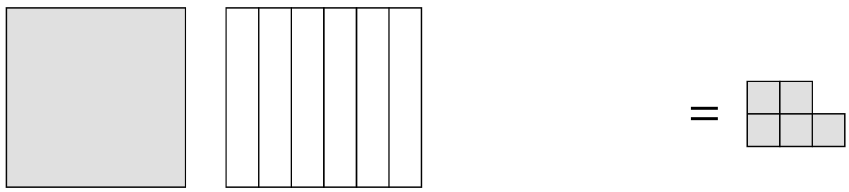


This expression can't be factored, so we work through the steps outlined above:

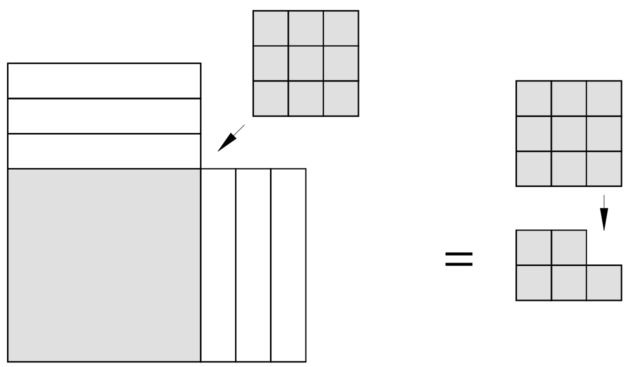
$$\begin{aligned} x^2 - 6x - 5 &= 0 \\ +5 \quad & \;\; +5 \end{aligned}$$



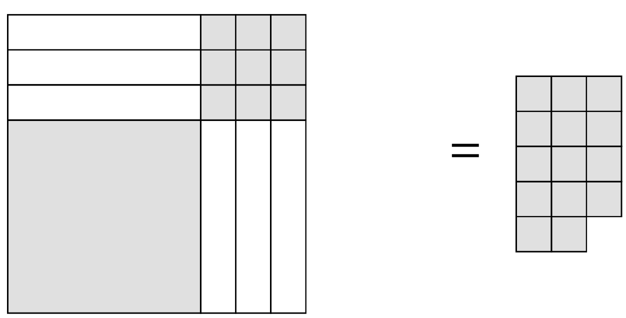
$$x^2 - 6x \quad = 5$$



$$x^2 - 6x + 9 = 5 + 9$$

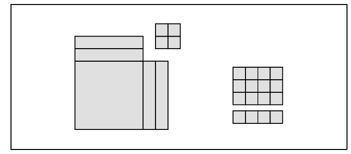


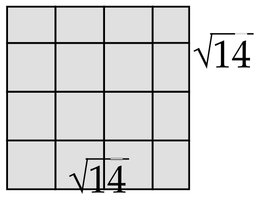
$$(x-3)^2 = 14$$



$(x - 3)$ on each side

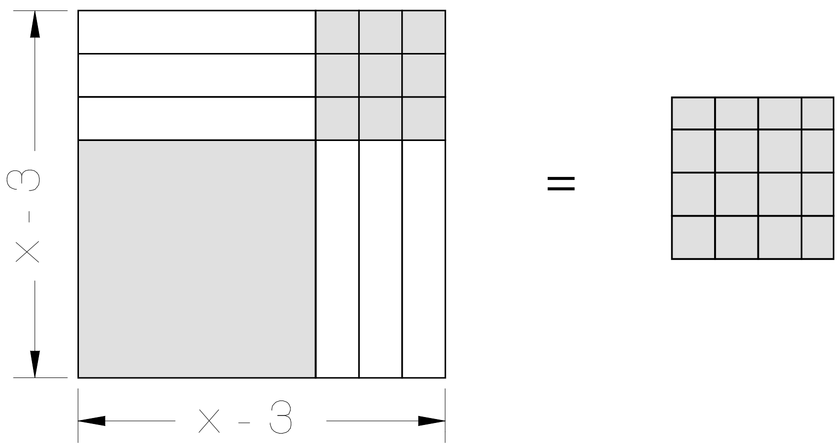
The units on the right cannot be arranged into a perfect square without cutting some of them into smaller pieces. But if we imagine trimming some of the unit squares so that we can make a nice square having exactly 14 units in it, we already know how to express the length of the side of that square—the side will be $\sqrt{14}$.



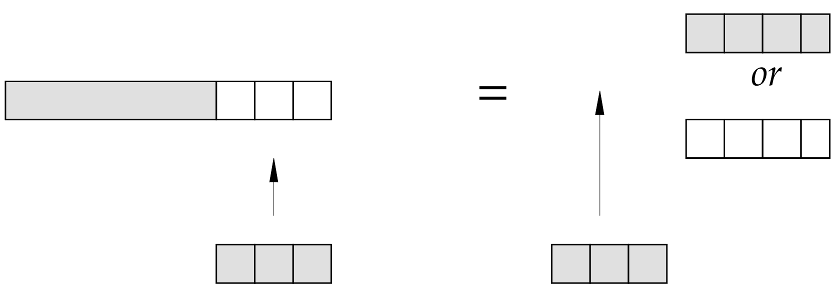


From the picture, we can estimate that the $\sqrt{14}$ is between $3\,\frac{5}{6}$ and $3\,\frac{5}{7}$.

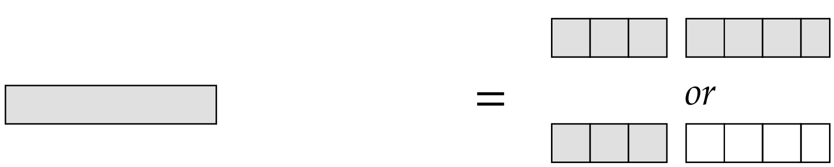
We continue with completing the square:



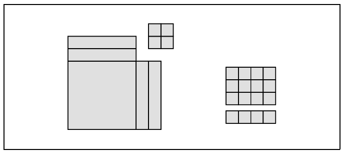
$$(x-3)^2 = 14$$



$$\begin{aligned} x - 3 &= \pm\sqrt{14} \\ +3 \quad & \quad +3 \end{aligned}$$

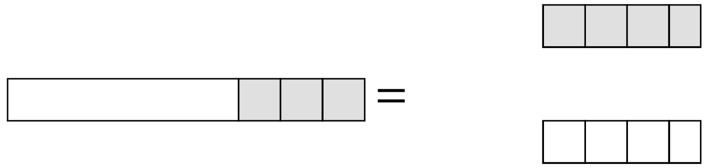


$$x = 3 \pm \sqrt{14}$$



Note that the diagram might be interpreted in a different way:

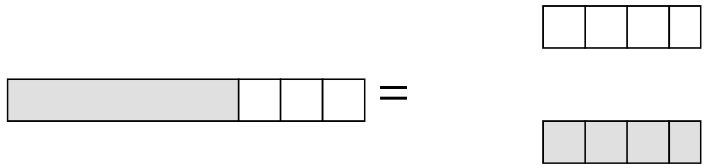
$$-x + 3 = \pm\sqrt{14}$$



Flipping both sides over (multiplying by $^{-}1$) gives:

$$^{-}1(-x + 3) = {^{-}1}(\pm\sqrt{14})$$

$$x - 3 = \pm\sqrt{14}$$



This will obviously have the same result as before.

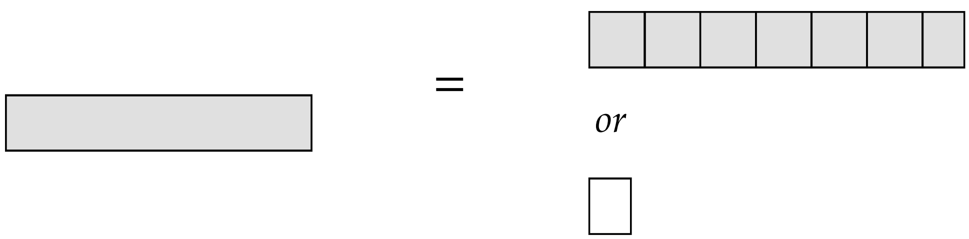
It may not feel totally comfortable to you to have answers like

$$x = 3 \pm \sqrt{14}$$

From the pictures we can see that this means approximately

x is about $6\frac{5}{7}$

or $-\frac{5}{7}$



For more exact values we can use a calculator to get a decimal value for 14:

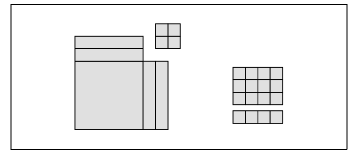
$$\sqrt{14} = 3.7417\ldots$$

This means that

$$x = 3 + \sqrt{14} = 3 + 3.7417 = 6.7417$$

or

$$x = 3 - \sqrt{14} = 3 - 3.7417 = {^{-}0.7417}$$



These answers are not nice round numbers. This shows that the method of completing the square can be used to find solutions for quadratic equations which don't have simple integral factors. To check our answers for this example, it is easiest to use a pencil and paper and a *calculator* to substitute these values into the original equation. (Be sure to write down everything you are doing as you are entering numbers into your calculator so that you don't get lost half-way through the operation.)

$x^2 - 6x - 5 = 0$	
$x = 6.7417$ *or*	$x = {}^-0.7417$
$(6.7417)^2 - 6(6.7417) - 5 = 0$	$({}^-0.7417)^2 - 6({}^-0.7417) - 5 = 0$
0.0003	0.0003

If we use even more accurate decimal values for $3 \pm \sqrt{14}$ we will get results which come out even closer to being exactly zero.

You have learned enough of the process of completing the square to have a good start. Try your hand at using this new method to do the following problems on your own. Use your chips and a scratch pad to do these exercises. Check your results, using a calculator if necessary.

Exercises

1. $x^2 + 6x - 7 = 0$

2. $x^2 - 4x - 5 = 0$

3. $x^2 + 8x = 9$

4. $x^2 = 10x + 16$

5. $x^2 + 15x = 3x - 20$

6. $x^2 + 10x + 18 = 0$

7. $x^2 + 6x - 9 = 0$

8. $x^2 + 14x = 2x - 11$

9. $x^2 - 5x = 3x + 3$

10. $x^2 - 10x = {}^-17$

11. $x^2 - 3x - 5 = 0$

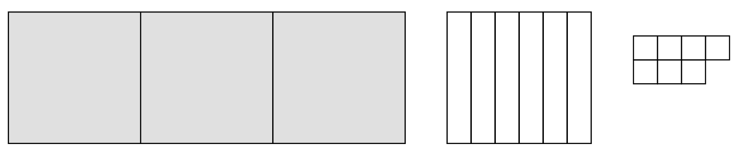
12. $x^2 - 5x + 2 = 0$

Section 6
Equations with More than One x^2

Beginning With More Than One x^2

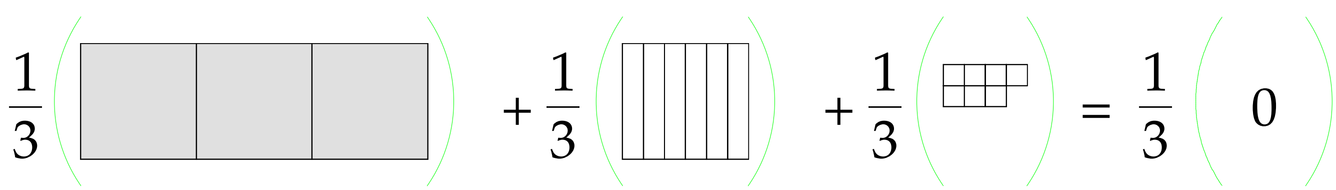
When we begin with a quadratic equation in standard form which has more than one x^2 we must add one additional step to our methods for completing the square. Let's begin with

$$3x^2 - 6x - 7 = 0$$

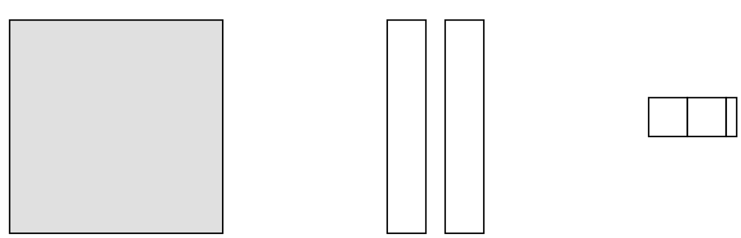


$$3x^2 \quad -6x \quad -7 \quad = 0$$

It will not be possible to make a perfect square using the x^2's and the x-bars, since there is no way to arrange three x^2's into a square. To deal with this we take ⅓rd of all the terms on both sides of the equation, so that we are left with only one x^2. (One way to describe this is to say that we *divide by the coefficient of* x^2.)

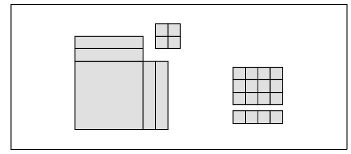


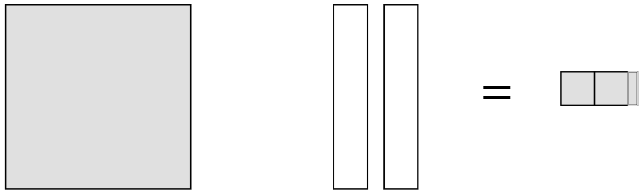
This gives:



$$x^2 \quad -2x \quad -2\frac{1}{3} \quad = \quad 0$$

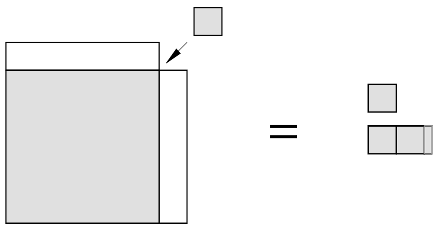
Don't let the fraction scare you; from here we proceed just as we did in the last example. Since we can't factor, we move the unit chips to the other side of the equation (remembering to change their sign when they cross the equal sign).

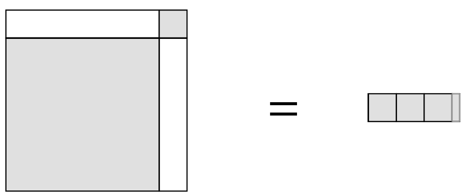




$$x^2 - 2x = 2\frac{1}{3}$$

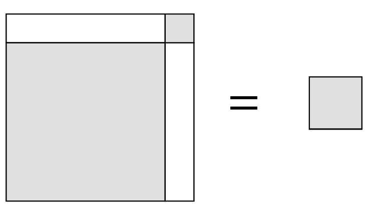
Now make the square





$$(x-1)^2 = 3\frac{1}{3}$$

Taking the square root of both sides we get:



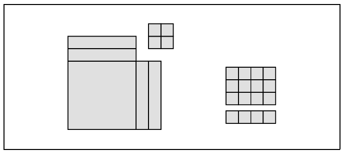
$$x - 1 = \pm\sqrt{3\frac{1}{3}}$$

This is more commonly written as

$$x - 1 = \pm\sqrt{\frac{10}{3}}$$

(It is hard to draw a picture of the square root of $^{10}/_{3}$, but it is just the side of a square having $3^1/_3$ units of area inside it.) Using a calculator we find that

$$\sqrt{^{10}/_{3}} = 1.8257$$



so

$$x - 1 = \pm 1.8257$$

Adding one to both sides gives

$$x = 1 + 1.8257$$
or
$$x = 1 - 1.8257$$

$$x = 2.8257$$
or
$$-0.8257$$

Use your calculator and check these results in the original equation.

Summary of Steps

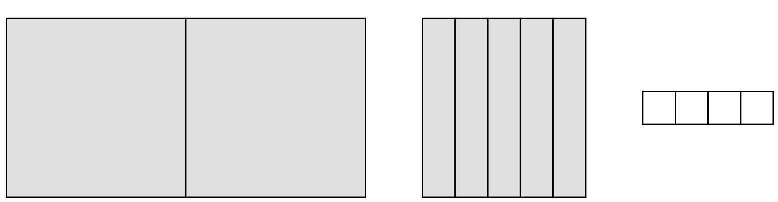
To accommodate equations which begin with more than one x^2 when in standard form, we must add one more step to our procedure for completing the square. Now the procedure will read:

- **Arrange the quadratic equation in standard form.**
- **If the first term has more than one x^2, divide all terms on both sides of the equation by the coefficient of x^2, leaving only one x^2.**
- **Move the units (chips) across the equal sign by adding their opposite to both sides.**
- **Proceed as before.**

A Final Example

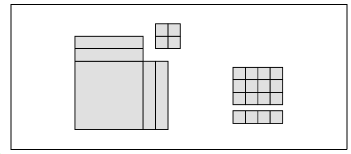
Here is one final example:

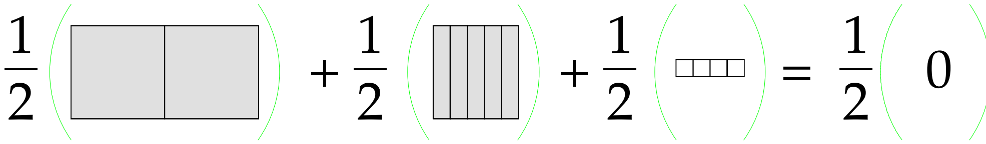
$$2x^2 + 5x - 4 = 0$$



= 0

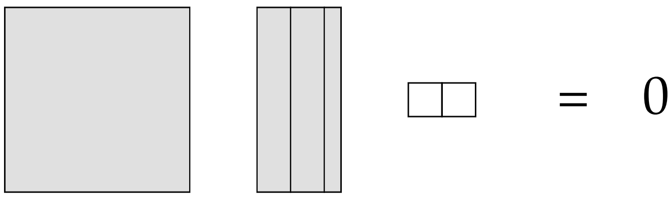
First divide each term by 2 (multiply by ½), to leave just a single x^2.





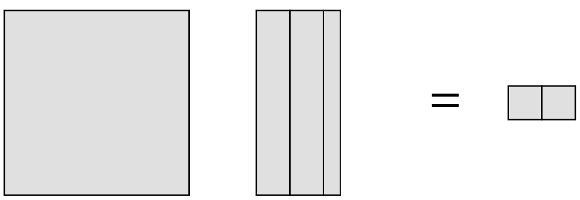
$$\frac{1}{2}(2x^2) + \frac{1}{2}(5x) + \frac{1}{2}(-4) = \frac{1}{2}(0)$$

This leaves:



$$x^2 + \frac{5}{2}x - 2 = 0$$

Moving the units across the equal sign we have:

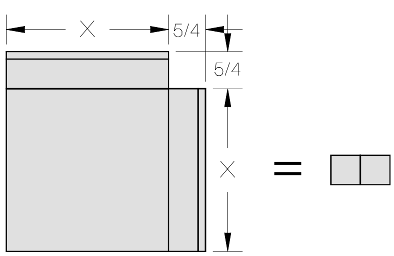


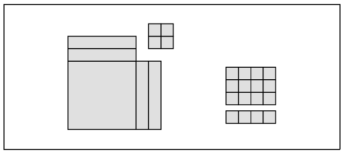
$$x^2 + \frac{5}{2}x = 2$$

To make a square from the left side of the equation, we have to cut the (5⁄2)x into two equal pieces, so that one can go above the large square and the other can go beside the large square. Again, don't let the fractions scare you. Half of (5⁄2)x is easy to figure out:

$$\frac{1}{2}\left(\frac{5}{2}x\right) = \frac{5}{4}x$$

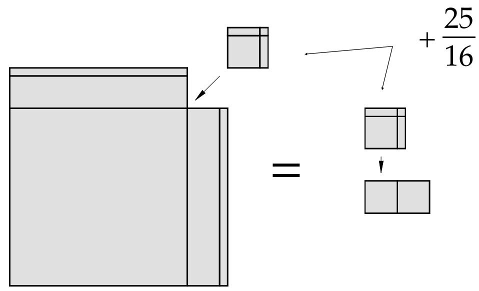
The figure which results looks like this:





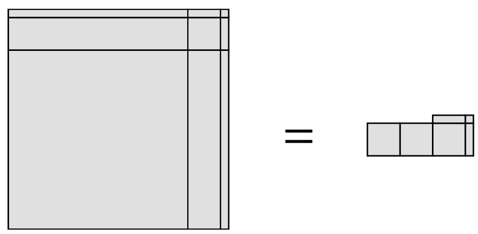
This time the number of unit chips required to fill in the corner of the square will be a fraction. The square corner to be filled in is ($\frac{5}{4} \cdot \frac{5}{4}$), so the required amount will be

$$\frac{5}{4} \cdot \frac{5}{4} = \frac{25}{16}$$



This gives

$$\left(x + \frac{5}{4}\right)^2 = 3\,\frac{9}{16}$$



Although our results will be fractions, the process of taking the square root is no different than before.

$$\left(x + \frac{5}{4}\right)^2 = 3\,\frac{9}{16} = \frac{57}{16}$$

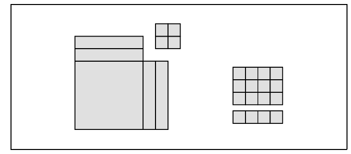
$$x + \frac{5}{4} = \pm\sqrt{\frac{57}{16}} = \pm\frac{\sqrt{57}}{4}$$

Adding -$\frac{5}{4}$ to both sides gives

$$x = -\frac{5}{4} \pm \frac{\sqrt{57}}{4}$$

which can also be written (since the two fractions have the same denominator) as:

$$x = \frac{-5 \pm \sqrt{57}}{4}$$



Using a calculator this comes out to be

$$x = \frac{-5 \pm 7.5498}{4}$$

$$x = +0.6375$$

or

$$-3.1375$$

Use your calculator to check these results as shown below, and then congratulate yourself for working through such a challenging problem.

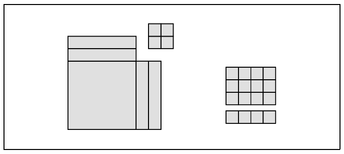
Check:

$2x^2 + 5x - 4 = 0$	
$x = 0.6375$ *or*	$x = -3.1375$
$2(0.6375)^2 + 5(0.6375) - 4 = 0$	$2(-3.1375)^2 + 5(-3.1375) - 4 = 0$
0.0003	0.0003

Does it check out?

Now we will review the steps in this process, then you can try some more problems on your own. Have chips, a scratch pad and your calculator close at hand and work carefully.

- **Arrange the quadratic equation in standard form.**
- **If the first term has more than one x^2, divide all terms on both sides of the equation by the coefficient of x^2 which will leave only one x^2.**
- **Move the units (chips) across the equal sign by adding their opposite to both sides.**
- **Now that the x^2 piece and the x-bars are isolated, use them together to make a figure as close to a square as possible while having no units, by putting half of the bars above the square and half of the bars beside the square.**



- Add enough units to both sides of the equation to *complete the square* begun by the x^2 and the x-bars. Across the equal sign also make the units into a square, or as close to a square as possible.

- **Take the square root of both squares by noting the length of their sides. Remember that the side length of the units square can be either + or – its square root.**

- **Set the square roots of the expressions equal to each other.**

- **Isolate the unknown (x) on the left side of the equation by adding the necessary positive or negative units to each side of the equation. This gives the two solutions for x.**

Exercises

Solve for x by completing the square:

1. $x^2 + 10x - 24 = 0$

2. $x^2 + 6x - 4 = 0$

3. $2x^2 - 20 = x^2 - 8x$

4. $2x^2 - 6x = 2x - 11$

5. $2x^2 - 8x - 6 = 0$

6. $3x^2 - 12x + 9 = 0$

7. $3x^2 - 11x = x - 5$

8. $4x^2 - 9x = x^2 + 6x - 6$

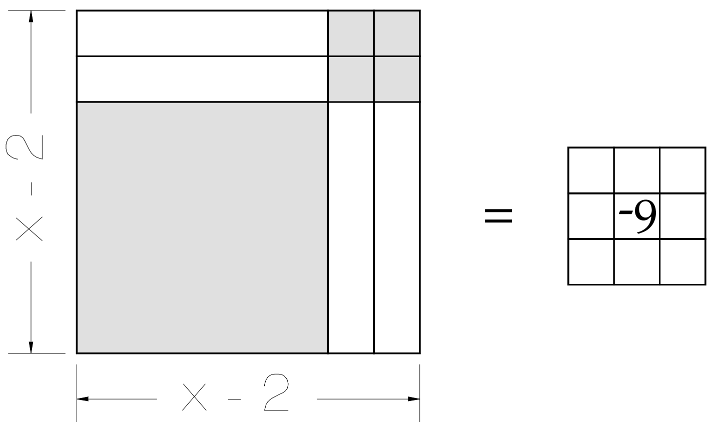
9. $2x^2 + 7x - 8 = 0$

10. $x^2 - 8x - 10 = x - x^2$

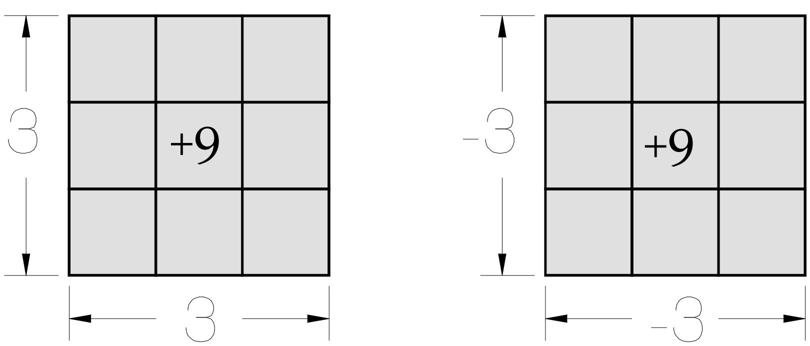
Section 7
Imaginary Solutions

The Square Root of a Negative Number

There are still some quadratic equations which we cannot solve, even when using our new methods. These equations involve situations where we have a perfect square equaling a negative number. For example:



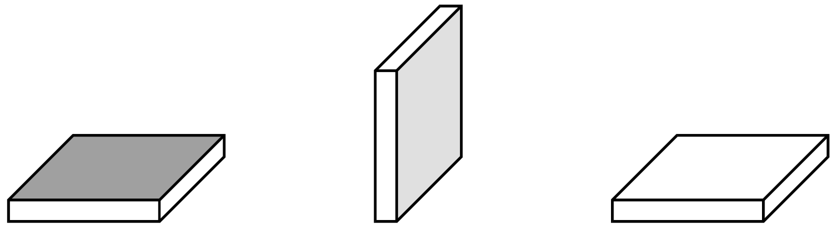
The reason we cannot solve equations like these is that we have not defined any square roots of negative numbers. Square roots (the sides of squares) must be equal, by definition, and there are no two equal numbers that multiply together to give a negative area.



$$(+3) \cdot (+3) = +9$$
$$(-3) \cdot (-3) = +9$$

Because of this problem, we are going to invent a new type of number which we will only use for the very special purpose of solving these equations. (Later math courses will have more uses for these special numbers.) We will call these new numbers **imaginary**, because they won't be positive and they won't be negative and they won't be zero; in fact they may not seem to really exist at all except within our imaginations, hence the name.

Imaginary numbers will be like chips flipped only half of the way over, or standing on edge; they aren't plus and they aren't minus.

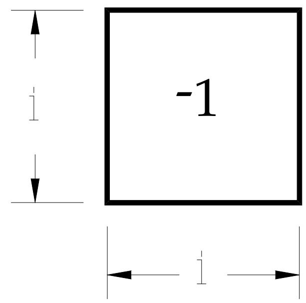


We call the unit imaginary number ***i***, and we define ***i*** by the equation

$$i \cdot i = -1$$

or

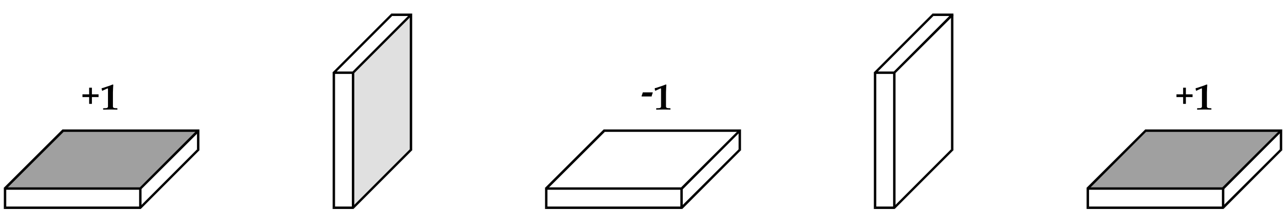
$$i^2 = -1$$



Another way of saying this is

$$\boldsymbol{i = \sqrt{-1}}$$

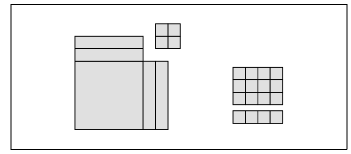
Two imaginaries multiplied together will give two half flips, or one whole flip, which is the same as a minus (-) sign.



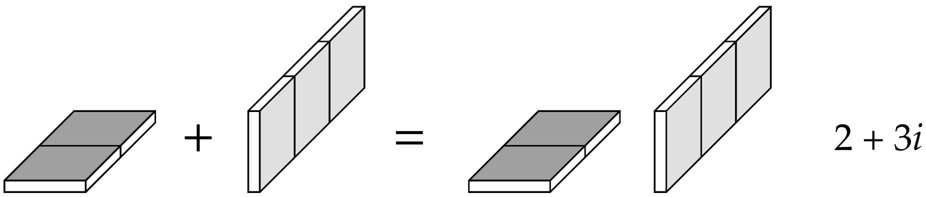
+1 (no flip)	+1i (half	$(1i)(1i) = {}^-1$ (whole flip)	$({}^-1)i = -i$ (1 ½ flips)	$(-i)(i) = {}^-({}^-1) = +1$ (two flips)

Compared to imaginaries, the other regular numbers (which we have been working with up until now) are called **real numbers**. Real numbers and imaginary numbers can only interact in certain ways.

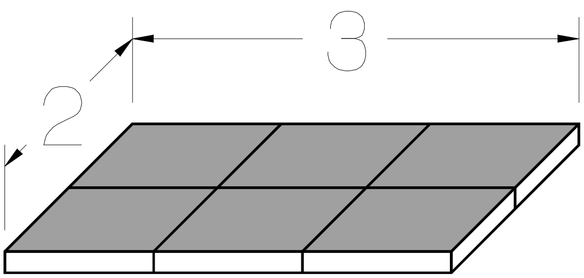
Adding and Subtracting with Imaginaries



We cannot combine real and imaginary numbers because they are different kinds of chips (in geometry, we would say that they are on different planes). *A real number and an imaginary number cannot cancel or add together. They must stay as separate terms.*

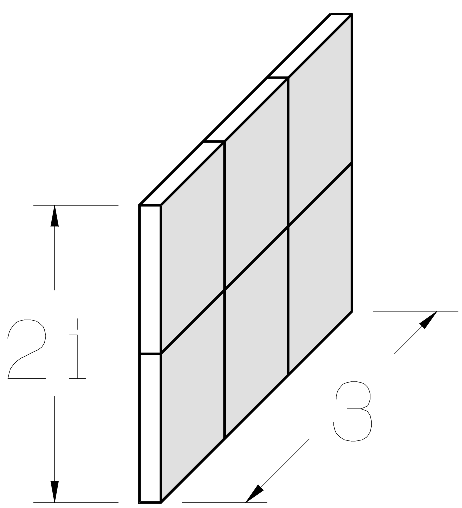


If we multiply real and imaginary numbers, the unit imaginary (i) acts very much like a sign (plus or minus) rather than like a number. The numbers multiply together, making a rectangle as usual; and the imaginaries (i) and negative signs (–) tell how many half flips or whole flips the rectangle goes through. *Each **i** makes one half flip; each negative sign makes one whole flip.*



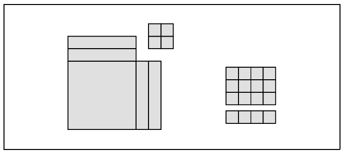
$(2)(3) = 6$

No flips



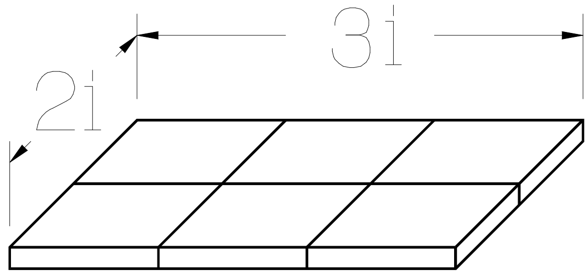
$(2i)(3) = 6i$

(half flip) (no flip) → (half flip)



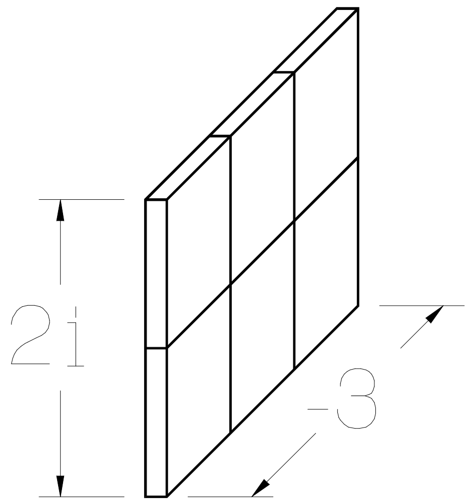
$$(2i)(3i) = -6$$

$$\begin{pmatrix}\text{half}\\ \text{flip}\end{pmatrix}\begin{pmatrix}\text{half}\\ \text{flip}\end{pmatrix} \rightarrow \begin{pmatrix}\text{whole}\\ \text{flip}\end{pmatrix}$$



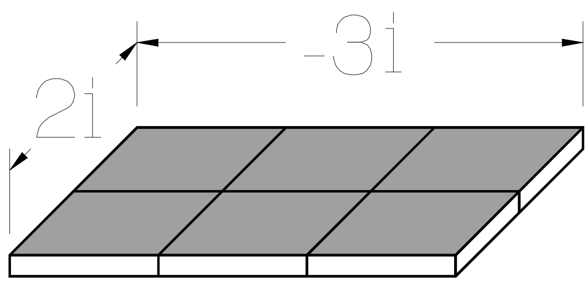
$$(2i)(-3) = -6i$$

$$\begin{pmatrix}\text{half}\\ \text{flip}\end{pmatrix}\begin{pmatrix}\text{whole}\\ \text{flip}\end{pmatrix} \rightarrow \begin{pmatrix}1\ \tfrac{1}{2}\\ \text{flips}\end{pmatrix}$$



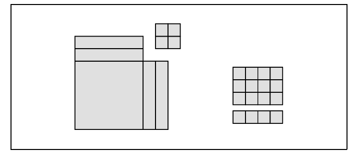
$$(2i)(-3i) = -(-6) = +6$$

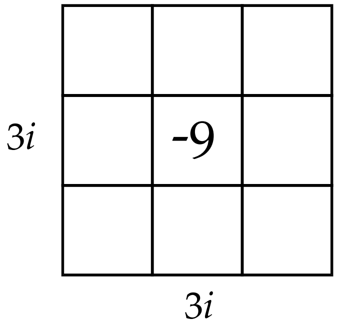
$$\begin{pmatrix}\text{1 half}\\ \text{flip}\end{pmatrix}\begin{pmatrix}\text{1 whole}\\ +\text{ 1 half flip}\end{pmatrix} \rightarrow \begin{pmatrix}\text{2 whole}\\ \text{flips}\end{pmatrix}$$



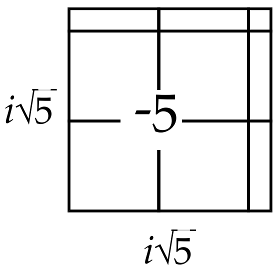
At this stage we will just introduce the idea of imaginary numbers, but you will learn more about them and work with them more in Intermediate Algebra.

For now we will use imaginary numbers only to describe the square root of negative numbers. Such results will be written as shown below:





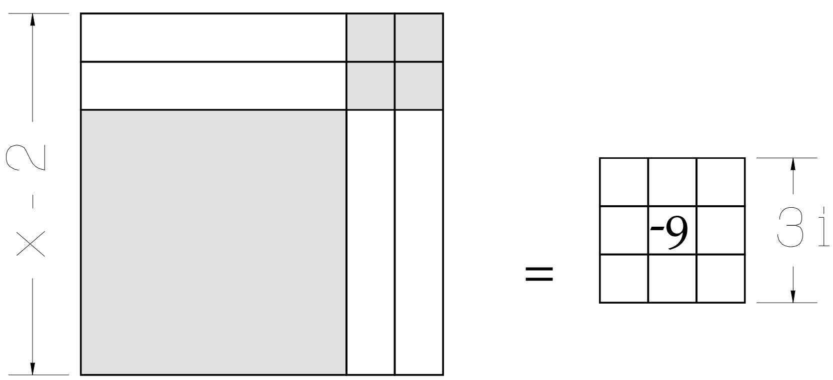
$$\sqrt{-9} = 3i$$



$$\sqrt{-5} = i\sqrt{5}$$

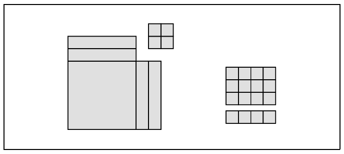
Notice that we write the symbol *i* after a number but before a root.

This notation will allow us to give solutions in situations such as the example shown earlier:



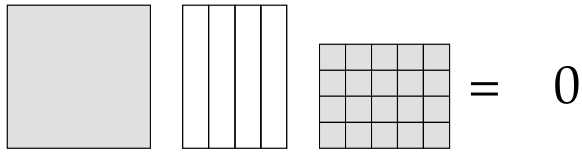
The solution will look like this:

$$(x+2)^2 = -9$$
$$x-2 = \pm\sqrt{-9}$$
$$x-2 = \pm 3i$$
$$x = 2 \pm 3i$$



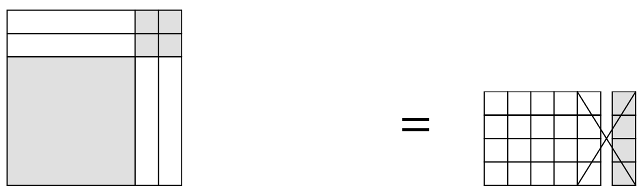
Worked out from the beginning, a quadratic equation of this type will look something like this:

$$x^2 - 4x + 20 = 0$$

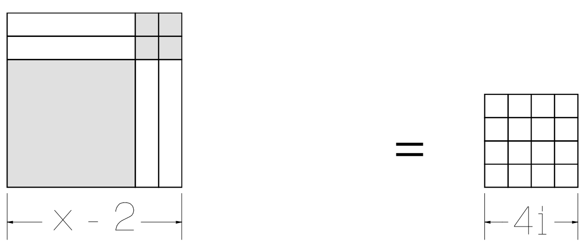


$$x^2 - 4x = -20$$

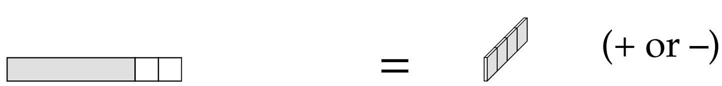
$$x^2 - 4x + 4 = -20 + 4$$



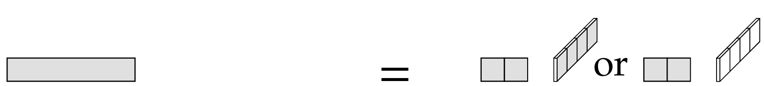
$$(x - 2)^2 = -16$$



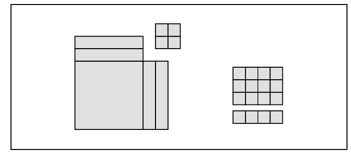
$$x - 2 = \pm 4i$$



$$\boldsymbol{x = 2 \pm 4i}$$



It is often more of a challenge to check a solution having real and imaginary terms than it is to find the solution. One method of checking your results is to work carefully backwards through the problem. To check by substitution, the procedure is the same as we have outlined before; just substitute the solution values one at a time into the original equation and simplify. Here we will show the check for one solution of this example. You may wish to try checking the other solution.



$$x^2 - 4x + 20 = 0 \qquad x = 2 - 4i$$

$$(2 - 4i)^2 - 4(2 - 4i) + 20 =$$

$$(2 - 4i)(2 - 4i) + (-4)(2 - 4i) + 20 =$$

$$4 - 8i - 8i + (-4i)(-4i) - 8 + (-4)(-4i) + 20 =$$

$$4 - 16i - 16 - 8 + 16i + 20 =$$

$$4 - 16 - 8 + 20 - 16i + 16i =$$

$$-20 + 20 + 0i = 0$$

We will avoid more difficult examples using imaginary numbers in this text. If you continue with further studies of mathematics, you will work with imaginaries again and this introduction will help guide you then.

Exercises

Solve for x:

1. $x^2 - 6x + 25 = 0$

2. $x^2 - 10x + 34 = 0$

3. $x^2 + 4x + 5 = 0$

4. $x^2 + 8x + 20 = 0$

5. $x^2 + 6x + 15 = 0$

6. $x^2 - 2x + 12 = 0$

7. $2x^2 + 6x = x^2 - 6x - 37$

8. $23 - x^2 = 8x - 2x^2$

9. $2x^2 = x^2 + 4x - 11$

10. $x^2 + 6x + 12 = 0$

11. $x^2 - 5x = 5x - 33$

12. $x^2 + 3x + 1 = {}^{-}5(x + 4)$

Section 8
The Quadratic Formula

Introduction

We have learned to use the method of **completing the square** to solve *any* quadratic equation. Once the equations are written in standard form we simply follow the list of specified steps until we reach the solution. Since these steps are always the same, we can use them to write a *formula* for the solution to any quadratic equation written in standard form. We can use this formula, called **the quadratic formula**, to solve any quadratic equation without going through all the steps of completing the square every time.

Deriving the Formula

To derive the **quadratic formula** we will begin with the standard form for all quadratic equations:

$$Ax^2 + Bx + C = 0$$

A, B, and C represent the number or coefficients in the equation; we will perform the steps of completing the square with these variables instead of with the numbers. Reviewing the steps for completing the square, the first thing we do with an equation in standard form is divide through all terms by the coefficient of x^2; so in our formula we first divide each term by A.

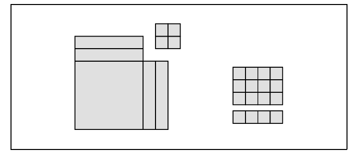
$$x^2 + \left(\frac{B}{A}\right)x + \left(\frac{C}{A}\right) = 0$$

We will call this form of the quadratic equation the **simplified standard form**. Equations which begin with only a single x^2 start out in this form, and all other quadratic equations can be put into this form as soon as we begin working toward the solution. To make the simplified standard form easier to read, we will substitute new letters for the fractions:

$$\frac{B}{A} = D$$

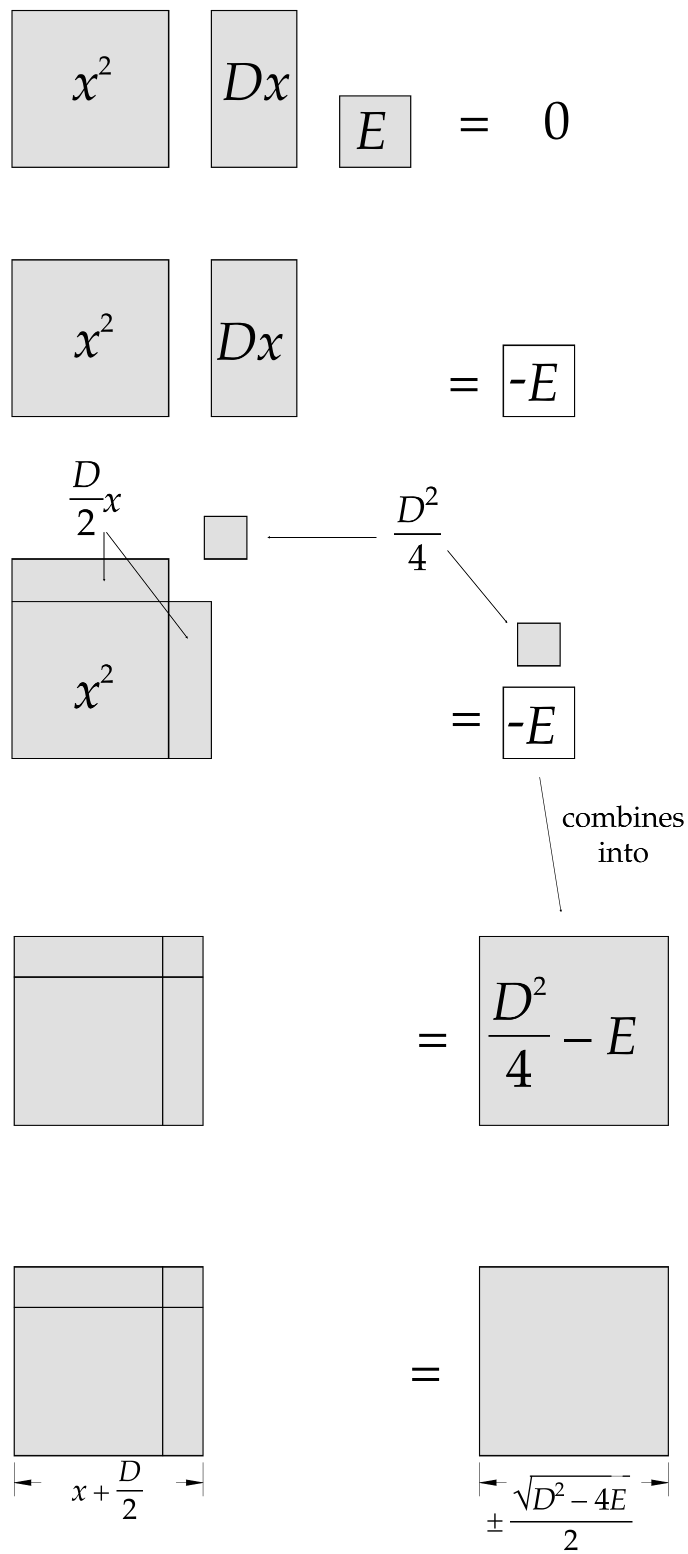
$$\frac{C}{A} = E$$

Now the simplified standard form of the equation reads:



$$x^2 + Dx + E = 0$$

Now we can easily use both pictures and symbols to derive the *quadratic formula*.



$$x^2 + Dx + E = 0$$

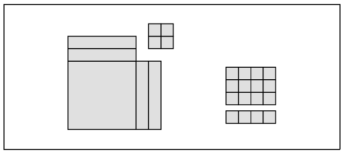
$$x^2 + Dx = -E$$

$$x^2 + Dx + \frac{D^2}{4} = \frac{D^2}{4} - E$$

$$\left(x + \frac{D}{2}\right)^2 = \frac{D^2}{4} - E$$

$$= \frac{D^2 - 4E}{4}$$

$$x + \frac{D}{2} = \pm\frac{\sqrt{D^2 - 4E}}{2}$$



Isolating x, we get our solution in its general form.

$$x = \frac{-D}{2} \pm \frac{\sqrt{D^2 - 4E}}{2}$$

Since both fractions have the same denominator, we can combine the fractions into one, as follows:

$$x = \frac{-D \pm \sqrt{D^2 - 4E}}{2}$$

$$\text{where } D = \frac{B}{A}, \text{ and } E = \frac{C}{A}$$

This formula will give the solution to **any** quadratic equation if the equation is written in simplified standard form or standard form, and then the values of D and E (or A, B and C) are substituted into the solution shown here.

Let's use the formula to do a few examples. Solve the quadratic equation

$$x^2 + 6x + 5 = 0$$

In this equation $D = 6$ and $E = 5$. Substituting these values into the quadratic formula, we find

$$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(5)}}{2}$$

$$x = \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$x = \frac{-6 \pm \sqrt{16}}{2}$$

$$x = \frac{-6 \pm \sqrt{16}}{2}$$

$$x = \frac{-6 \pm 4}{2}$$

$$x = \frac{-6 + 4}{2} \quad or \quad x = \frac{-6 - 4}{2}$$

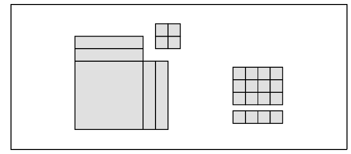
$$x = \frac{-2}{2} \quad or \quad x = \frac{-10}{2}$$

$$\mathbf{x = -1 \quad or \quad x = -5}$$

Check these solutions for yourself to verify that they are accurate.

For quadratic equations where the solutions are integers, the quadratic formula may not save much time in getting to the answer, but for more difficult solutions the formula can save a lot of time.

Solve for the unknown in this equation. Use your calculator to reduce your answers to decimal numerical form:



$$x^2 - 5x - 3 = 0$$

This time $D = {}^{-}5$ and $E = {}^{-}3$.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(-3)}}{2}$$

$$x = \frac{5 \pm \sqrt{25 + 12}}{2}$$

$$\mathbf{x = \frac{5 \pm \sqrt{37}}{2}}$$

$$\mathbf{x = 5.54138 \quad \textit{or} \quad x = -0.54138}$$

Here is one final example:

$$2x^2 + 3x + 8 = 0$$

This time $D = \frac{3}{2} = 1.5$, and $E = \frac{8}{2} = 4$.

$$x = \frac{-(1.5) \pm \sqrt{(1.5)^2 - 4(4)}}{2}$$

$$x = \frac{-1.5 \pm \sqrt{2.25 - 16}}{2}$$

$$x = \frac{-1.5 \pm \sqrt{-13.75}}{2}$$

$$\mathbf{x = \frac{-1.5 \pm i\sqrt{13.75}}{2}}$$

Exercises

Solve the following equations using the quadratic formula. Do some of the problems by factoring or completing the square and compare your results.

1. $x^2 - 5x - 3 = 0$

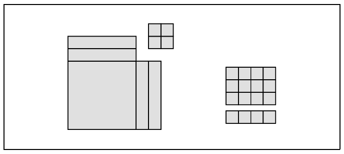
2. $2x^2 - 7x = x^2 + 6$

3. $2x^2 + 6x + 3 = 0$

4. $x^2 - 5x - 2 = -x^2 + 3x - 1$

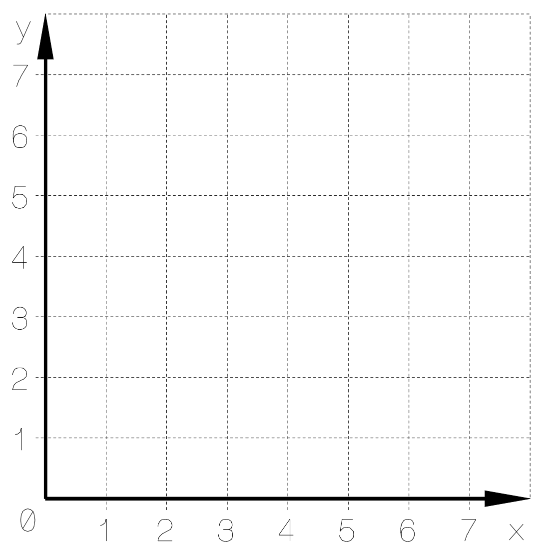
5. $x^2 - 3x + 5 = 0$

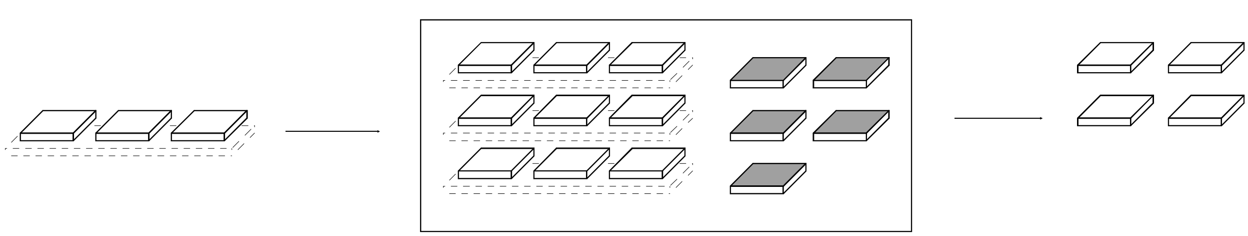
6. $x^2 - x + 6 = 0$



7. $3x^2 - 5x - 2 = 0$
8. $5x^2 - 6x - 4 = 0$
9. $3x^2 - 8x + 5 = 0$
10. $2x^2 - 7x + 6 = 0$
11. $(x + 3)^2 = 12$
12. $x^2 + 7x + 3 = x(2 - x)$
13. $x^2 - 4x = {}^-8$
14. $3x^2 + 2x - 1 = 0$
15. $x^2 - 5x = x - 13$
16. $4x(x + 2) = {}^-1$
17. $2x(x - 1) = 5$
18. $2x^2 + x = 7$
19. $x^2 - 8x = {}^-25$
20. $x^2 - 4x + 5 = 0$

Chapter 12
Rules and Graphs

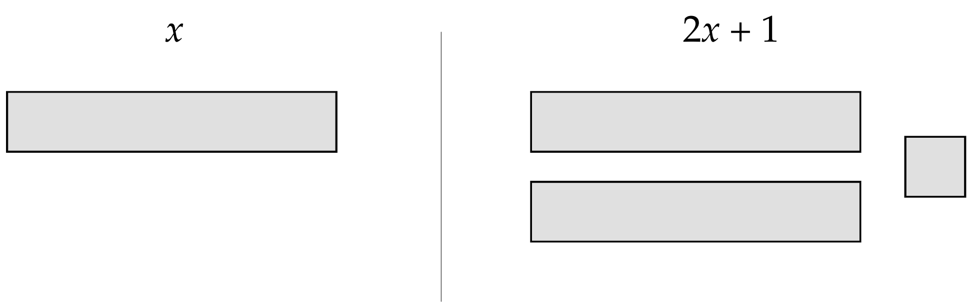




Section 1
Related Numbers

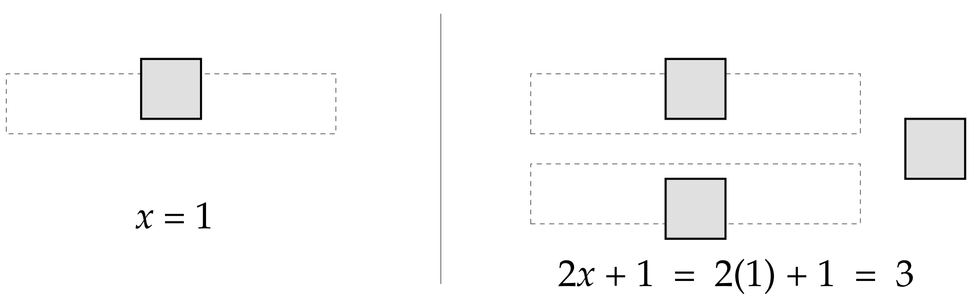
Related Pairs

In this chapter and the next we are going to discuss pairs of variables which are related to each other. For example, consider two variables where the first variable is x and the second variable is $2x + 1$.

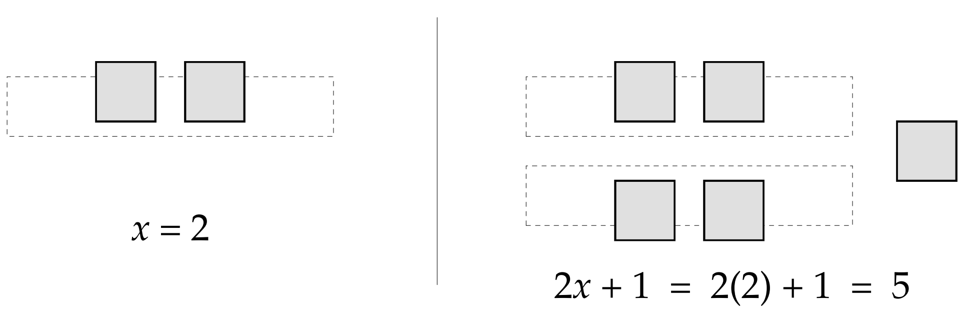


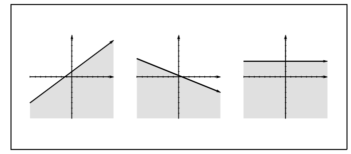
If we agree that the x-bar from the variable on the left has the same value as the x-bar found in the variable on the right, we can see that the values of these two variables will be different, but related to each other.

If the x has a value of 1, then $2x + 1$ will have a value of 3:

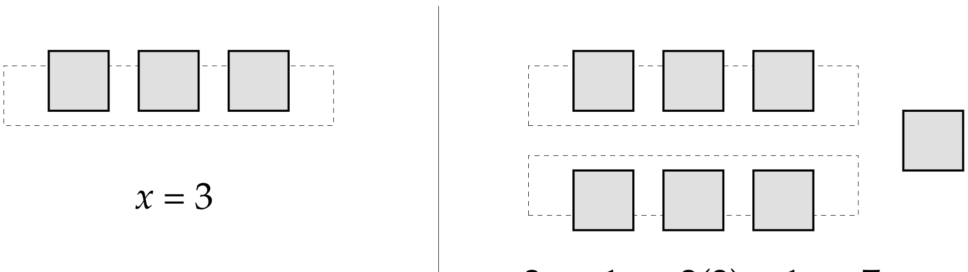


If $x = 2$, then $2x + 1 = 5$.



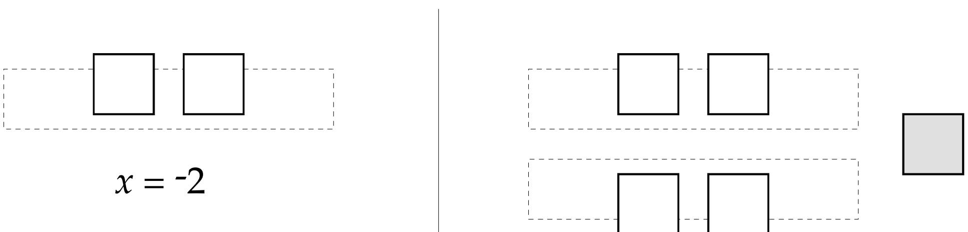


Every time we replace the x-bar with a certain number of chips, each of the other x-bars will also represent that same number of chips, and the two variables will take on values which are specifically related to each other. Using the above example again, if $x = 3$ then $2x + 1 = 7$.



$$2x + 1 = 2(3) + 1 = 7$$

If $x = {}^-2$ then $2x + 1 = {}^-3$.

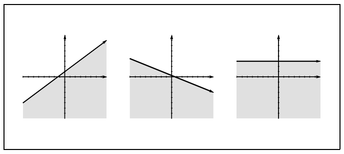


$$2x + 1 = 2({}^-2) + 1 = {}^-3$$

We often show the related values of two related variables by making a **table of values**. To make the table, we set up two columns: one representing the values of x, and the other representing the related values for $2x + 1$.

x	$2x + 1$
1	3
2	5
3	7
$^-2$	$^-3$

When filling in the table, we can choose any values we wish for x, but once an x-value is selected we must use that same x-value in calculating the related value for $2x + 1$. (Normally when choosing x-values we choose simple ones, like 1, 2, 3, 0, $^-1$, $^-2$, but we could choose any other values we desire). We call x the **independent variable** and $2x + 1$ the **dependent variable**; the first number can be chosen *independently*, but the second value *depends* on the first one.



In the expanded table below, some new values for x have been suggested. How can we calculate the related values of $2x + 1$?

x	$2x + 1$
1	3
2	5
3	7
$^{-}2$	$^{-}3$
$^{-}3$	
0	
5	

To find the missing values, we substitute the value of x into the expression $2x + 1$ as we did in the chapter on EXPRESSIONS.

For $x = {}^{-}3$:

$$2x + 1 \;=\; 2({}^{-}3) + 1 \;=\; {}^{-}6 + 1 \;=\; {}^{-}5$$

For $x = 0$:

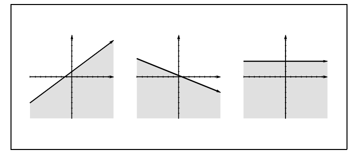
$$2x + 1 \;=\; 2(0) + 1 \;=\; 0 + 1 \;=\; 1$$

For $x = 5$:

$$2x + 1 \;=\; 2(5) + 1 \;=\; 10 + 1 \;=\; 11$$

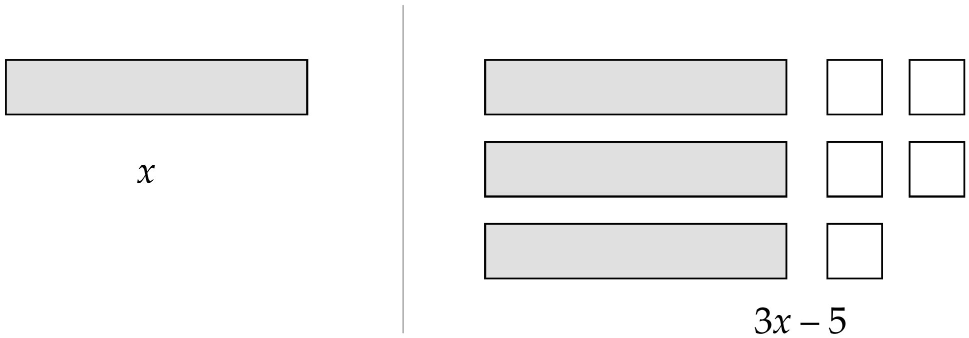
The completed table looks like this:

x	$2x + 1$
1	3
2	5
3	7
$^{-}2$	$^{-}3$
$^{-}3$	**$^{-}5$**
0	**1**
5	**11**



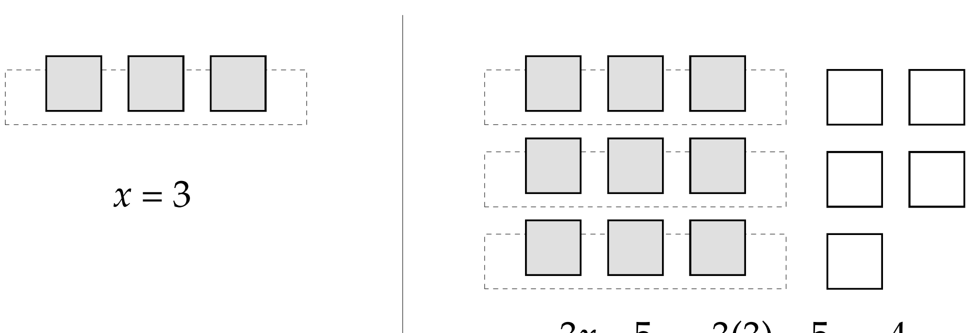
You can see that there will be no end to the number of values we might pick for x; the table could go on forever. Each line of the table represents one entry—one pair of related numbers which are specific values of the two related variables. Each line of the table may be referred to as an **ordered pair**, or a pair of numbers where the first number refers to the value of the independent variable, and the second number represents the related value of the dependent variable.

Now let's use what we have discussed to make a table of values for a different pair of related variables. We will always start with x as the independent variable. This time let's consider the dependent variable $3x - 5$. Use your chips and substitution techniques to complete the following table of values.

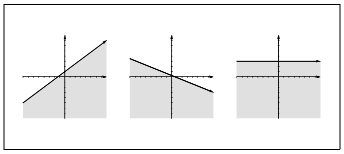


x	$3x - 5$
1	-2
2	1
3	
4	
0	
-1	
-2	

When $x = 3$ the picture will look like this:



$$3x - 5 = 3(3) - 5 = 4$$

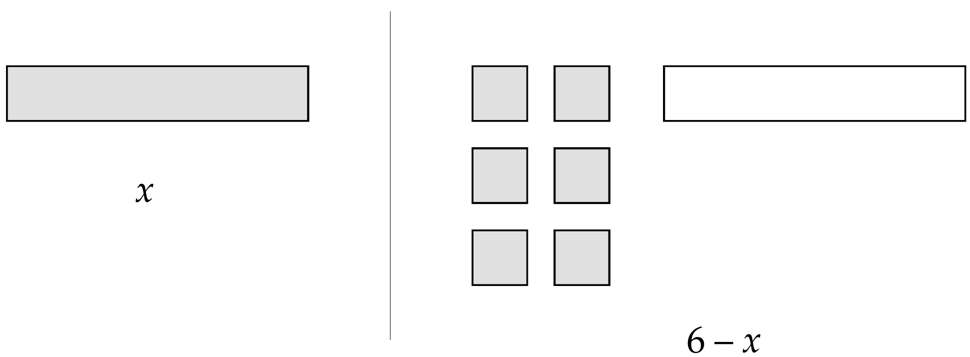


The completed table will be:

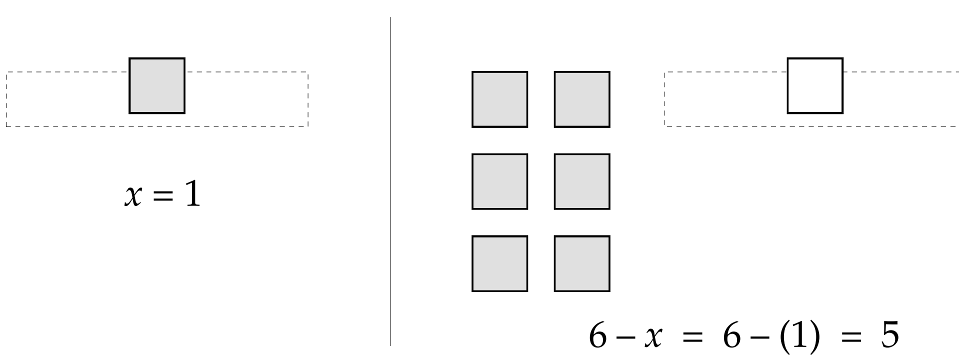
x	$3x - 5$
1	$^{-}2$
2	1
3	**4**
4	**7**
0	**$^{-}5$**
$^{-}1$	**$^{-}8$**
$^{-}2$	**$^{-}11$**

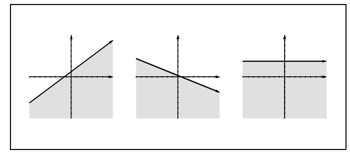
In the picture you can easily see that every time we add one more unit chip to the value of the independent variable x, the dependent variable $3x - 5$ will increase by 3 unit chips: one for each of its x-bars. This is an important idea which we will use later for graphing.

Let's make one more table where the dependent variable will have a negative x-bar.



In this case, when we let $x = 1$, the -x in the dependent expression will be flipped over to $^{-}1$.





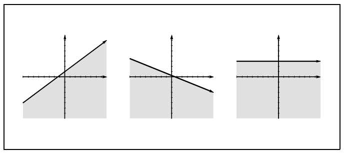
When substituting, remember to put the value for x inside a parentheses with the negative sign out in front of the parentheses. This will help you to calculate the correct values for $6 - x$.

Complete this table.

x	$6 - x$
1	
2	
3	
0	
-1	
-2	8
-3	
-4	
-5	
-6	

The completed table looks like this.

x	$6 - x$
1	**5**
2	**4**
3	**3**
0	**6**
-1	**7**
-2	8
-3	9
-4	**10**
-5	**11**
-6	**12**



Exercises

Complete the following tables: When x-values aren't listed, choose your own values for x.

1.

x	$3x - 4$
1	$^-1$
2	
3	
0	
$^-1$	
$^-2$	

2.

x	$2x + 3$
1	
2	
3	
0	
$^-1$	
$^-2$	

3.

x	$5 - x$
1	
2	3
3	
0	
$^-1$	
$^-2$	

4.

x	$3 - 2x$
1	
2	$^-1$
3	
0	
$^-1$	
$^-2$	

5.

x	^-3x
1	
2	$^-6$
?	
0	0
?	
?	

6.

x	$5x - 3$
?	
?	
?	
?	
?	
?	

Make your own table of values for the following related variables. (Use x for the independent variable.) Choose at least five values for x.

7. x and $4x - 5$

8. x and $^-3x + 2$

9. x and $^-5 - x$

10. x and $7 - 2x$

11. x and $3x + 1$

12. x and ^-4x

Section 2
Rules, Machines, and a Second Variable

Using *x* and *y*

In the previous section, two related variables were both represented using the same letter; one variable was x, the other was represented in terms of x. In this section we will expand the idea slightly by calling the two variables by different letters — x and y — and by using a rule. The rule will be an equation showing the relationship between y and x.

For example, rather than saying that our two related variables are x and $3x + 5$, we now will say that the two variables are x and y, and the rule relating them is

$$y = 3x + 5$$

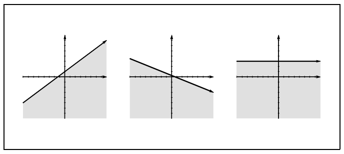
A table of values expressing this would be
Sometimes, rather than listing a rule and a whole table of values for x and y, we may want to talk about just the pairs of values. If we list only the pairs, *it is convenient to agree that we will list the pair in parentheses with x first and y second, separated by a comma.*

Each pair of x and y is called an **ordered pair**, and when we see an ordered pair written in this way we know that the two values are usually connected by some rule.
The rule above can now be shown as a list of pairs:

$$y = 3x + 5: \quad (0, 5), (1, 8), (2, 11), ({}^{-}1, 2)$$

Machines and Rules

Another way to illustrate a rule is to imagine a factory or machine that takes x's and manufactures y's. The individual machine is the rule; the input is the x and the output is the y.
The machine uses the rule as a pattern. Each y is set up as an expression using x-bars and unit squares. If x is 2, then 2 units are put in the place of each x-bar in the pattern and the total is sent out as y:
If x is $^{-}3$, then $^{-}3$ chips are put on each x in the pattern:
If the rule is $y = 2x - 3$ then the machine would take each x and pair it with three less than twice x. The machine would look like this:
If y = $-x + 4$ then the machine would perform the operation of taking the opposite of x and adding four to it. The machine would look like this:
Note that each of these machines involves only two basic operations—multiplying x by a quantity and then adding another quantity to the result.



Some rules can even be constants which don't depend on x. An example would be $y = 3$. The machine would be:

The value of y would be 3 for any choice of x. Here is a list of ordered pairs:

$$(1, 3), (2, 3), (0,3), ({}^-1, 3), ({}^-2,3)$$

Working Backwards to Find *x*

If we are given a rule like $y = 2x - 5$, it is possible to take any given value of y and use the rule to calculate the corresponding related value for x.

For example, using $y = 2x - 5$, we can find the x-value corresponding to $y = 9$ by substituting 9 for y and then solving for x in the remaining equation:

$$y = 2x - 5$$

$$\begin{aligned} \text{Since } y = 9, \text{ then } \quad (9) &= 2x - 5 \\ 9 + 5 &= 2x - 5 + 5 \\ 14 &= 2x \\ \frac{14}{2} &= \frac{2x}{2} \\ 7 &= x \end{aligned}$$

Summary

Two related variables and their values can be illustrated in several ways:

- **An equation defining y in terms of x using a rule:**

$$\mathbf{y = 3x + 5}$$

- **A *table of values*:**

Rule: $\mathbf{y = 3x + 5}$

x	y
0	5
1	8

2	11
-1	2

- **A list of *ordered pairs*:**

(0, 5), (1,8), (2, 11), (-1, 2)

- **A *machine* with a clear rule to determine y from x:**

Exercises

Finish the tables:

1. $y = 3x$

x	y
-5	-15
5	?
10	?
3	?
1	?
-1	?

2. $y = -2x + 1$

x	y
-1	3
0	1
1	?
3	?
2	?
-5	?

3. $y = -x$

x	y
6	-6
-4	4
0	?
2	?
-3	?
17	?

Using the given rule, complete the list of ordered pairs by filling in the missing y's:

4. $y = x - 17$: (2, _), (5, _), (-5, _), (17, _)

5. $y = 3x + 1$: (2, _), (5, _), (-5, _), (17, _)

6. $y = 17 - x$: (2, _), (5, _), (-5, _), (17, _)

Using the given rules, choose four x's and then calculate the matching y's:

7. $y = 17x - 17$ (_ , _), (_ , _), (_ , _), (_ , _)

8. $y = x + 17$ (_ , _), (_ , _), (_ , _), (_ , _)

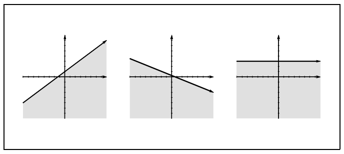
9. $y = 17 - x$ (_ , _), (_ , _), (_ , _), (_ , _)

10. $y = 3x - 3$ (_ , _), (_ , _), (_ , _), (_ , _)

11. $y = 3x - 4$ (_ , _), (_ , _), (_ , _), (_ , _)

12. $y = 4x - 5$ (_ , _), (_ , _), (_ , _), (_ , _)

13. $y = -x - 1$ (_ , _), (_ , _), (_ , _), (_ , _)



14. $y = -x + 1$ (_ , _), (_ , _), (_ , _), (_ , _)

15. $y = 7$ (_ , _), (_ , _), (_ , _), (_ , _)

Take the given rule and the given value of y and work backwards to find the related value of x:

16. $y = 17x - 17$ $y = 17$

17. $y = x + 17$ $y = 21$

18. $y = 17 - x$ $y = 0$

19. $y = 3x - 3$ $y = {}^-9$

20. $y = 3x - 4$ $y = {}^-10$

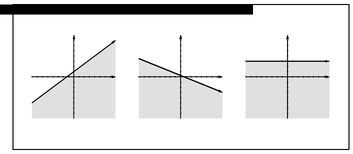
21. $y = 4x - 5$ $y = 15$

22. $y = -x - 1$ $y = {}^-6$

23. $y = -x + 1$ $y = {}^-4$

24. $y = {}^-3x + 1$ $y = 1$

25. $y = {}^-3x - 1$ $y = 2$



Section 3
Graphs and Coordinates

Chips and Ordered Pairs

One value from an ordered pair can easily be represented by a column of unit chips. For the pair (1, 3), we line up a column of 3 chips to represent y: If we want to show a list of pairs from an equation, we first prepare a column for the y -value in each ordered pair:

$$y = 2x + 1: \quad (1, 3), (2, 5), (3, 7)$$

We then take a horizontal number line and arrange the columns (y) at the place on the line which matches that pair's x. The position on the line represents x and the height of the column represents y for each ordered pair.

If negative numbers are included, we expand the x number line to the left to represent negative values of x; if the y value is negative, we arrange the chips down instead of up.

In summary, we can show a list of pairs as a group of columns:

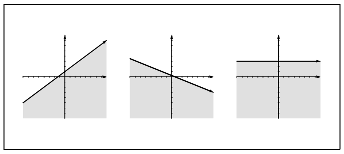
From Chips to Points

To simplify the picture, we can use the position of the center of the end of the column (top or bottom) to represent the related values of both x and y; the chips are no longer needed. This means that an ordered pair can be shown as a single point:

To represent several pairs at once, we simply draw each point separately in the same picture:

The Coordinate System

The idea we have just developed is called the **coordinate system**. The number lines showing the values for x and y are called **axes**; individually they are called the ***x* axis** and the ***y* axis**. An ordered pair now represents two



numbers—the first is the *x* **coordinate** and the second is the *y* **coordinate**. The point where *x* and *y* are both zero — (0, 0) — is known as the origin.

Each ordered pair can be shown as a single point. When we illustrate a list of ordered pairs as points on the coordinate system, we call this a **graph**.
Here are two examples of graphs:

Graphing Points

To graph a point, we start at the origin. We then move *x* units to the right (if *x* is positive) or to the left (if *x* is negative). The second step is to move up *y* units (if *y* is positive) or down (if *y* is negative). The ending point represents that ordered pair (*x*, *y*):

Did you notice that *x* and *y* can be both positive, both negative, or one positive and the other negative?

Alternative Methods

An ordered pair can be graphed in several different ways; the result is of course the same. The procedure shown above is to begin at the origin, move horizontally (right or left) for *x*, then vertically (up or down) for *y*:
It is obvious that we can accomplish the same thing by first locating the *y* by moving up or down and then locating the *x* by moving right or left:
Finally, we can locate the *x* and *y* separately; the point is graphed at the intersection of the lines on the grid:

These methods are essentially the same—the choice is yours. You must keep our agreements:

- **In an ordered pair, the *x* is listed first and the *y* second.**
- **On the graph, the *x* axis is horizontal (left to right) and the *y* axis is vertical (up and down).**
- **On the coordinate system, we count from the origin (0, 0).**
- **On the *x* axis, positive numbers are to the right of the center and negative numbers are to the left of the center.**
- **On the *y* axis, positive numbers are up from the center and negative numbers are down from the center.**

These agreements are a sensible method of ensuring that we are all talking about the same things; the choice of positive and negative directions is natural. If we were inventing mathematics ourselves, we could alter the order of *x* and *y* or the direction of the axes. But since this system is already in use, we will abide by the rules so that we are all speaking the same language.

Coordinates that are Fractions or Irrationals

The coordinates of a point can be fractions or irrational numbers. The method of finding the proper position of such a point is the same, but you must determine the approximate locations on the axes.

Fractions such as $^{3}/_{2}$ should be written as mixed numbers ($1\,^{1}/_{2}$) to help you find the locations. Square roots should computed on a calculator or approximated (as in the chapter on POWERS AND ROOTS).

Exercises

Set up each pair as a column of chips at the correct *x* position:

1. (5, $^-1$)

2. ($^-1$, 5)

3. (3, 1)

4. (0, 3)

5. ($^-5$, 0)

Graph each ordered pair:

6. (1, 6)

7. ($^{7}/_{2}$, 3)

8. ($^-3$, $^{7}/_{2}$)

9. ($^-3$, - $^{7}/_{2}$)

10. ($^-5$, 4)

Graph each list of ordered pairs on the same coordinate graph.

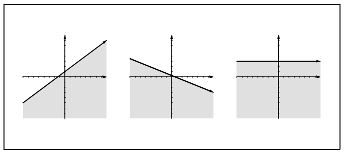
11. (0, 0), (3, 3), (5, 5), ($^-1$,$^-1$), ($^-3$, $^-3$)

12. (0, 7), ($^-1$, 8), (2, $^-5$), (3, 4), (4, 3)

13. (2, 1), (1, 2), (2, 6), (6, 2), (3, 4), (4, 3), (5, $^{12}/_{5}$)

14. (0, 1), (1, 8), ($^-1$, 8), (2, 6), ($^-2$, 6), (3, 1), (4, $^-6$), ($^-3$, 1), ($^-4$, $^-6$)

15. (0, 0), (1, $^-1$), ($^-2$, 2), (2, $^-2$), ($^-5$, 5), (7, $^-7$), ($^-4$, 4)



Section 4
Graphs of Lines

Lines on the Coordinate Graph

In the previous section, we graphed individual points and lists of points. Now we will learn to graph lists of points from a rule. Each rule will have a specific shape.

In this chapter, we will limit ourselves to rules that include x's, numbers, adding/subtracting, and multiplying/dividing. We will *not* consider rules or equations that include x^2, x^3, $1/x$, and other more complicated formulas.

To graph a rule, follow these simple steps:

- **Choose several x values. Zero and small numbers are usually best.**
- **Calculate the matching y for each x, making an ordered pair (x, y).**
- **Put each ordered pair onto the graph.**
- **Connect the points.**

Here are the graphs of two examples:

Lines and Rules

The graphs shown above are in the form of a **line**. A rule that is a line on the graph is called **linear**. The following rules are linear:

$$y = 3x + 4$$
$$y = {}^{-}2x + 5$$
$$y = 10 - x$$
$$y = 2x$$

These rules have an x term and/or a number term and are in the form

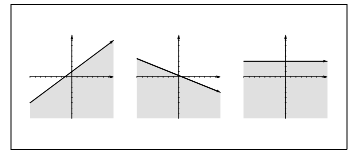
$$y = mx + b$$

where m and b are any numbers. Such a rule will be a line. As you might expect, the reverse is also true—any graph that is a line has a rule which can be written in the form of $y = mx + b$.

Not all equations graph as lines. The following equations are *not* lines; they involve higher powers of x (x^2, x^3), division by x, or products of x and y:

$$y = x^2 + \frac{1}{x}$$

$$xy + 3 = x$$



To graph a line, we can follow the steps in Section 3, but since we know the graph will be a line, we will only need three (x, y) pairs. Here are the steps:

- **Confirm the rule is a line and has the form** $y = mx + b$.
- **Choose any three *x* values.**
- **Calculate the *y* values.**
- **Graph each ordered pair as a point.**
- **Draw a line through the three points.**

If we never made errors, two points would be enough. Using three points helps to ensure that we graph the correct line; if the three points are not in a line, than it is obvious that we have made a mistake in calculating the *y* values or in placing the values on the graph.
Here are two more examples of graphing a linear rule:

A Special Case

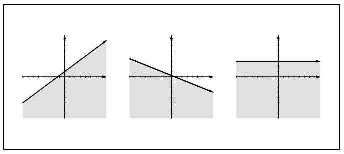
What is the meaning of the following rule?

$$y = 5$$

If *x* is 2, what is *y*? If *x* is $^-4$, what is *y*? The answer is very simple—*whatever x you choose, the answer is that y is 5*! As a table, it looks like this:

As a machine, it looks like this:
The graph of $y = 5$ is a simple horizontal line. We choose several ordered pairs where the *y*-value is 5:
Notice that this is a rule—it is just a very simple one. As a machine, we can see that we can send in anything that we want, but it always sends out three chips.

As a graph, $y = 5$ is a horizontal line because as *x* changes, *y* stays level at exactly five. The line travels along from left to right, but *y* remains constant.



The *y* bar.

We will now create a chip for *y*. Since *y* represents some unknown number of unit squares, we will show it as a bar that is 1 unit wide and unknown number of units long:
The bars for *y* and *x* are similar but stand for different unknown quantities:
We will use the symbol **-*y*** to stand for the opposite of *y*. The other (white) side of the bar will represent -*y*.
The following cautions may be helpful:

- ***x* is *shown a*s longer than *y* to help us tell the two apart; *x* is not necessarily greater than *y*.**
- ***x* may be greater, less than or equal to *y* in any given problem.**
- ***x* and *y* are shown with their shaded side up, but they may stand for numbers that are negative, zero, or positive.**
- **The symbols -*x* and -*y* represent the opposites of *x* and *y*.**
- **The symbols -*x* and -*y* may stand for numbers that are negative, zero, or positive.**

Equations and Rules

We will often encounter an equation involving both *x* and *y*. An **equation** is defined as any two expressions that are given as equal to each other. This is not quite the same as a rule because a rule gives *y* as some expression including numbers and *x*'s.

Some equations can be rewritten to represent rules by the use of our previous techniques for solving equations. Our goal is to rearrange the equation to "isolate" *y*. For example, begin with:

$$x + y = 12$$

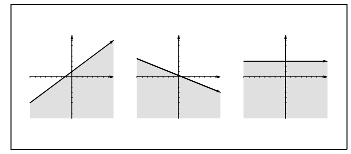
Isolate *y*:

$$x - x + y = -x + 12$$
$$y = -x + 12$$

Notice that we are only rearranging the equation to show the rule. In general, *you cannot solve the equation to determine one x or y.* (You can find the rule for *y* in terms of *x*.) Here are several more examples:

The same problem shown with symbols only:

$$4x + 2y = 6$$
$$4x - 4x + 2y = {}^{-}4x + 6$$



$$
\begin{aligned}
2y &= {}^-4x + 6 \\
\frac{1}{2}(2y) &= \frac{1}{2}({}^-4x + 6) \\
y &= {}^-2x + 3
\end{aligned}
$$

A more difficult example:

The same problem shown with symbols only:

$$
\begin{aligned}
3x + 6 - 2y &= y + 3 + 2(x - 6) \\
3x + 6 - 2y &= y + 3 + 2x + {}^-12 \\
3x + 6 - 2y &= y + 2x + {}^-9 \\
3x + 6 - 2y - y &= y - y + 2x + {}^-9 \\
3x + 6 - 3y &= 2x + {}^-9 \\
3x + {}^-3x + 6 + {}^-6 + {}^-3y &= 2x + {}^-3x + {}^-9 + {}^-6 \\
{}^-3y &= -x + {}^-15 \\
-\frac{1}{3}({}^-3y) &= -\frac{1}{3}(-x + {}^-15) \\
y &= \frac{1}{3}x + 5
\end{aligned}
$$

Summary

If a rule is given as an equation in x and y, here are the steps to graph the line:

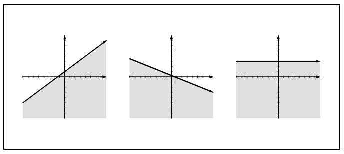
- **Solve the equation for y:**
  Multiply out any quantities in parentheses.
  Add y's to both sides to "isolate" the y's on one side.
  Add x's to both sides to leave x's only on the other side.
  Add units to both sides to leave units and x's on the other side.
  Multiply (or divide) both sides to leave only one y.

- **Pick three values of x. Calculate the matching y values to give ordered pairs (x, y).**

- **Plot the points on the graph.**

- **Draw a line through the points.**

It should be obvious that the first steps are almost identical to the steps for solving equations with only x. The difference is that we solve for y as equal to an expression containing x's and numbers. As before, it does not matter which side you pick for y; you may choose, however, to write the final rule with the y on the left.



Exercises

Which of the following rules are linear? (a linear rule can be written in the form $y = mx + b$)

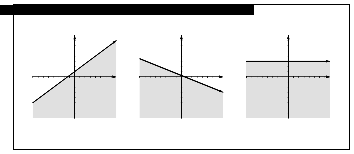
1. $y = 999x + 1234$
2. $y = 125 - .532x$
3. $y = x$
4. $y = 0$
5. $y = 2x^2 + 1$
6. $y = 0x^2 + 2x + 1$

Graph the following rules:

7. $y = x + 1$
8. $y = x - 3$
9. $y = {}^-3x + 7$
10. $y = 2x - 5$
11. $y = {}^-2x - 5$
12. $y = 6$

Change the following equations to rules by solving for y. Graph the result by plotting at least three points for each equation:

13. $x + y = 3x + 2y + 3$
14. $2x + y = 3x - 5$
15. $2x + y = {}^-3x + 7$
16. $3y + x = 2x - 2y + 15$
17. $2x + y - 6 = 3x - 5$
18. $3x + 2y + 5 = -x + 9$
19. $2(x - 3) = -x + y$
20. $x + 3(2 - x) = x + 2(y + 1) - y$
21. $2(y + x) = 2x + y + 6$
22. $3[2 - 4(y + 3)] = 6(x - 1)$
23. $y + \frac{1}{2} = 2x - \frac{3}{2}$
24. $5y + 6 = 3x + 1$
25. ${}^-1[{}^-1 - 1(y - 1)] = -x - 1$



Section 5
Slopes and Intercepts

Uphill and Downhill

Consider the following lines as side views of roads in the real world. We will imagine traveling the roads from *left to right*.
Like a road, a line on the graph is either uphill, downhill, or flat. We say the line has **slope**; as we move to the right we say that an uphill line has positive slope, a downhill line has negative slope, and a flat line has zero slope.

If we want to assign a number to represent the slope, it makes sense to call a flat line "0" and to give higher numbers to steeper uphill slopes. It also makes sense to give increasing downhill slopes values which are increasingly negative (${}^{-}1$, ${}^{-}2$, ${}^{-}3$, …):
The formal definition is as follows:
On the graph, you can see that the slope measures how far the line moves up for each unit that it moves to the right.

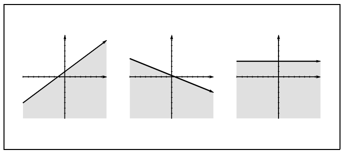
The slope can be measured at any point. It is useful to think of a line as a series of stair steps that move 1 to the right, then up or down, 1 more to the right, then up or down again, etc.
Here are some other examples of slopes:

Another Definition

We can also define the slope as a fraction or ratio. In the line shown on the next page, the slope is clearly 2 when we measure over 1 and up 2. On the same line, if we measure over 2 or 3 on the x axis, the y value increases at a rate that is in the same *proportions.* This means that the y increase *relative* to the x increase is still the same:
We write this as a fraction with the change in y (rise) on top and the change in x (run) on the bottom. *The slope can also be defined as the rise over the run.*
Here are some other examples:

Two Definitions Compared

We now have two seemingly different definitions of slope:
It would not be useful if the definitions were actually different; in fact they are really the same:
The first definition is in fact a ratio of the distance up relative to 1, while the



second definition is the ratio of rise to run over any distance. The two results are identical, so in each problem we can use the definition that is more convenient or more comfortable.

The Slope Between Two Points

It takes time to graph points and lines. If we wish to know the slope of the line between two points, we can calculate the slope *without* making the graph. First, look at the slope of the line between (2, 1) and (5,7):

The rise is 6 and the run is 3. We can find this by taking the difference of the y coordinates (rise) and the difference of the x coordinates (run):
Because the individual x or y coordinate represents the distance of a given point from one axis of the graph, subtracting the x or y coordinates of two points gives the horizontal or vertical difference between the two points. The slope is the vertical difference divided by the horizontal difference:
As seen above, you can use either point as the first point, as long as you use the same point first when subtracting for both x and y; you do not need to memorize this formula; if you understand the concept of a slope, you will be able to construct the formula any time you need it.

Intercepts

A place where a line hits the y axis or x axis is called an **intercept**. As you might expect, if the line hits the x axis we call the point an ***x* intercept** and if it hits the y axis we call the point a ***y* intercept**.
In this book, we will use the term **intercept** loosely; it will mean either the ordered pair (point) at the intersection or the coordinate on the axis. For example:
We can see that x intercepts always have coordinates of $(\_, 0)$ and y intercepts always have coordinates of $(0, \_)$. In addition, we can see that a linear rule may have a y intercept or it may have both a y intercept and an x intercept. (Can it have only an x intercept?). The intercepts may be positive, negative, or zero.

Finding Intercepts

Intercepts can be found by graphing the rule and then examining the graph. This can be slow; if the intersection points are not exact integers, it can also be inaccurate:

If we use the idea that the x intercept is a point where $y = 0$, then we can solve the equation algebraically as follows:

$$y = 2x - 5$$

$$\text{for } x \text{ intercept, } y = 0$$
$$(0) = 2x - 5$$
$$5 = 2x$$
$$\frac{5}{2} = x$$

To find the y intercept, we use the idea that points on the y axis have $x = 0$:

$$y = 2x - 5$$

Since $x = 0$

$$y = 2(0) - 5$$
$$y = 0 - 5$$
$$y = {}^{-}5$$

To find the intercepts:

- **For the x intercept:**
  1. On the x axis, y is 0.
  2. Substitute $y = 0$ in the formula for the line.
  3. Solve for x. This is the value of the x intercept.

- **For the y intercept:**
  1. On the y axis, x is 0.
  2. Substitute $x = 0$ in the formula for the line.
  3. Solve for y. This is the value of the y intercept.

It is not necessary (and not helpful) to attempt to memorize this process. For each intercept, decide from your understanding of graphing which coordinate is 0, then substitute and solve the equation.

Exercises

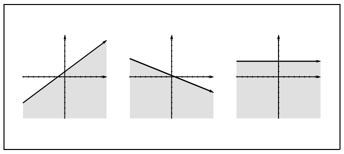
Find the slopes of these lines by examining the graphs:

Find the slope of the line between each pair of points without graphing. Then graph the line to see if you are correct:

3. $(2, 1), ({}^{-}2, 5)$

4. $({}^{-}1, {}^{-}1), (5, 6)$

5. $(1, 1), (4, 4)$



6. (2, 1), (7, 1)

7. (12, 1), (1, 12)

8. $(^-1, ^-2)$, $(4, ^-5)$

Find the slope and intercepts by graphing:

9. $y = -x - 7$

10. $y = {}^-3x - 12$

11. $2y + 3 = 4x + 9$

Find the intercepts by using the formula alone, without graphing:

12. $y = x - 6$

13. $y = {}^-2x - 2$

14. $3y + 3 = -x + 9$

Find the slope between the two intercepts. Remember that each intercept is an ordered pair.

Example: y intercept is 5, x intercept is 2:

Solution: Points are (0, 5) and (2, 0). Slope is $\frac{0-5}{2-0} = \frac{^-5}{2}$

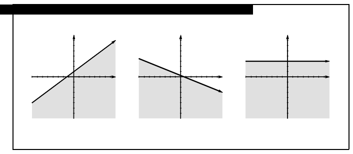
15. The y intercept is (0, 3) and the x intercept is (1, 0).

16. The y intercept is $(0, ^-3)$ and the x intercept is (6, 0).

17. The y intercept is (0, 6) and the x intercept is $(^-3, 0)$.

18. The y intercept is $^-1$ and the x intercept is $^-1$.

19. The y intercept is $^-5$ and the x intercept is $\frac{3}{2}$.



Section 6
Graphing with Slopes and Intercepts

The Slope-Intercept Method

Graphing a line by making a table of points can be lengthy and inefficient. This section will cover a better method that allows you to merely *look* at the rule and then immediately graph it. In addition, you will be able to look at any graph of a line and then immediately know the rule that it represents. To accomplish these feats, we will use the ideas of slope and intercept from the previous section.

First, examine the following graphs, their slopes, and their y intercepts:

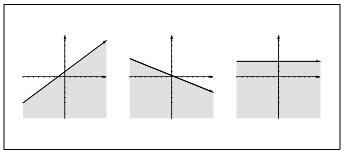
We can see that when the rule is written in the form of $y = mx + b$, m is the slope and b is the y intercept. This is true for all linear rules, even if there is a negative slope or a zero slope.

You may want to rearrange the terms to put the rule in the form of $y = mx + b$. You can add missing ones and zeros to make the equation fit the format:

$y = 2x$	becomes	$y = 2x + 0$
$y = 10 - x$	becomes	$y = {}^{-}1x + 10$
$y = 5$	becomes	$y = 0x + 5$
$y = \frac{x}{2}$	becomes	$y = \frac{1}{2}x + 0$
$y = 3x - 5$	becomes	$y = 3x + {}^{-}5$
$y = \frac{{}^{-}2x}{5}$	becomes	$y = \frac{{}^{-}2}{5}x + 0$

To graph a rule or equation using this method:

- **If necessary, solve the given equation for y.**



- **Rewrite the rule in the form of $y = mx + b$. Write all terms as adding negatives rather than subtracting.**

- **Determine the slope (m) and the y intercept (b).**

- **On the graph, start at the y intercept and go to the right at the correct slope.**

- **Draw the line.**

You have two methods to draw the line when the slope is a fraction such as ⅔ — either go over one and then up ⅔, or go over 3 (run) and up 2 (rise).

Here is how to graph

$$y = \frac{1}{2}x + 3$$

Here is how to graph

$$y = {}^{-}2x \qquad (y = {}^{-}2x + 0)$$

The Slope-Intercept Method—Why it Works

The slope-intercept method is not just a way to memorize how to make graphs quickly—there is a simple reason *why* the m turns out to be the slope and the b turns out to be the y intercept. As we can see from the next two diagrams, m is the number of x's; it controls how much y goes up each time we increase x by 1. If there are 3 x's, then y goes up 3 each time x goes up 1.

We can also see that b is the number of unit chips we have when x is zero. This is the "starting point" on the y axis or the y intercept.

Below, we can see a machine for $y = 3x + 5$. If we think of x as first 0, then 1, then 2, we are moving along the x axis from the origin to the right. Because the machine has 5 units ($b = 5$), we start at 5 when x is 0. Because the machine has 3 x's ($m = 3$), y goes up 3 each time x goes up 1. This is a slope of 3.

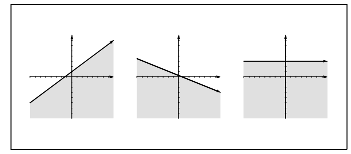
Special Lines

We have already looked at lines such as

$$y = 3 \quad \text{or} \quad y = {}^{-}2$$

If you use the slope-intercept method, first rewrite the equation in the $y = mx + b$ form as:

$$y = 0x + 3 \quad \text{and} \quad y = 0x + {}^{-}2$$



We now can see that these lines have a slope of 0 (they are flat) and have y intercepts of 3 and $^{-}2$:
What about an equation such as

$$x = 5\,?$$

First, this is *not* really a rule because it gives no instructions to determine y from x. We can graph all the ordered pairs with $x = 5$ and we see that we get a vertical line. Using (5, 1), (5, 2), and (5, $^{-}1$):

The slope of this line is also hard to define because it is infinitely steep. By our previous definition, it has a run of zero and division by zero is not defined. We must then agree that the slope is not defined.

Exercises

Graph the following lines by using the slope and y intercept:

1. $y = 2x + 1$

2. $y = 2x + 3$

3. $y = {}^{-}2x - 1$

4. $y = -\frac{1}{2}x + 3$

5. $y = x - 2$

Write the rule for the following lines. (Hint: Determine the slope and y intercept, then write the equation in the format of $y = mx + b$ by filling in m and b.:

Solve the following equations for y and then graph using the slope-intercept method:

8. $y - 2 = x + 1$

9. $y + 5 = 2x + 4$

10. $2y - 1 = x - 7$

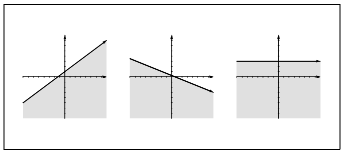
11. $2y + 6x = 10$

12. $3y + 2x = 12$

13. $2y - 2x = x - 4$

14. $y + 4 = -x - 2$

15. $2y = {}^{-}6 + y$



16. $x + y + 11 = 5 + 3x + 6$

17. $3y - 5 = 7 + 2x$

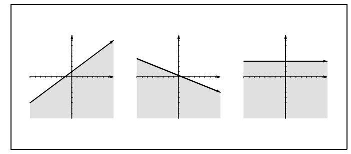
Which of the following equations have undefined slopes?:

18. $0 = y$

19. $y + x = y + 2x + 3$

20. $3(x + 2y) - 13 = 2x + 6(y - 2)$

n 7
ing With Two Intercepts



Standard Form

Equations involving x and y do not have to be written in the slope-intercept form to generate a straight line graph. Any equation which has both x and y to the first power will generate a straight line. More specifically, for the graph to be a straight line, neither variable can have exponents higher than one (like x^2 or y^3); no two variables can be multiplied together (like xy) and neither variable can appear in the denominator of a fraction (like $3/x$ or x/y). These examples will help to illustrate.

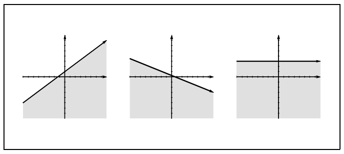
$3x + 2y = 6$	Straightline graph
$y = 2x - 3$	Straightline graph
$x^2 + 2 = 5y$	NOT a straight line
$2xy - 5x = 10$	NOT a straight line
$\frac{3}{x} + 5 = 2y$	NOT a straight line

Any equation which does generate a straight line graph can be written in many ways. Other than the slope-intercept form, another common way of writing linear equations is called **standard form**. Standard form always shows the x-term added to the y-term equaling a constant. For example, below are two linear equations written in standard form.

$$3x + 2y = 6$$

$$x - 5y = 10$$

To graph linear equations which are written in standard form we could rearrange each equation, solving it for y and thus putting it into slope-intercept form before graphing it. This method would work, but there is another way that is usually easier.



Two-Intercept Graphing

We know that we can find the y-intercept of a graph (where the graph crosses the y-axis) by letting $x = 0$ in the equation and solving for y. Similarly, we can find where the graph crosses the x-axis (the x-intercept), by letting $y = 0$ and solving for x in the equation.

Finding both the x and y intercepts of a graph is particularly easy when the equation is written in standard form, as these examples will illustrate:

Start with the equation:

$$3x + 2y = 6$$

To find the y-intercept, let $x = 0$:

$$\begin{aligned} 3(0) + 2y &= 6 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

The y-intercept is (0,3)

Now to find the x-intercept, let $y = 0$.

$$\begin{aligned} 3x + 2(0) &= 6 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

The x-intercept is (2,0)

Now that we know that where the graph crosses each axis, we can mark the two intercepts and draw the graph.
A second example would be:

$$2x - 6y = 12$$

To find the y-intercept let $x = 0$.

$$\begin{aligned} 2(0) - 6y &= 12 \\ {}^{-}6y &= 12 \\ y &= {}^{-}2 \end{aligned}$$

The y-intercept is $(0,{}^{-}2)$.

To find the x-intercept let $y = 0$:

$$\begin{aligned} 2x - 6(0) &= 12 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

The x-intercept is $(6, 0)$

The graph is:

When an equation is written in standard form, finding each of the intercepts is really a one step process requiring dividing the constant term by the coefficient of the letter whose intercept is being calculated.

Difficulties With the Two Intercept Method

Although the intercepts for an equation may be easy to calculate, they may not come out to be whole numbers. This may make the graph harder to draw accurately. For example, consider the equation

$$5x - 2y = 8$$

We can find the intercepts:

Let $x = 0$

$$\begin{aligned} 5(0) - 2y &= 8 \\ {}^{-}2y &= 8 \\ y &= {}^{-}4 \end{aligned}$$

The y-intercept is $(0, {}^{-}4)$

Let $y = 0$

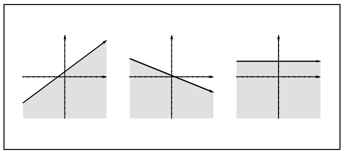
$$\begin{aligned} 5x - 2(0) &= 8 \\ 5x &= 8 \\ x &= \frac{8}{5} \end{aligned}$$

The x-intercept is $(8/5, 0)$

Exactly graphing a value like $8/5$ is more difficult than graphing a whole number. It may be difficult to make the graph accurate.

The problem of accuracy becomes particularly pronounced when one of the intercepts is a fraction and both intercepts are very near the origin $(0, 0)$ of the graph.

We can deal with these difficult cases by solving the equation for y and thus converting the equation into slope-intercept form.



$$5x - 2y = 8 \qquad \text{Add } {}^{-}5x$$
$$^{-}2y = {}^{-}5x + 8 \qquad \text{Divide by } {}^{-}2$$
$$y = \frac{5}{2}x - 4$$

The graph has a y-intercept of (0, $^{-}4$) and a slope of $^{5}/_{2}$. This can now be graphed more accurately:

Exercises

Graph these equations using the two-intercept method.

1. $2x - 3y = 6$

2. $x + 2y = 8$

3. $2x - y = 4$

4. $3x + 4y = 12$

5. $x - 5y = 5$

6. $^{-}3x + 2y = {}^{-}12$

7. $5x - 3y = 15$

8. $-x - 3y = 8$

9. $6x - 2y = 6$

10. $4x - y = 8$

Convert these equations to slope-intercept form by solving for y, and then graph.

11. $^{-}3x + 2y = 8$

12. $2x - y = 5$

13. $4x - 2y = {}^{-}2$

14. $5x + y = 4$

15. $^{-}4x + 3y = {}^{-}6$

Section 8
Summary

Related Variables

We now have many different meanings for the idea of two related variables. This concept is one of the most powerful and useful ideas in mathematics. Important ideas can usually be shown in many ways; we have presented more than one illustration in order to help you to understand, not to confuse you.

The ideas discussed in this chapter are tables, rules, lists of ordered pairs, machines, and graphs. Each is useful in different areas of mathematics, but each is essentially describing the same concept. To review, the different methods of showing two related variables are:

- **A *machine*:**

- **A *graph*:**

Note: See the APPENDIX for more information on the idea of a rule and the new concept of a **function.**

Exercises

Illustrate each of the following rules with a table, a list of ordered pairs, a machine, and a graph:

1. $y = -x + 6$

2. $y = 2x - 1$

3. $y = 2x - 3$

4. $y = {}^{-}3x + 5$

5. $y = \frac{1}{2}x - 3$

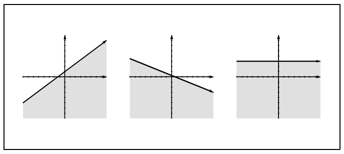
6. $y = 3x - 3$

7. $y = 0$

8. $y = -x$

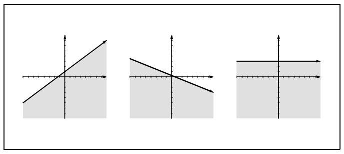
9. $y = 3$

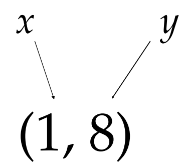
10. $y = x$

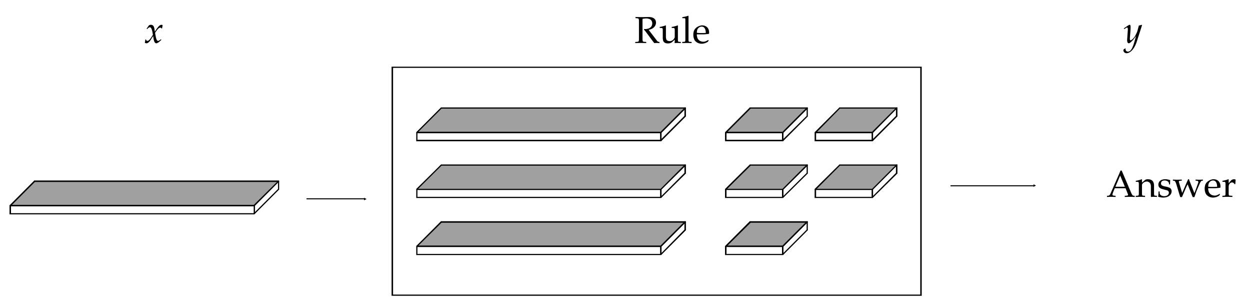


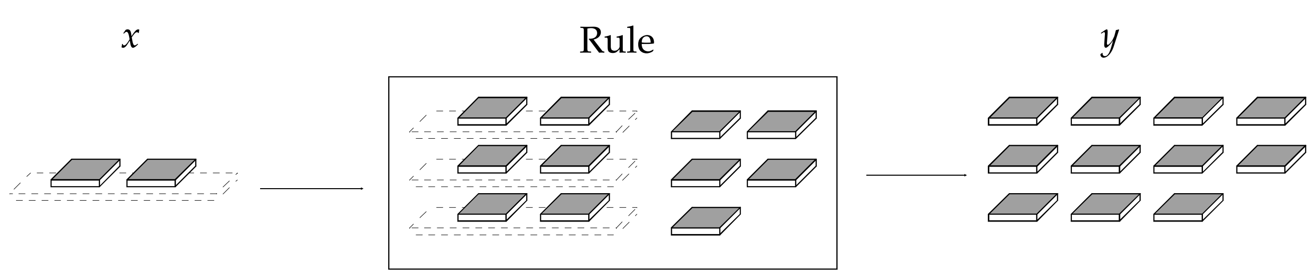
Rule: $y = 3x + 5$

x	y
0	5
1	8
2	11
-1	2
-2	-1
-3	-4
-4	-7

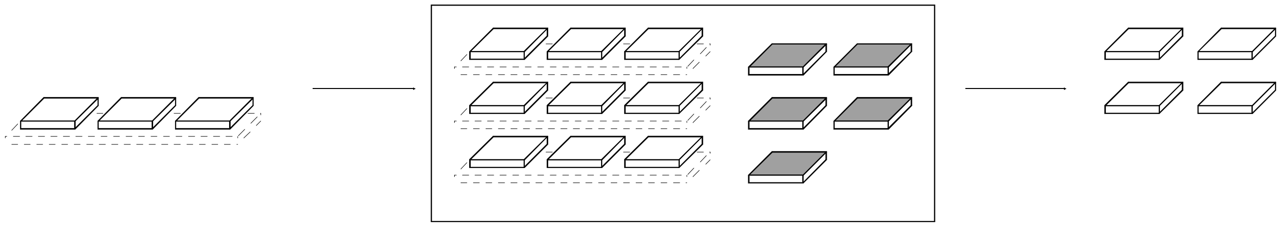


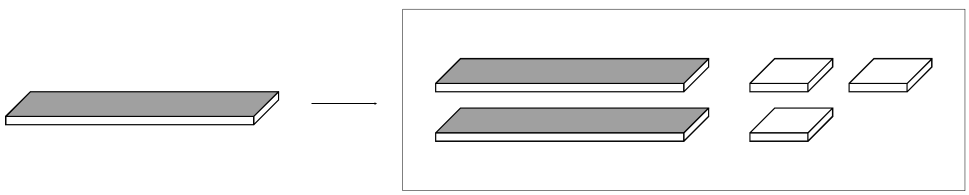

x
y
(1, 8)

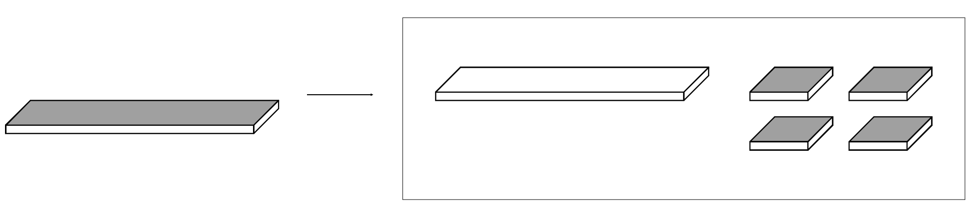

x
Rule
y
Answer

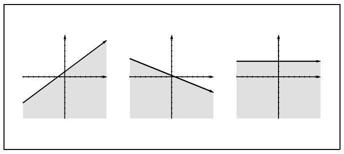

x
Rule
y

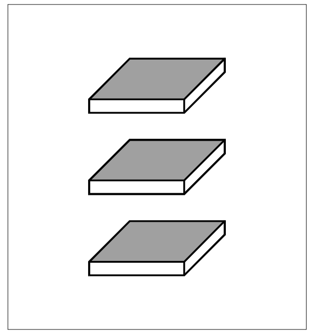
x Rule

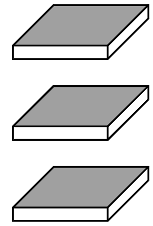


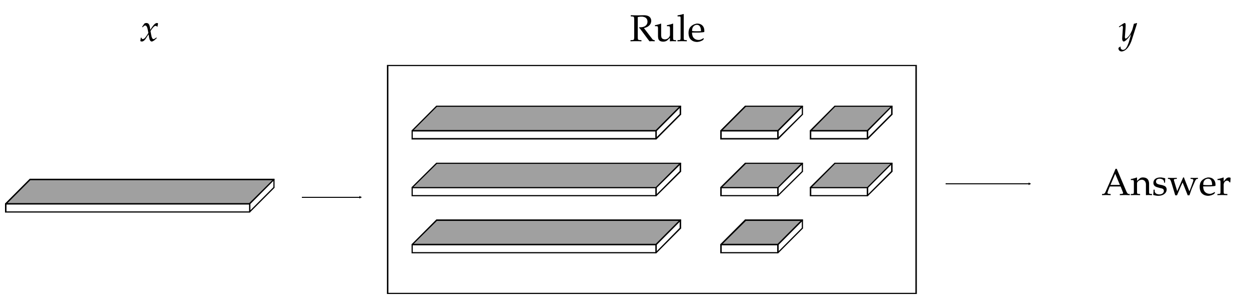




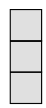


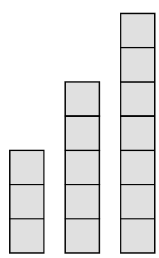


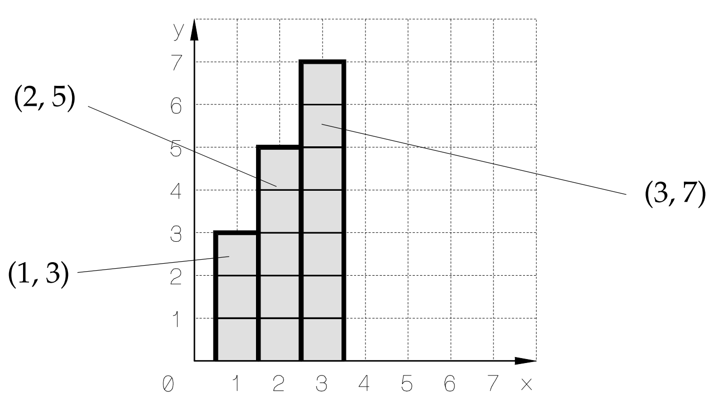


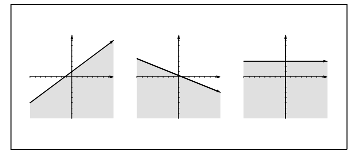


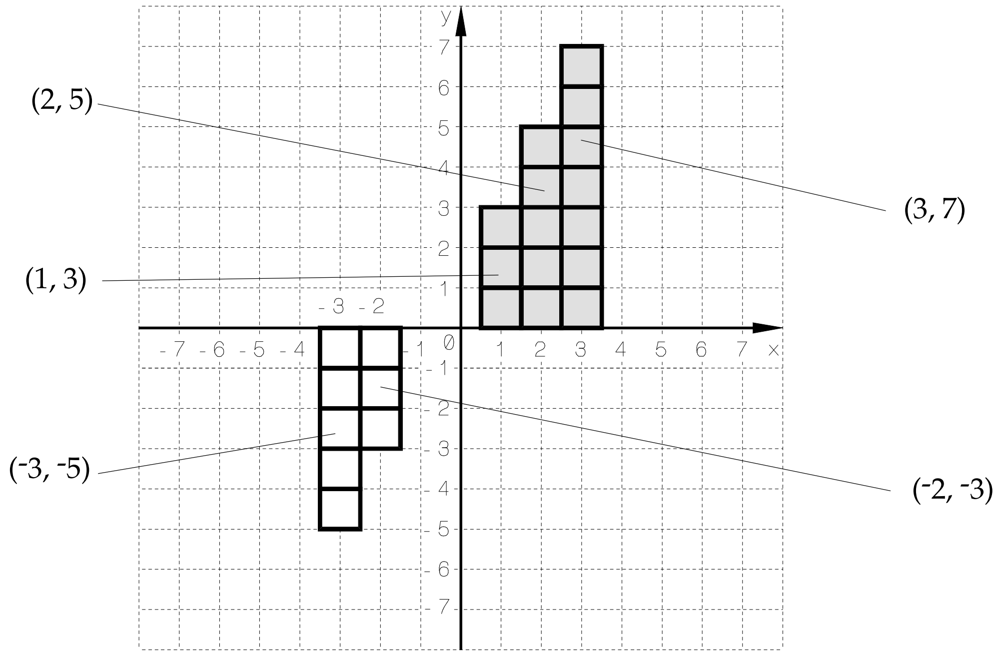
x
Rule
y
Answer

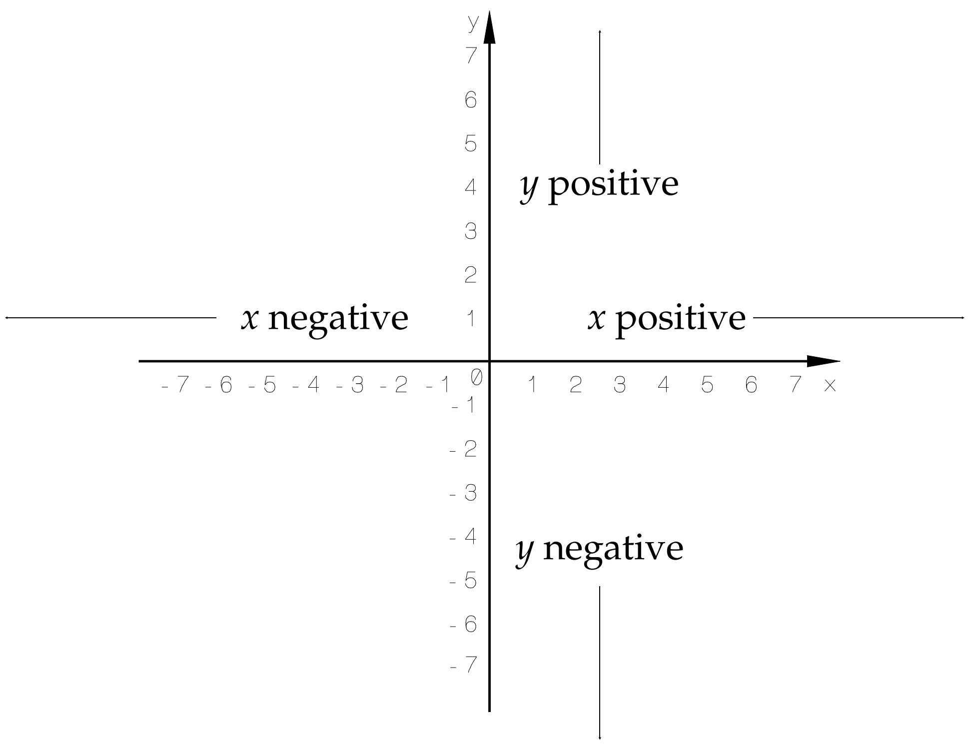


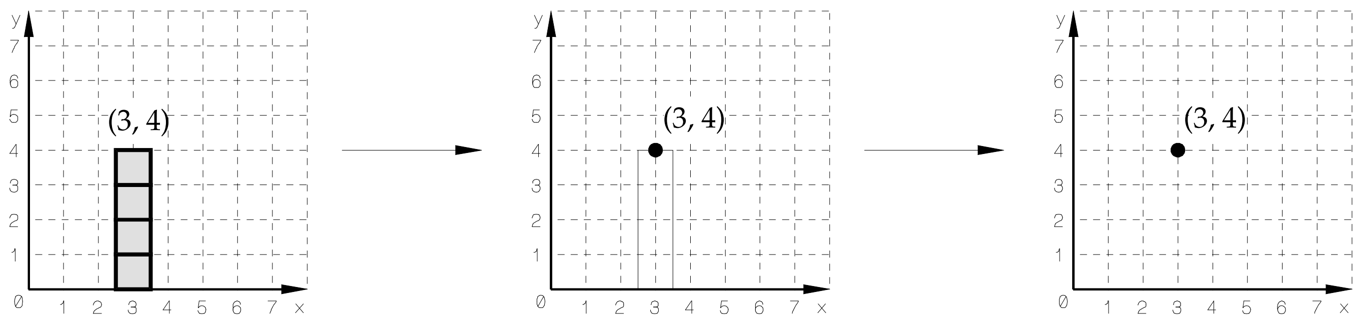



y
7
6
5
4
3
2
1
0
1 2 3 4 5 6 7 x
(2, 5)
(3, 7)
(1, 3)

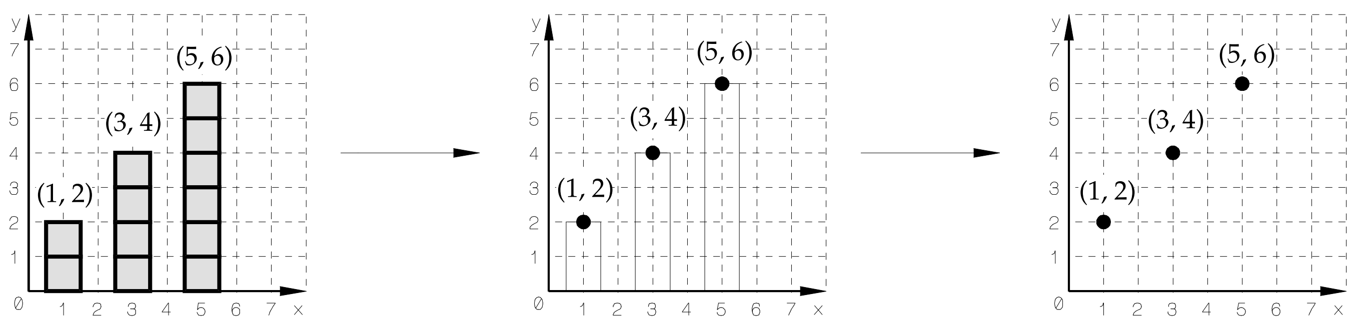



(2, 5)
(3, 7)
(1, 3)
(⁻3, ⁻5)
(⁻2, ⁻3)

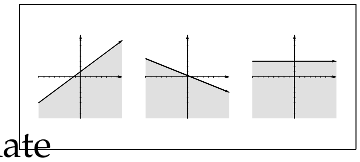

y positive
x negative
x positive
y negative

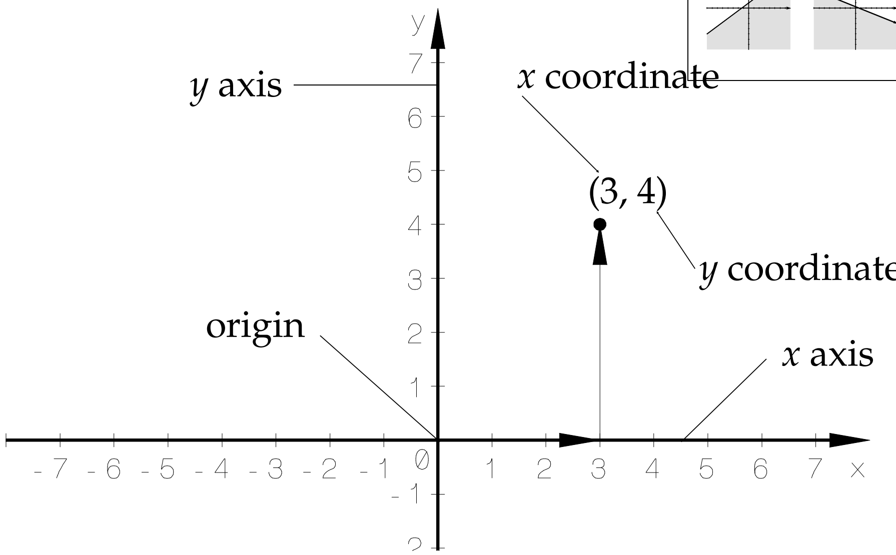


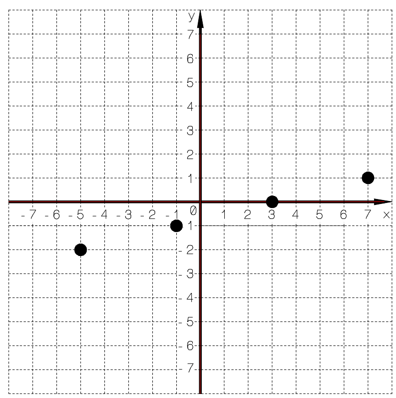
(3, 4)
(3, 4)
(3, 4)
y
x

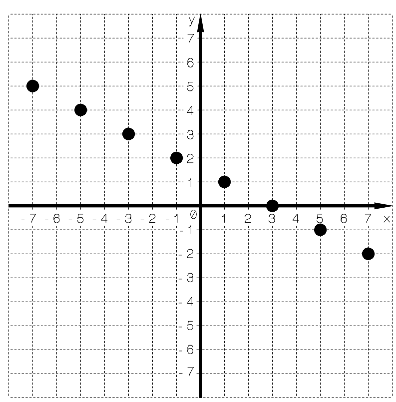


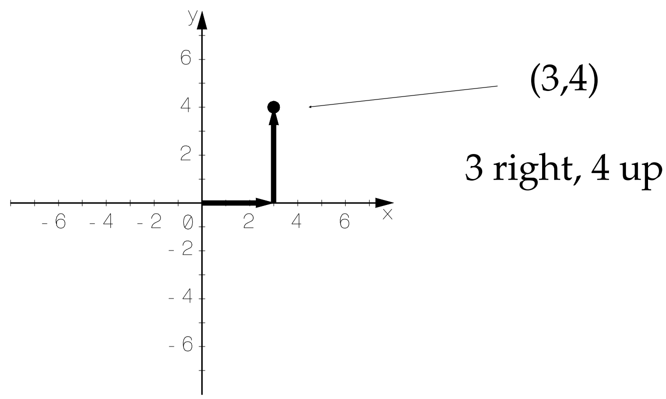
(1, 2)
(3, 4)
(5, 6)
(1, 2)
(3, 4)
(5, 6)
(1, 2)
(3, 4)
(5, 6)
y
x

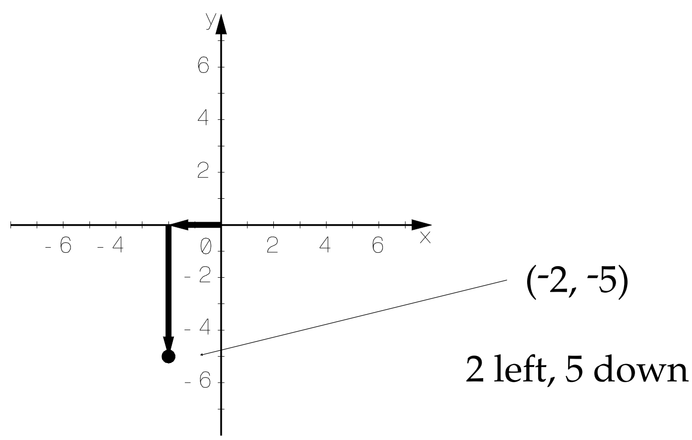


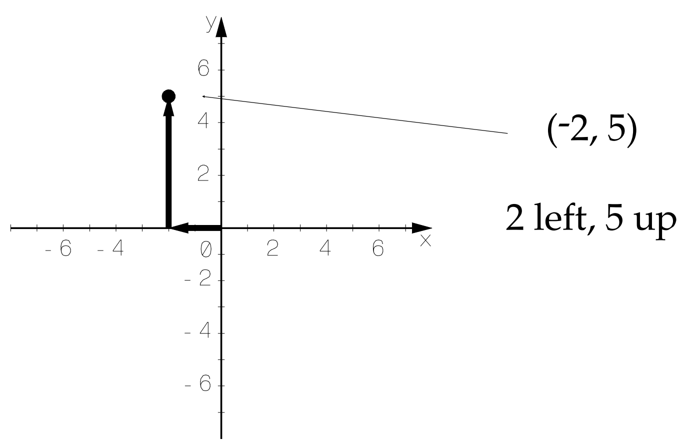

y axis
x coordinate
(3, 4)
y coordinate
origin
x axis

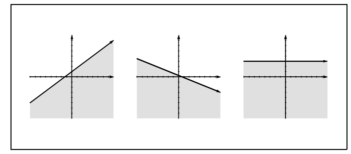


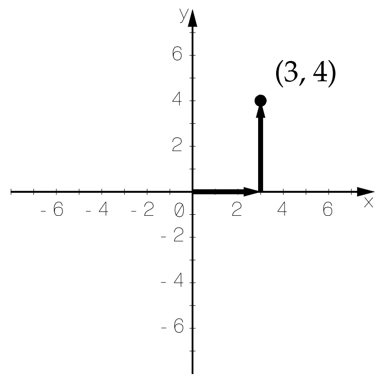


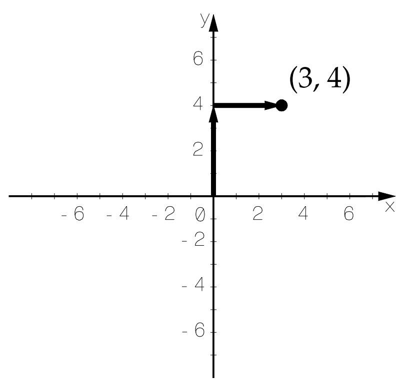

(3,4)
3 right, 4 up

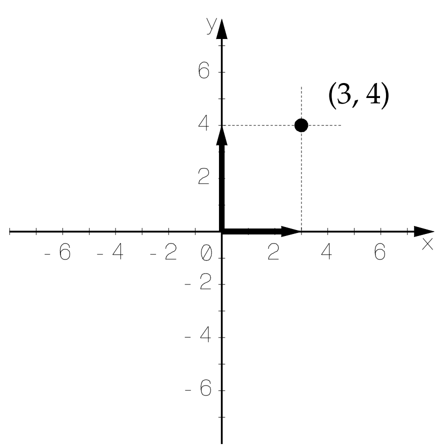

(⁻2, ⁻5)
2 left, 5 down

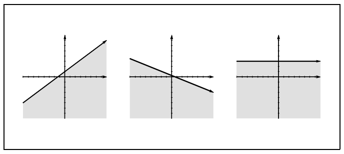

(⁻2, 5)
2 left, 5 up

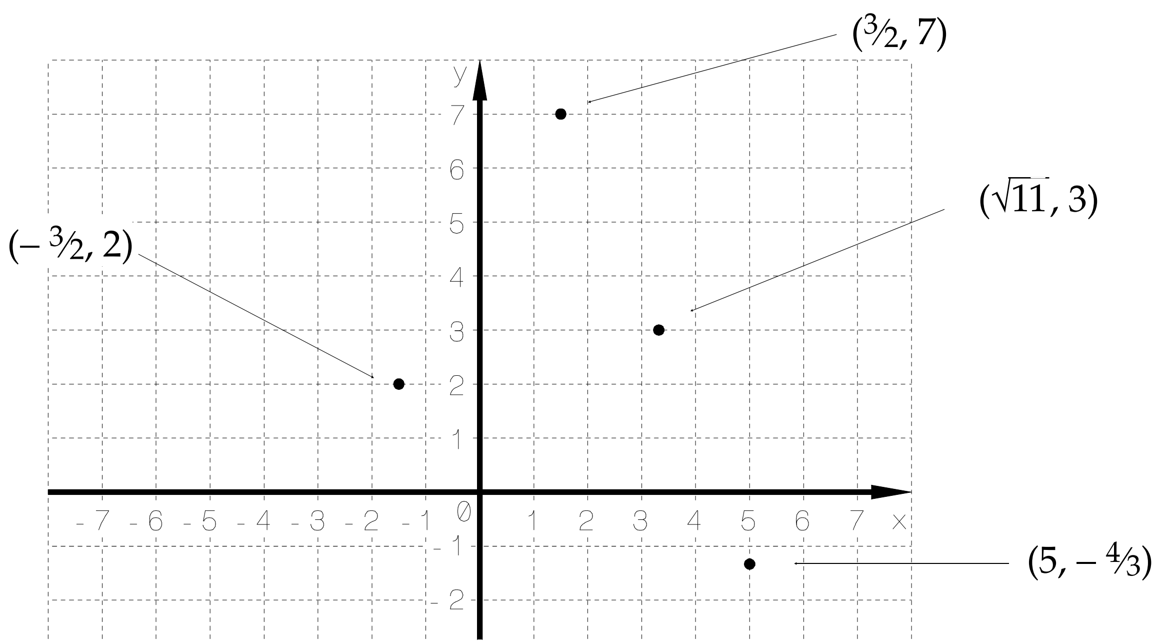



(3, 4)
x
y

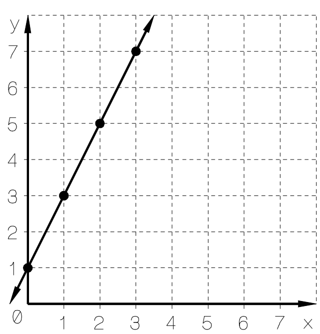

(3, 4)
x
y


(3, 4)
x
y



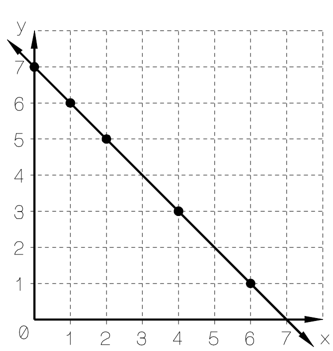


$(\frac{3}{2}, 7)$
$(\sqrt{11}, 3)$
$(-\frac{3}{2}, 2)$
$(5, -\frac{4}{3})$
y
7
6
5
4
3
2
1
0
-1
-2
-7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 x



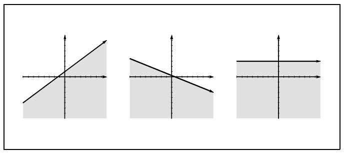
$y = 2x + 1$

x	y
1	3
2	5
3	7
0	1



$y = -x + 7$

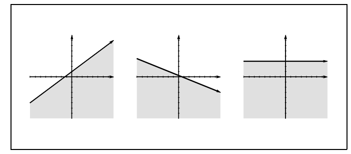
x	y
1	6
2	5
4	3
6	1
0	7

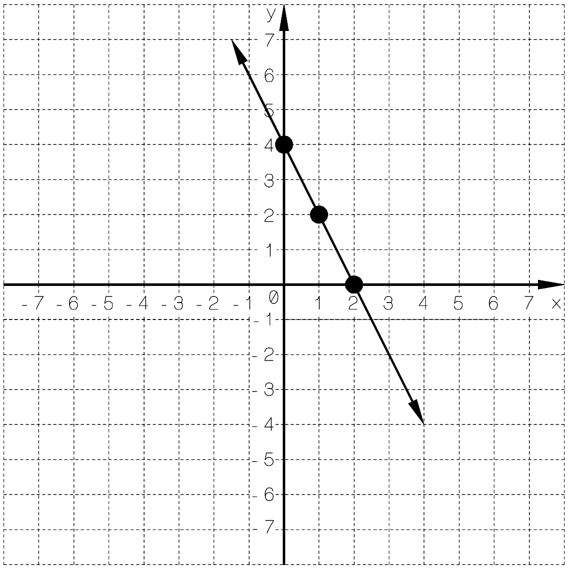


Any rule of form $y = mx + b$ is a line.

Any graph that is a line can be written in the form $y = mx + b$.

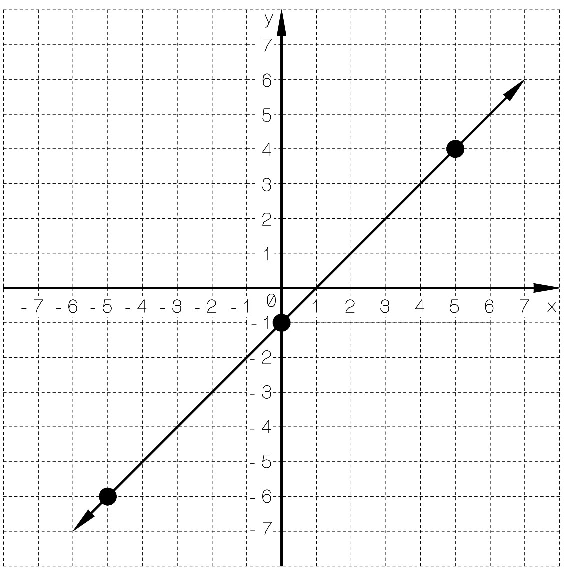
These equations do *not* graph as straight lines.





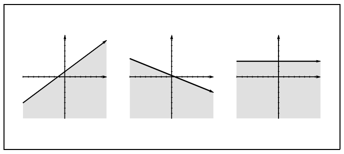
$y = {}^{-}2x + 4$

x	y
0	4
1	2
2	0



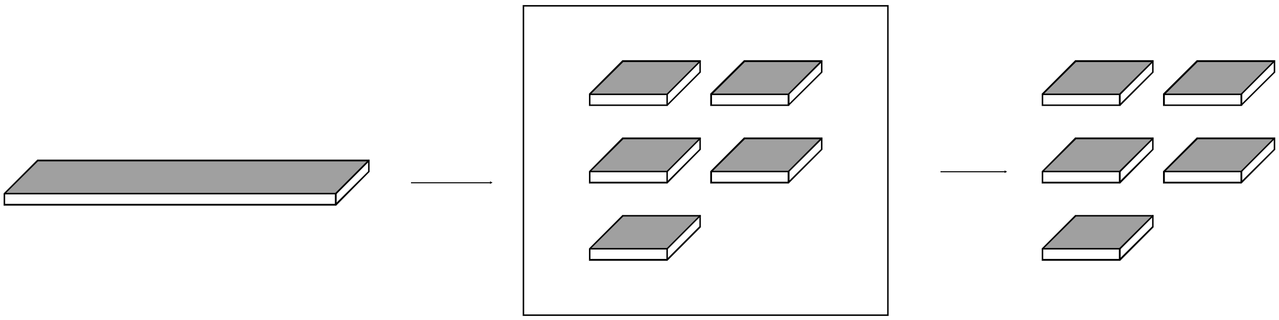
$y = x + {}^{-}1$

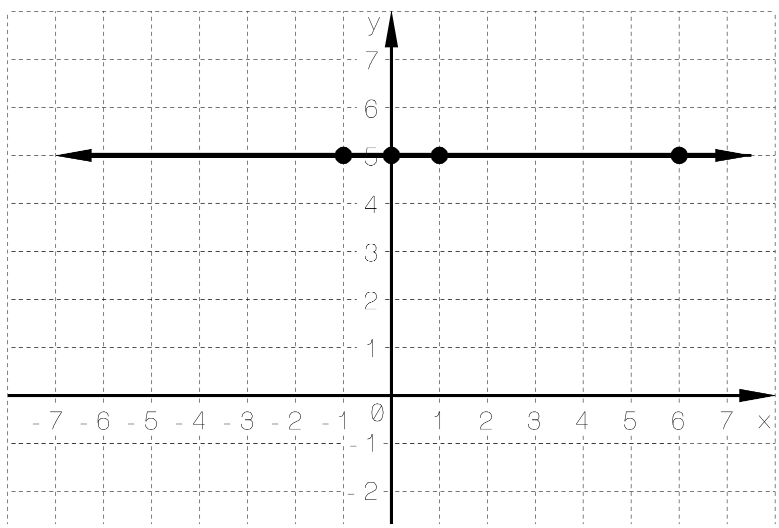
x	y
5	4
0	1
${}^{-}5$	${}^{-}6$

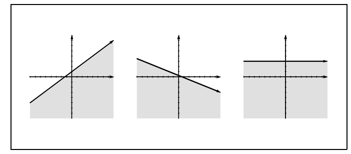


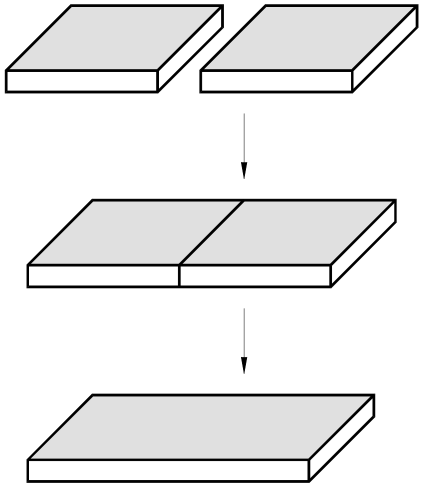
$y = 5$

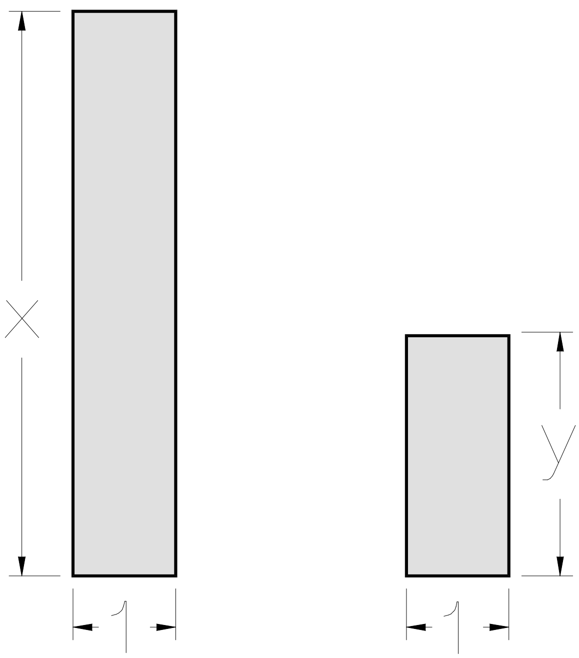
x	y
0	5
1	5
-1	5
6	5

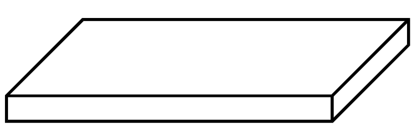


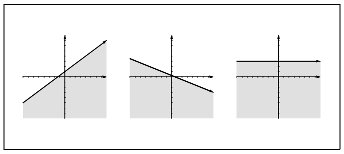




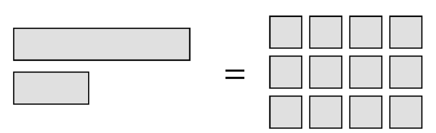



x
y
1
1

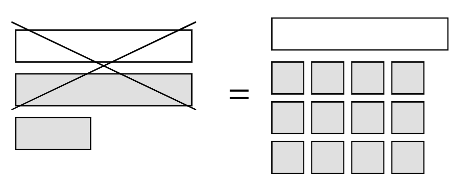




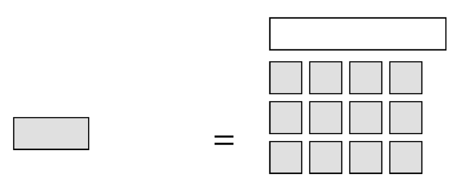
$$x + y = 12$$

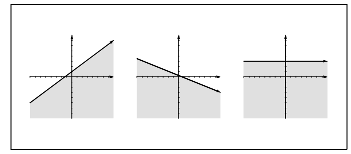


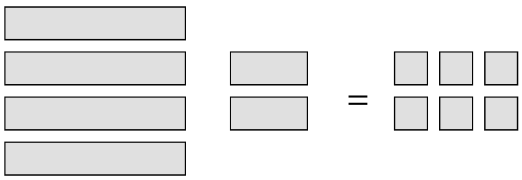
$$x - x + y = -x + 12$$



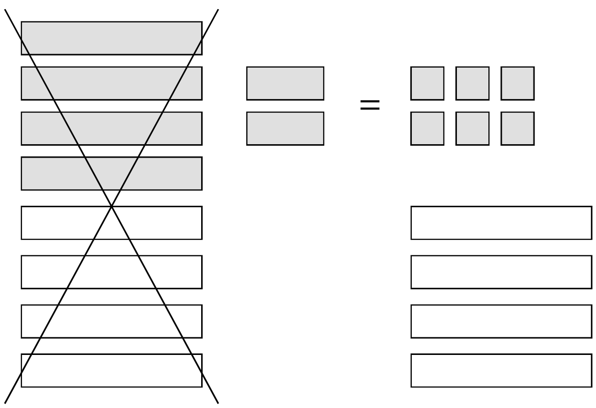
$$y = -x + 12$$





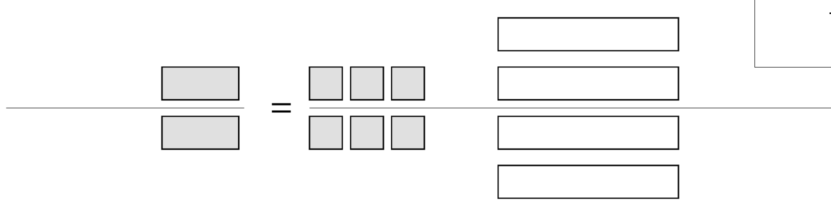


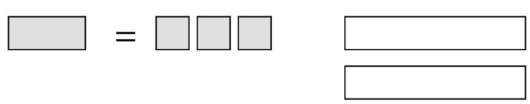
$$4x + 2y = 6$$



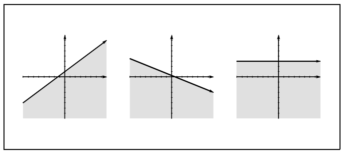
$$4x - 4x + 2y = {}^{-}4x + 6$$

$$\frac{1}{2}(2y) = \frac{1}{2}({}^{-}4x + 6)$$

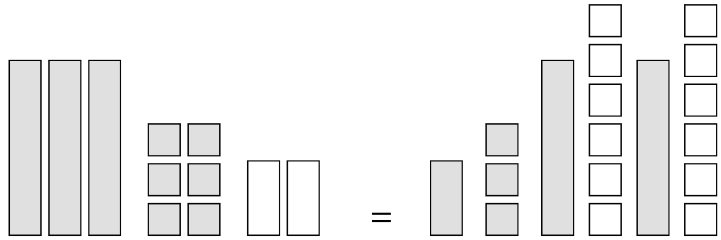




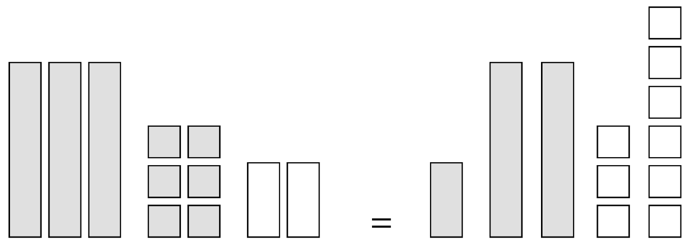
$$y = {}^{-}2x + 3$$



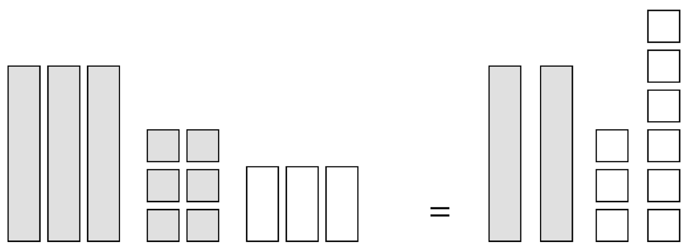
$$3x + 6 - 2y \;=\; y + 3 + 2(x - 6)$$



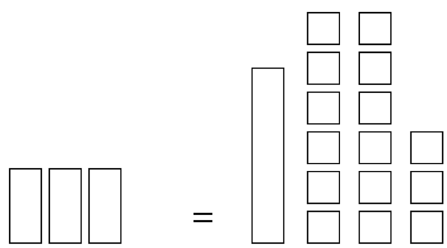
$$3x + 6 - 2y \;=\; y + 2x + {}^{-}9$$



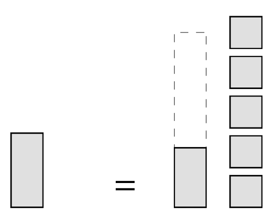
$$3x + 6 - 3y \;=\; 2x + {}^{-}9$$

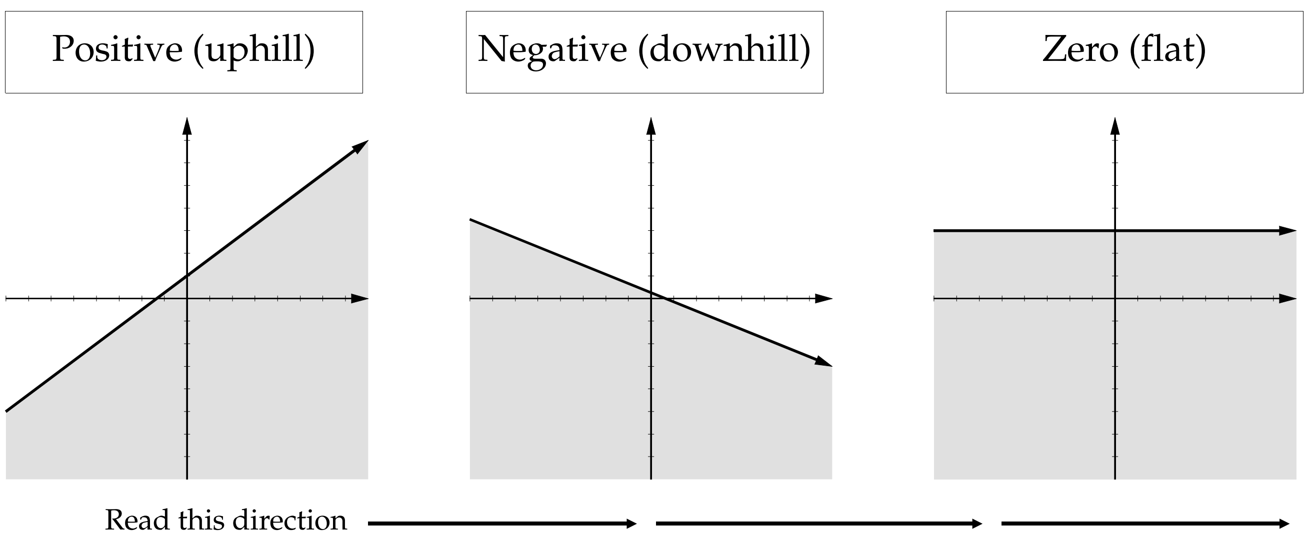


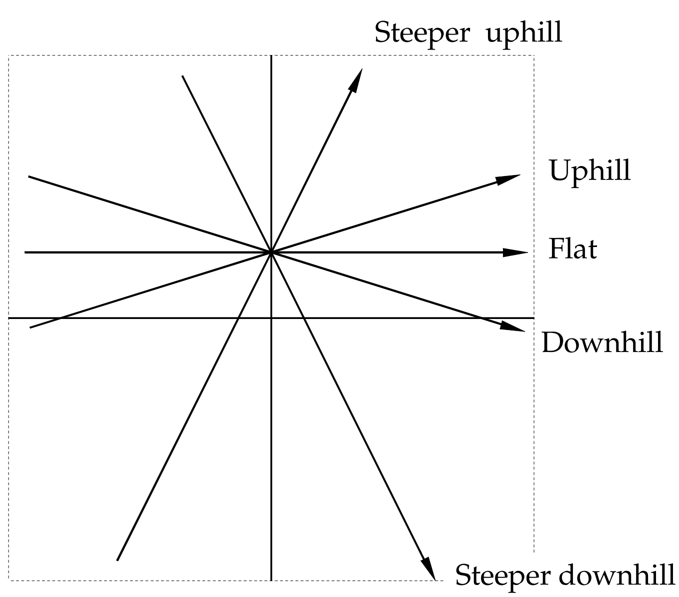
$${}^{-}3y \;=\; -x + {}^{-}15$$

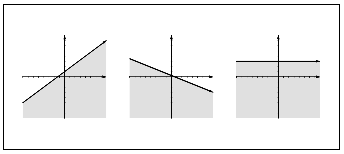


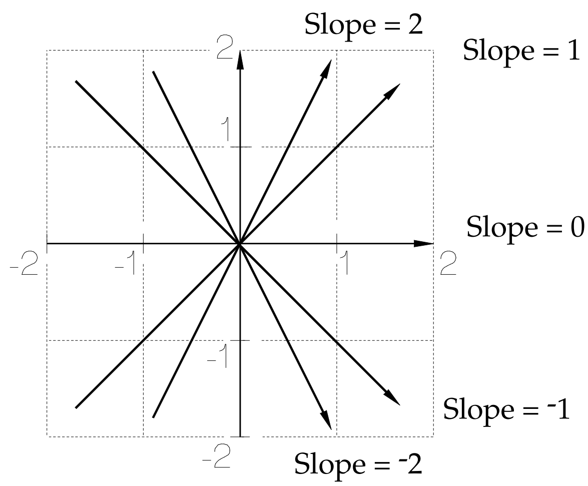
$$y \;=\; \frac{1}{3}x + 5$$




Positive (uphill)
Negative (downhill)
Zero (flat)
Read this direction


Steeper uphill
Uphill
Flat
Downhill
Steeper downhill

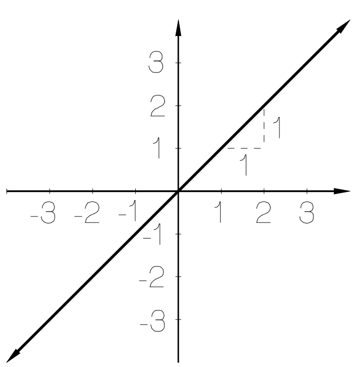




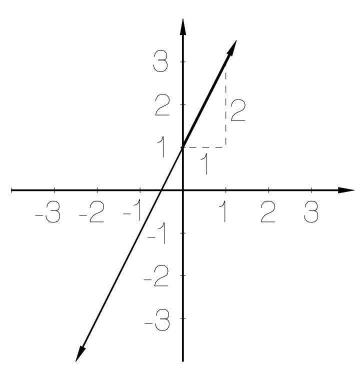
The Slope of a Line

Looking left to right, the slope is the distance travelled up or down for every 1 unit travelled to the right.

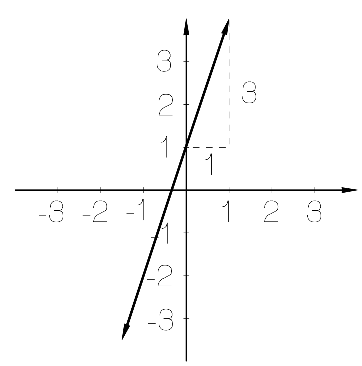
Slope = 1

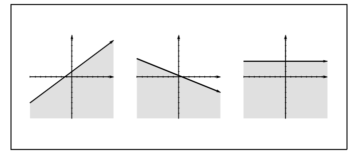


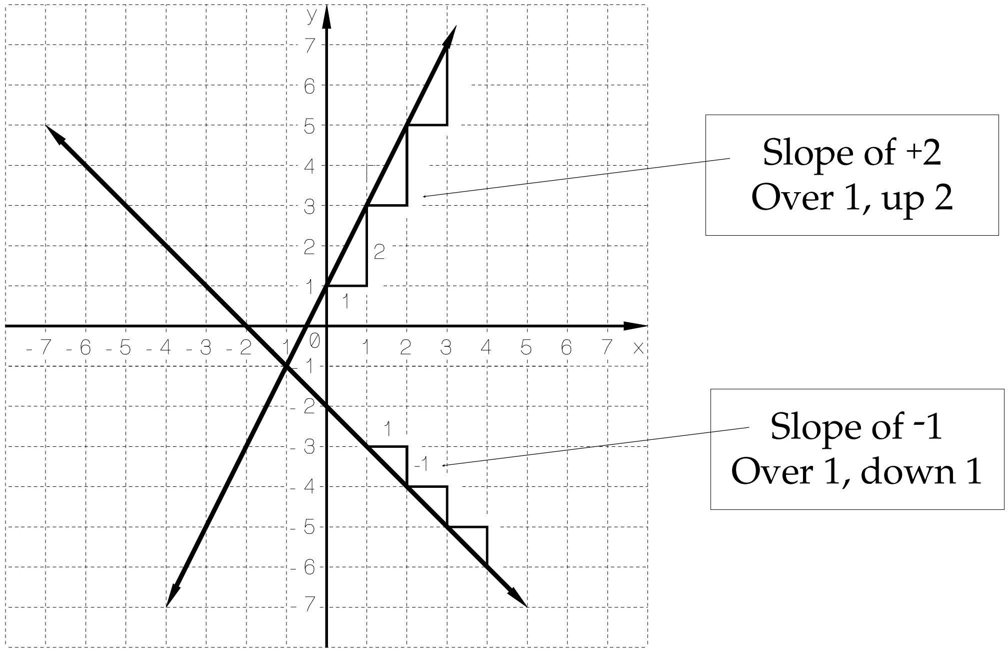
Slope = 2



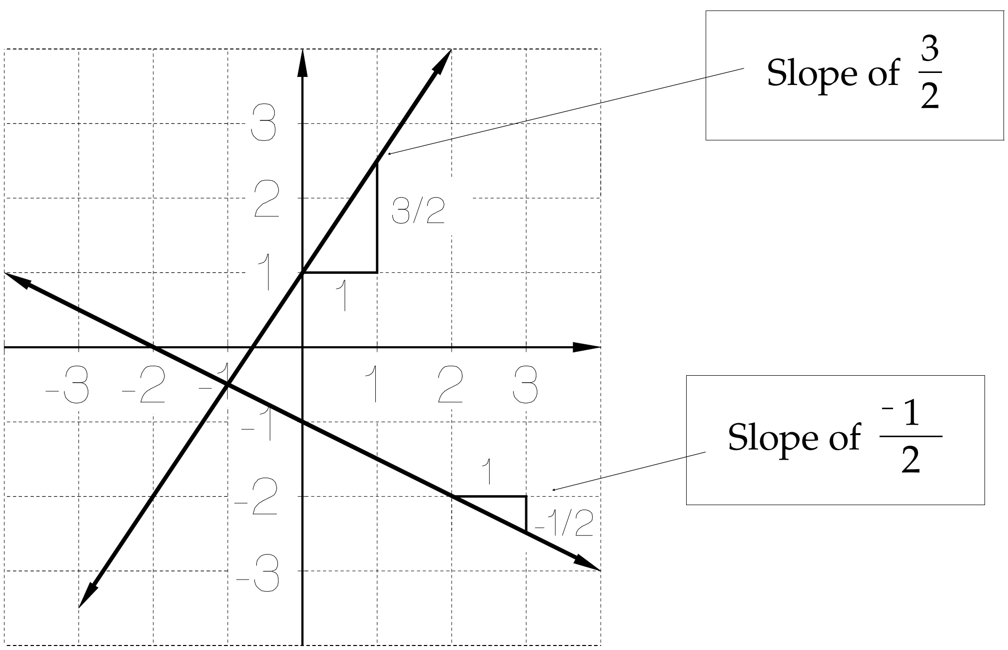
Slope = 3



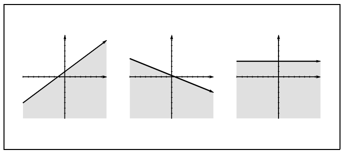


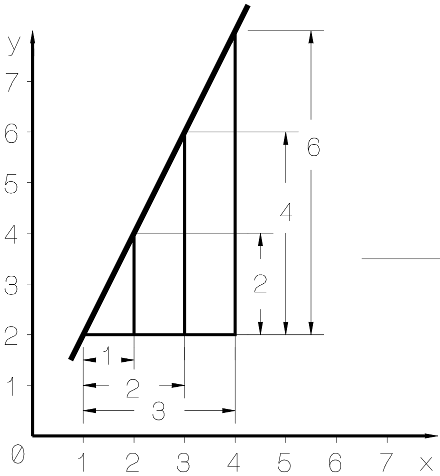


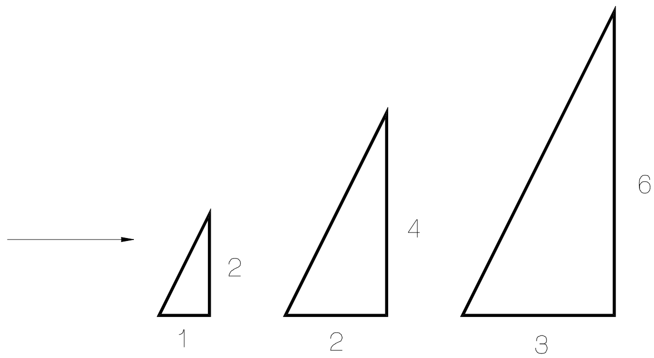
Slope of +2
Over 1, up 2
Slope of ⁻1
Over 1, down 1
y
x
2
1
-1



Slope of $\frac{3}{2}$
Slope of $\frac{-1}{2}$
3/2
1
-1/2





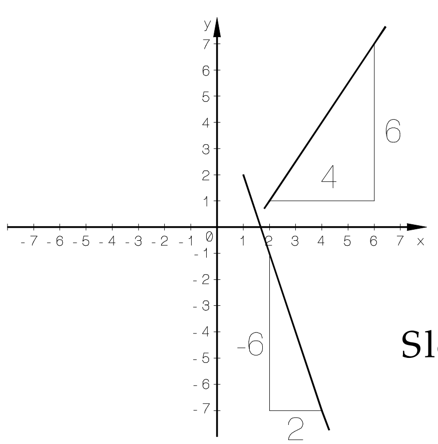


$$\text{slope} = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = 2$$

Slope of a Line (Alternate definition)

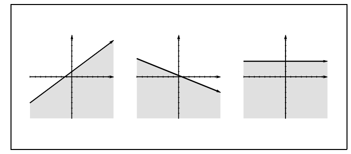
The ratio of the rise (change in *y*)
to the run (change in *x*)

$$\textbf{slope} = \frac{\textbf{change in } y}{\textbf{change in } x} = \frac{\textbf{rise}}{\textbf{run}}$$



$$\text{Slope} = \frac{6}{4} = \frac{3}{2}$$

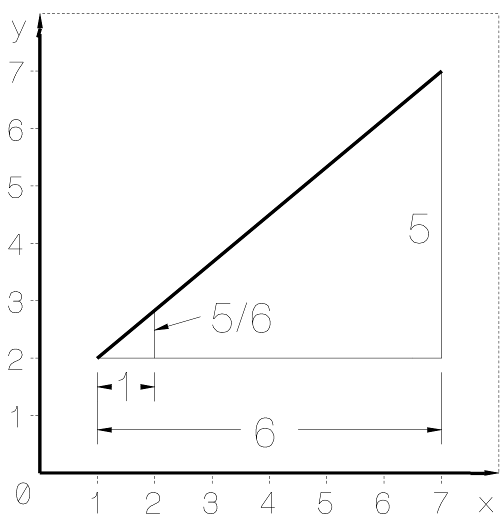
$$\text{Slope} = \frac{^{-}6}{2} = {^{-}3}$$



Definitions of Slope

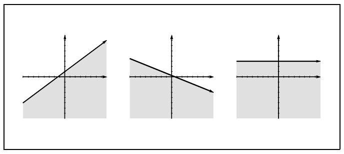
1. The distance traveled up or down for every 1 unit travelled to the the right.

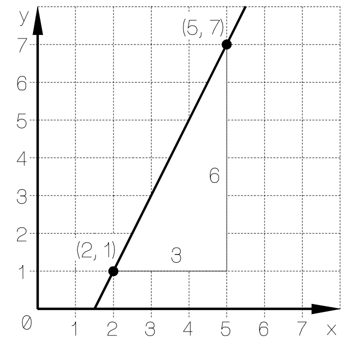
2. Slope $= \dfrac{\textbf{change in } y}{\textbf{change in } x} = \dfrac{\textbf{rise}}{\textbf{run}}$



Definition 1: $\frac{5}{6}$ up for every 1 over

Definition 2: Up 5 and over 6, slope = $\frac{5}{6}$

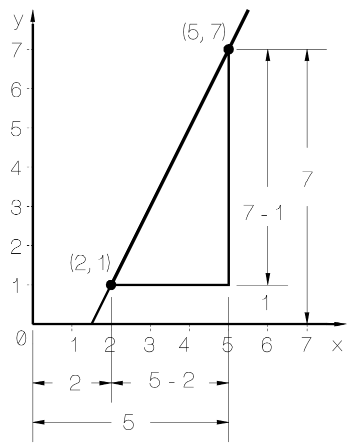




Slope between (2, 1) and (5, 7):

$$\frac{\text{Diff. of } y}{\text{Diff. of } x} = \frac{7-1}{5-2} = \frac{6}{3} = 2$$

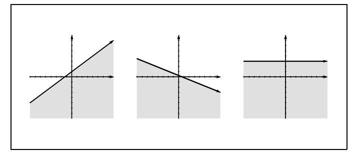
$$\text{or } = \frac{1-7}{2-5} = \frac{^-6}{^-3} = 2$$

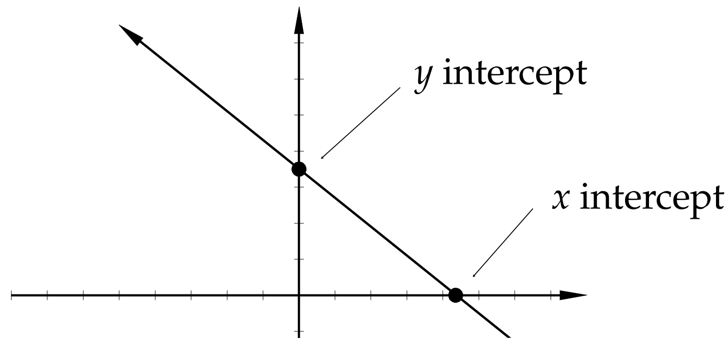


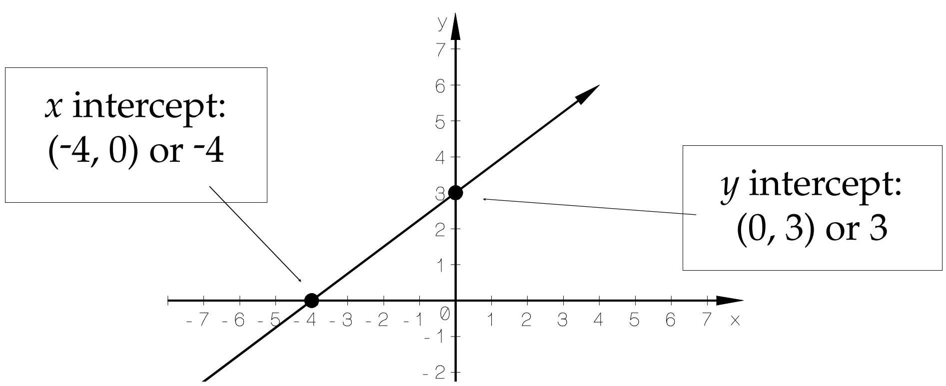
The Slope Between 2 Points (*a*, *b*) and (*c*, *d*)

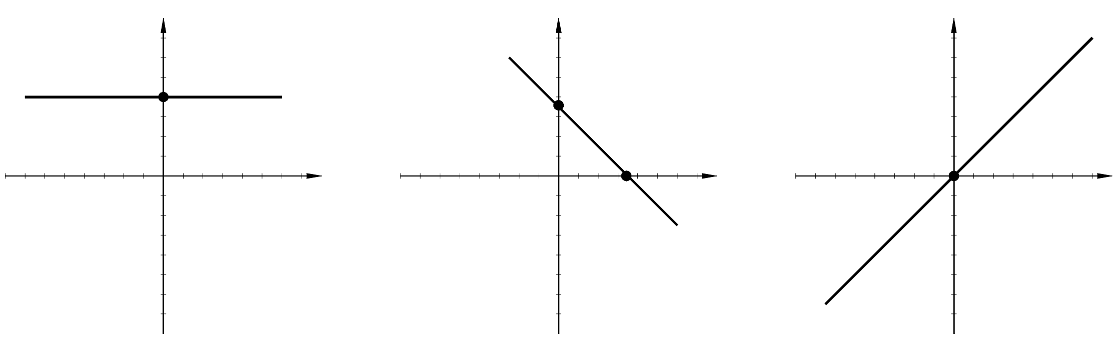
$$\textbf{Slope} = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } y}{\textbf{change in } x}$$

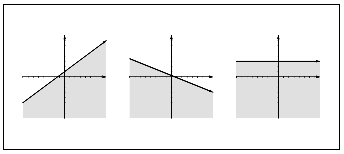
$$= \frac{d-b}{c-a} \textbf{ or } \frac{b-d}{a-c}$$

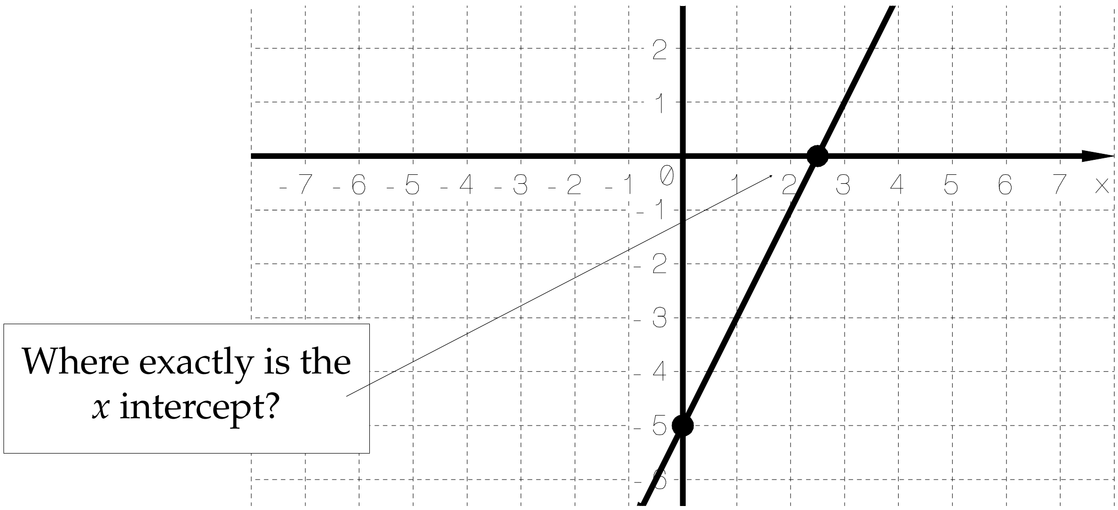



y intercept
x intercept

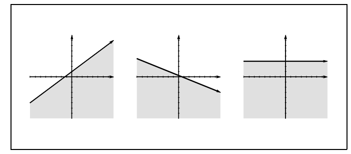

x intercept:
(⁻4, 0) or ⁻4
y intercept:
(0, 3) or 3
y
x



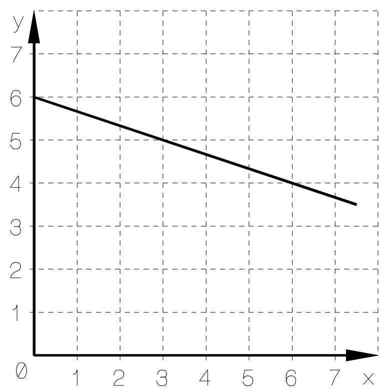




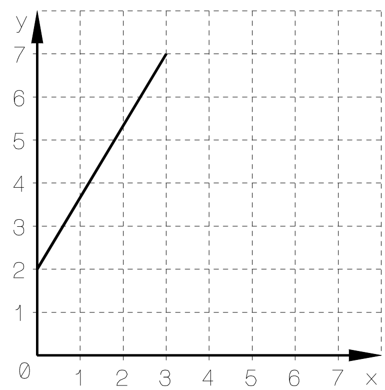
Where exactly is the x intercept?
- 7 - 6 - 5 - 4 - 3 - 2 - 1 0 1 2 3 4 5 6 7 x
2
1
- 1
- 2
- 3
- 4
- 5



1.

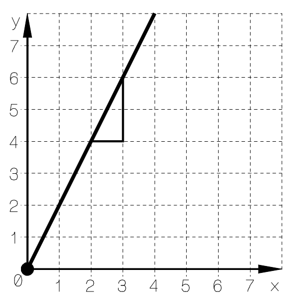


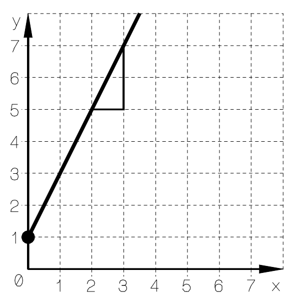
2.

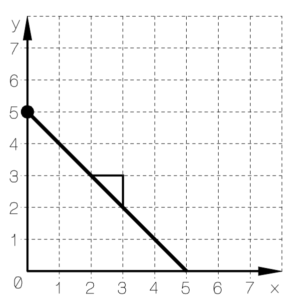


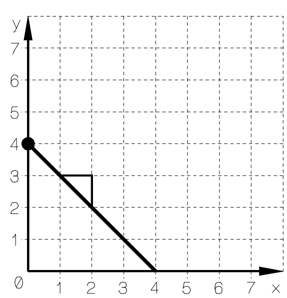
$y = 2x + 0$
Slope = 2
y intercept = 0

$y = 2x + 1$
Slope = 2
y intercept = 1



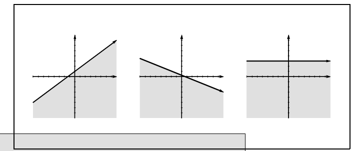






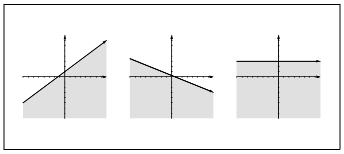
$y = {}^{-}1x + 5$
Slope = ${}^{-}1$
y intercept = 5

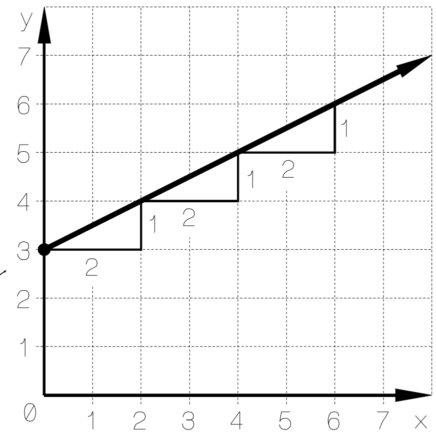
$y = {}^{-}1x + 4$
Slope = ${}^{-}1$
y intercept = 4



Graphing: Slope-Intercept Method

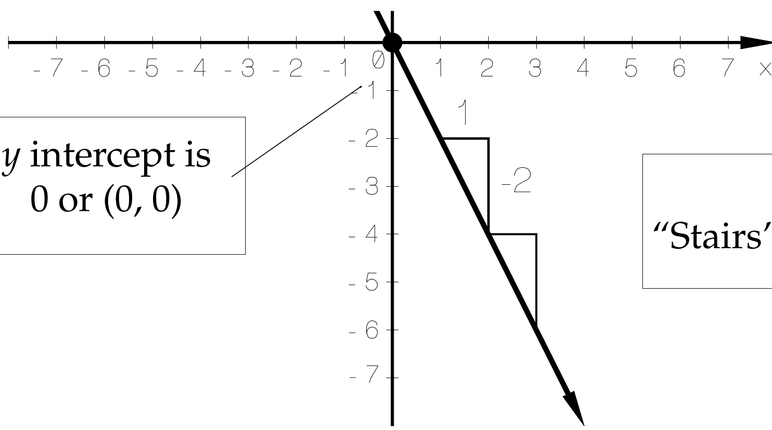
$y = mx + b$ m is slope, b is y intercept





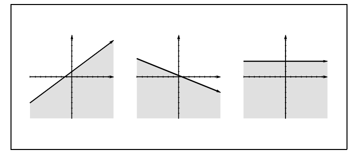
y intercept is
3 or (0, 3)

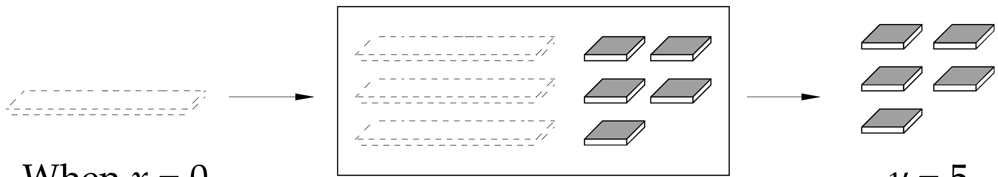
Slope is $\frac{1}{2}$
"Stairs" are 2 over
and 1 up



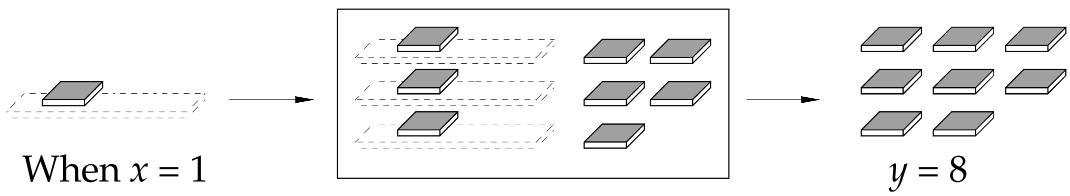
y intercept is
0 or (0, 0)

Slope is -2
"Stairs" are 1 over and 2 down

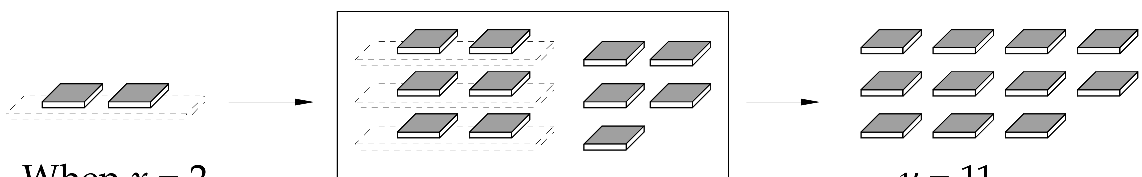




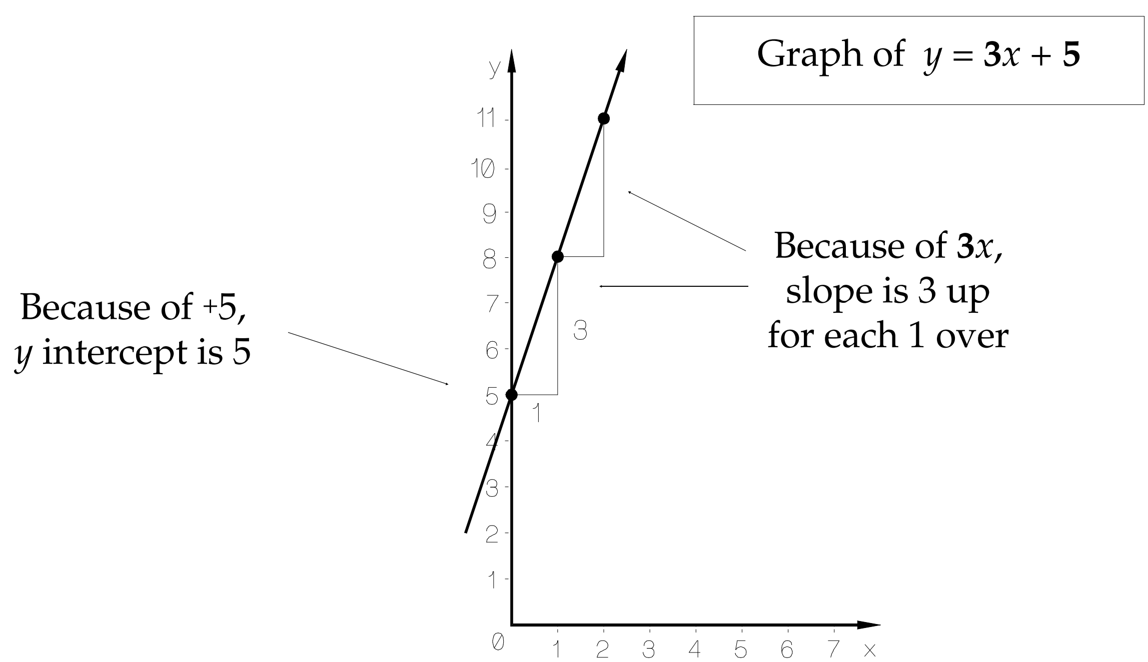
When $x = 0$ $y = 5$

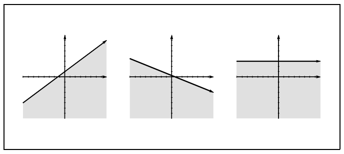


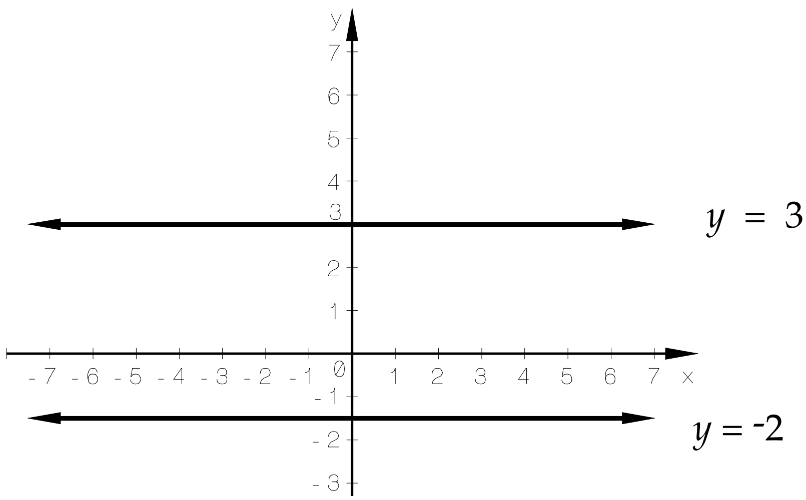
When $x = 1$ $y = 8$

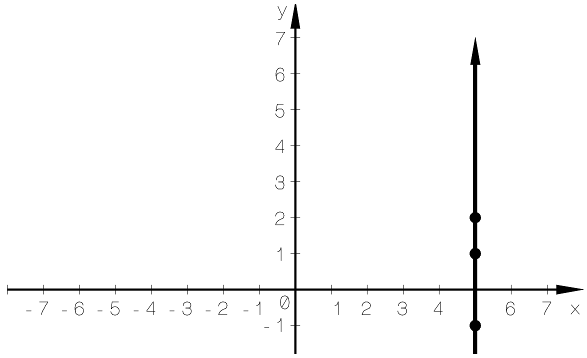


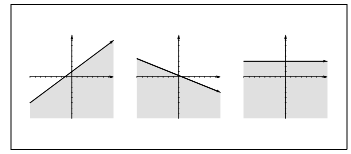
When $x = 2$ $y = 11$






y
7
6
5
4
3
2
1
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 x
-1
-2
-3
y = 3
y = -2

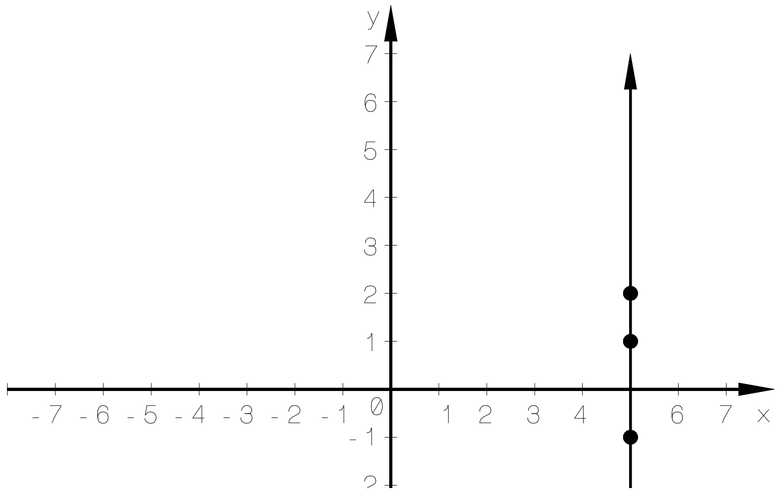

y
7
6
5
4
3
2
1
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 6 7 x
-1



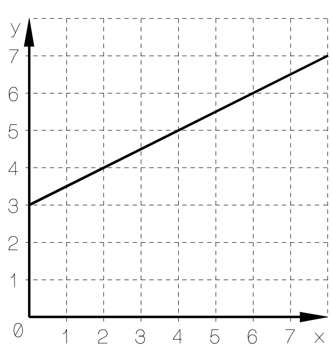
Vertical Line:

Rise is 3.
Run is 0.

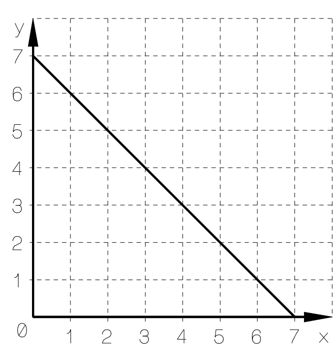
Slope is $\frac{\text{rise}}{\text{run}} = \frac{3}{0} = \textbf{?}$

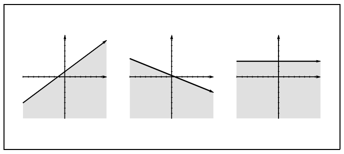


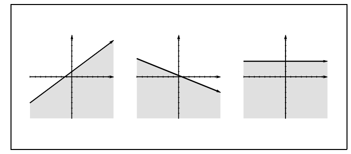
6.

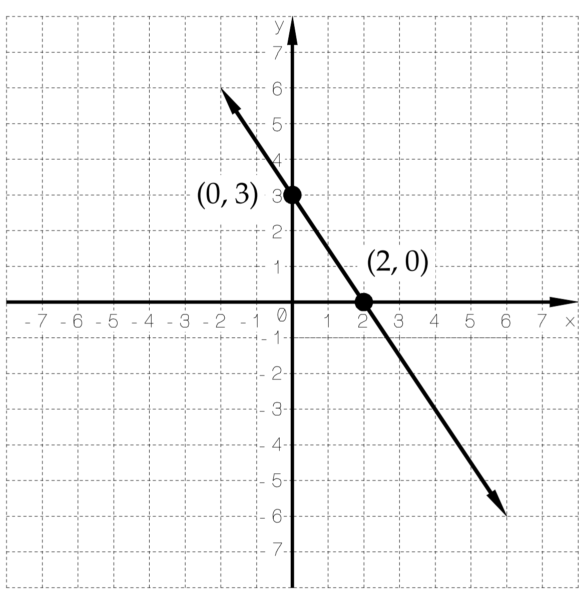


7.

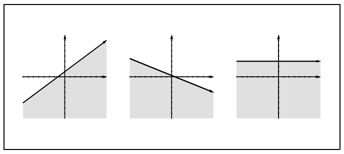




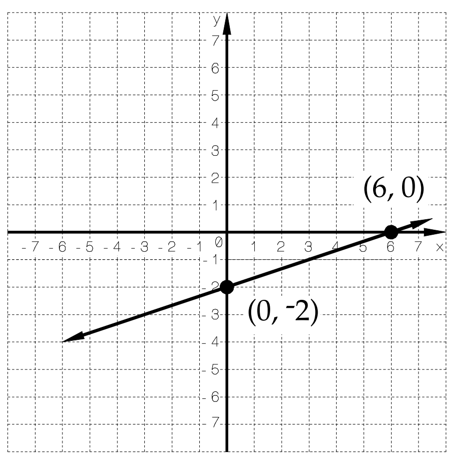


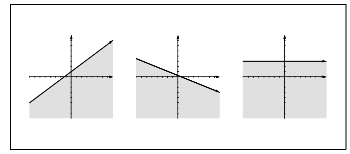


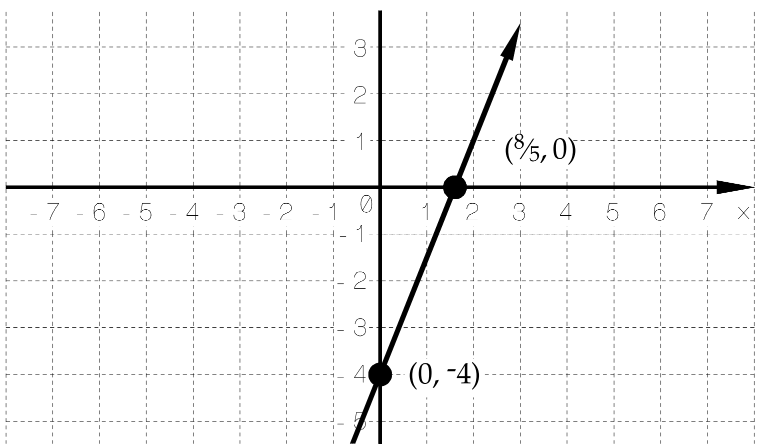
$$3x + 2y = 6$$



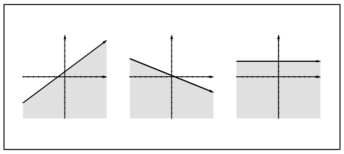
$$2x - 6y = 10$$

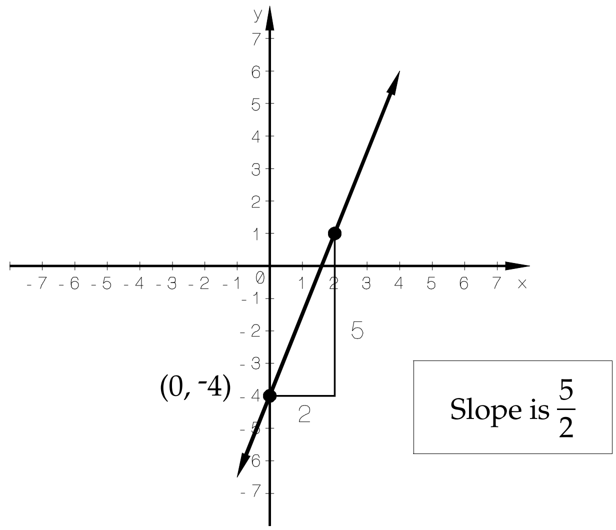






$5x - 2y = 8$





(0, -4)
5
2
Slope is $\frac{5}{2}$

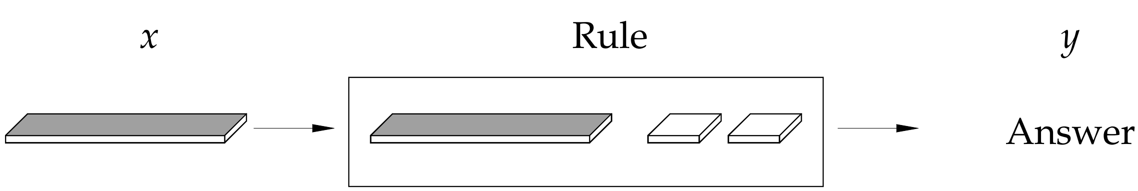
- **A *table* and *rule*:**

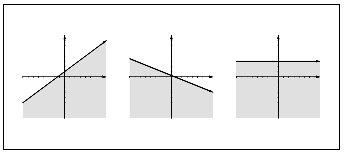
Rule: $y = x + {}^{-}2$

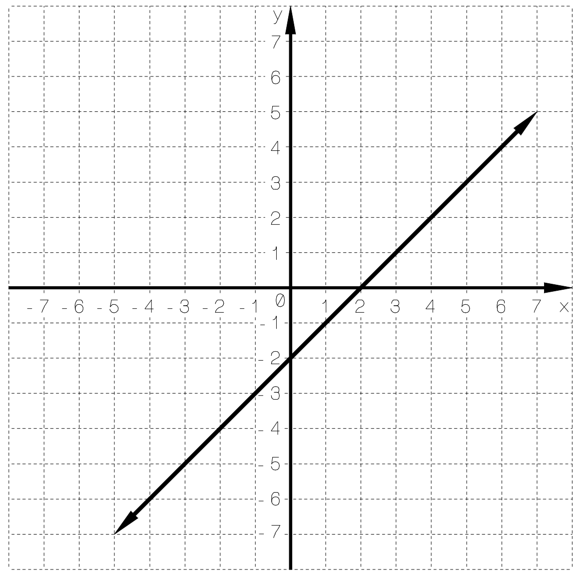
x	y
0	$^{-}2$
1	$^{-}1$
2	0
$^{-}1$	$^{-}3$

- **A list of *ordered pairs*:**

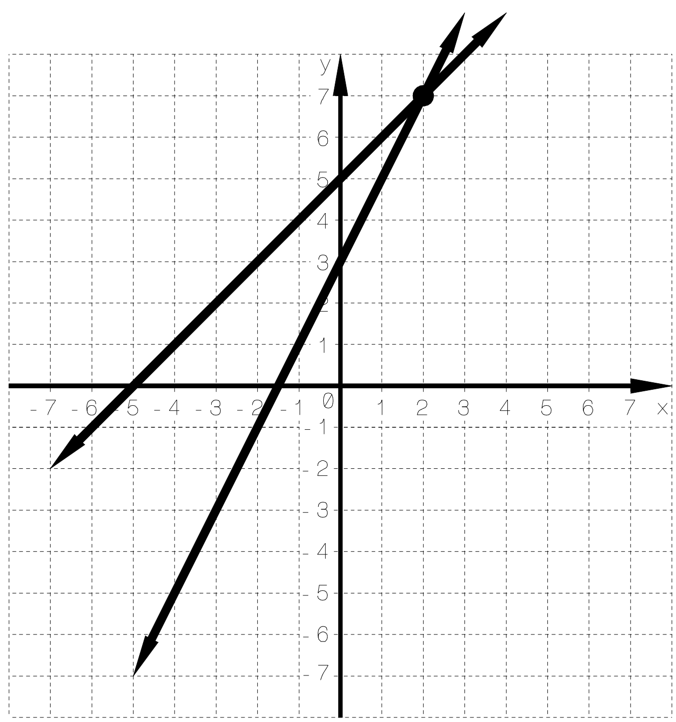
$(0, {}^{-}2), (1, {}^{-}1), (2, 0), ({}^{-}1, {}^{-}3)$






y
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 x

Chapter 13
Systems of Equations



Section 1
Equations and Solutions

Functions and Equations

In the last chapter, we looked at functions as charts, rules, lists of ordered pairs, maps, graphs, and machines. In this chapter, we will continue to develop ideas about ordered pairs, equations, and graphs.

An **equation** is a number statement having one or more unknowns and showing that two expressions are equal in value. For example:

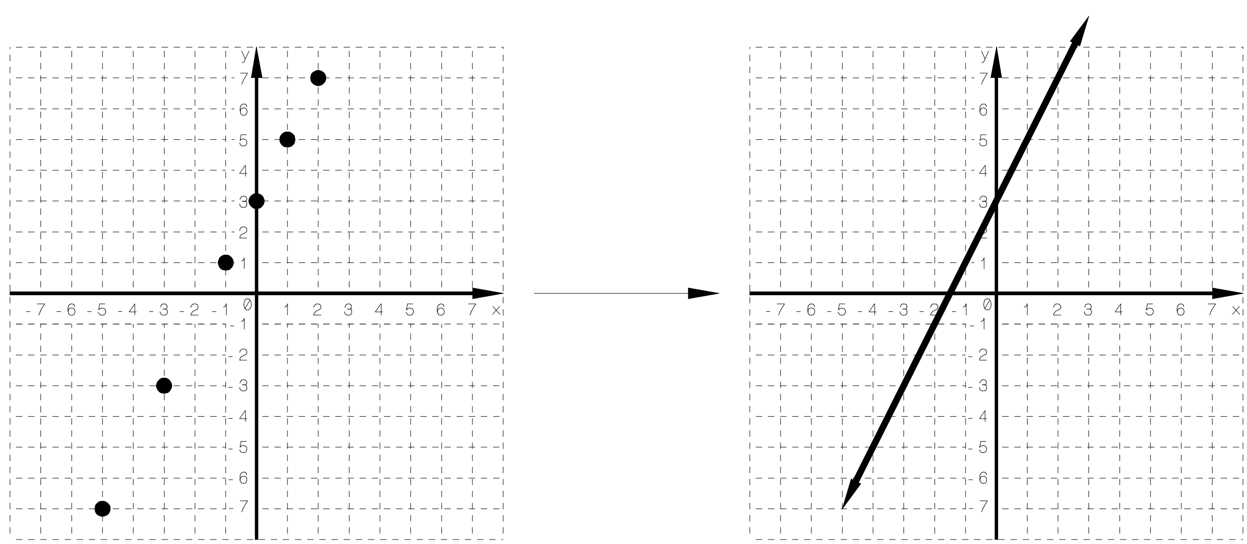
$$y = 2x + 3$$

This is a **linear equation in two unknowns.** The **solution** to this equation is not one number—it is an infinite list of ordered pairs:

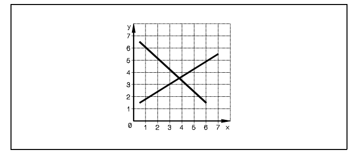
$$(0, 3), (1, 5), (^-1, 1), (1.5, 6), \ldots$$

On a graph, we can represent all of the answers by a line. Each point on the line is a solution to the equation.

Individual solutions | All solutions



A linear equation has x terms, y terms, or both. It may also have number terms. Equations having terms with x^2, y^2, $1/x$, $1/y$, or xy are not linear and will not have straight line graphs.



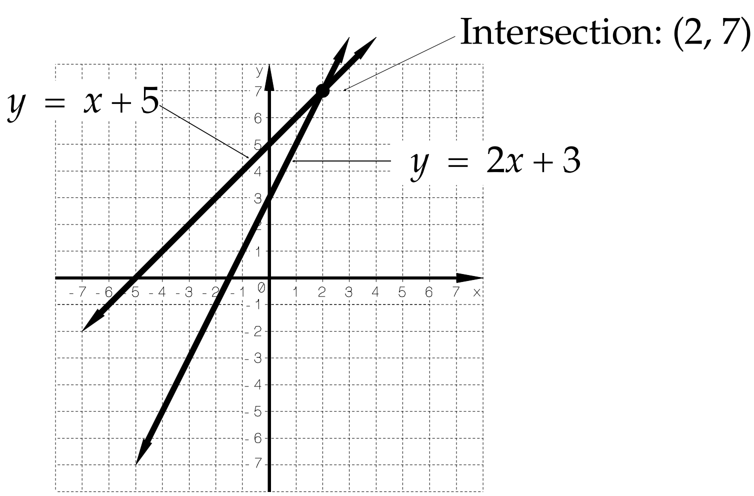
Linear	Not Linear
$y = 3x + 4$	$3 = 3$
$3x + y = 5 + x$	$xy + 3x = 4$
$y = 2$	$x^2 + 3x + 4 = 0$
$x - 2 = 0$	$y + \frac{1}{x} = 3$

Pairs of Equations

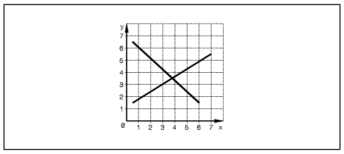
In many real-world situations, we encounter linear equations in related-pairs. A **system of equations** is a group of equations where we are looking for a common solution. The **solution** or **solution set** is one or more ordered pairs that satisfy *all* equations. Here is a system of two equations:

$$y = 2x + 3$$
$$y = x + 5$$

Since a single equation is an instruction to find all possible ordered pairs that make the equation true, *a group of two equations gives us the task of finding the ordered pair(s) that make* ***both*** *equations true at the same time*. Because the equations represent lines, we are looking for the point that is on both lines; this is the place where the graphs cross:



From the graph, we can see that *both* of these equations are satisfied at the point (2, 7) where the two lines cross. The values $x = 2$ and $y = 7$ will make both equations true, so this ordered pair is the solution of the system of equations.



We can confirm that (2, 7) is the solution by testing these x and y values in *both* equations:

$y = 2x + 3$	$y = x + 5$
$(7) = 2(2) + 3$	$(7) = (2) + 5$
$7 = 4 + 3$	$7 = 2 + 5$
$7 = 7$	$7 = 7$

Here is another example of a system of equations:

$$y = x$$
$$x + y = 6$$

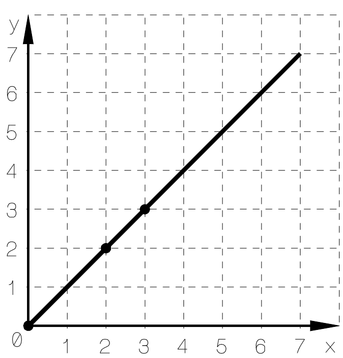
By trial and error, we could list some solutions (ordered pairs) for each equation and hope to find a pair that works in both equations:

For $y = x$, solutions are (0, 0), (2, 2), **(3, 3)**

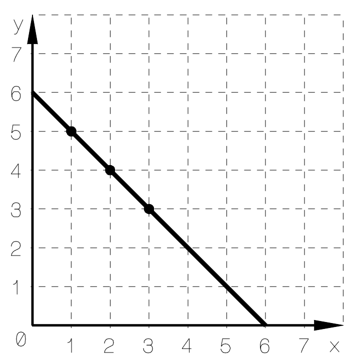
For $x + y = 6$, solutions are (1, 5), (2, 4), **(3, 3)**

The common solution is (3, 3). On the graph, we would see the answer as the intersection of the two lines:

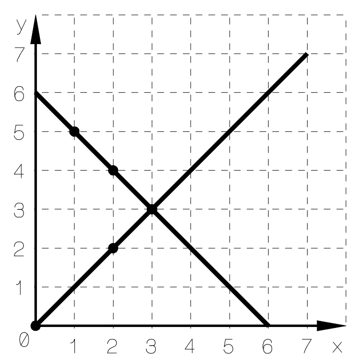
$y = x$



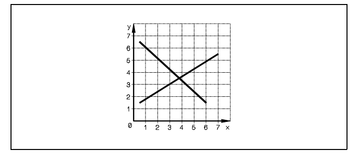
$x + y = 6$



Both equations

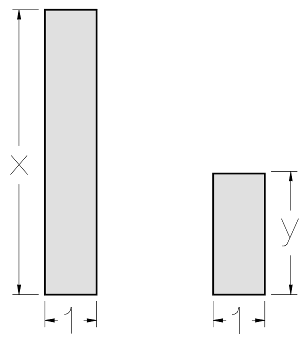


Although trial and error may work to find solutions for some easy situations, it will obviously be a poor way to find solutions for many system of equations; the rest of this chapter will cover several different ways to find the solution in a more efficient manner.



Bars for *x* and *y*

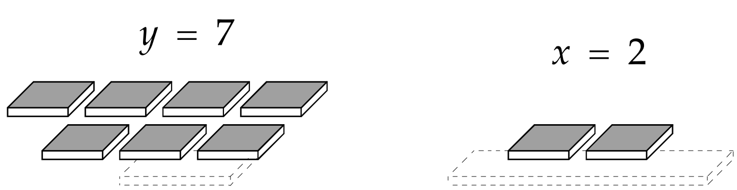
In the previous chapter, we created a new bar for y. We will continue to use the different bars for x and y to help us solve systems of equations. To review, the two bars look like this:

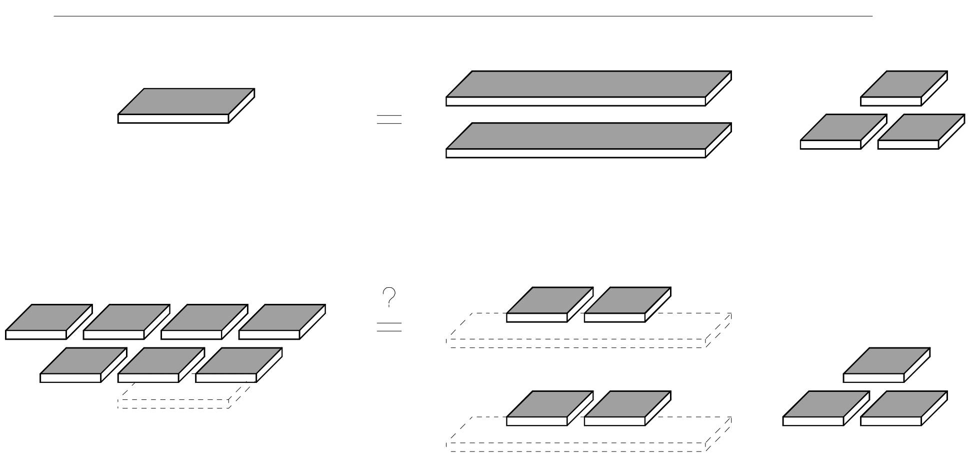


We will now use these bars to help us check the solutions we have already obtained for the two systems of equations given above. For the first system:

$$y = 2x + 3$$
$$y = x + 5$$
Solution: (2, 7) or $x = 2$, $y = 7$

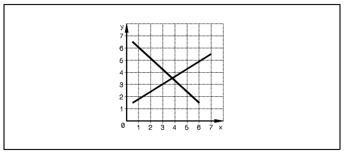
To verify the solution, take the number for x and put that many unit chips on each x bar. Do the same for y. If both sides balance in *each* equation, the solution is correct:



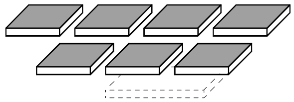


$$(y) = 2(x) + 3$$
$$(7) = 2(2) + 3$$

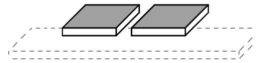
$$7 = 7$$



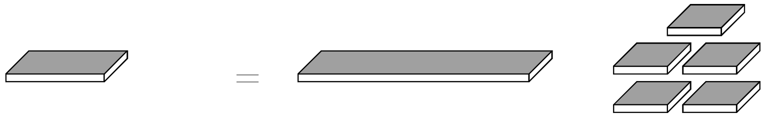
$y = 7$



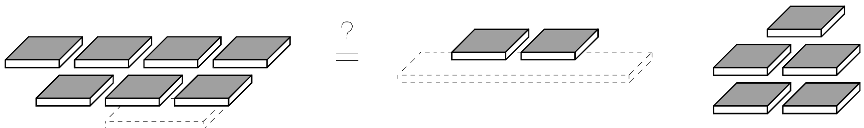
$x = 2$



$(y) = (x) + 5$
$(7) = (2) + 5$

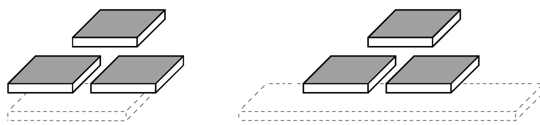


$7 = 7$



For the second system we have discussed ($y = x$ and $x + y = 6$):

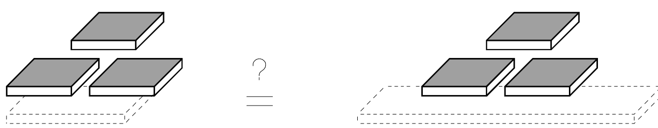
$y = 3$



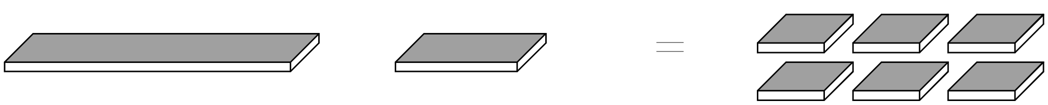
$x = 3$

$(y) = (x)$

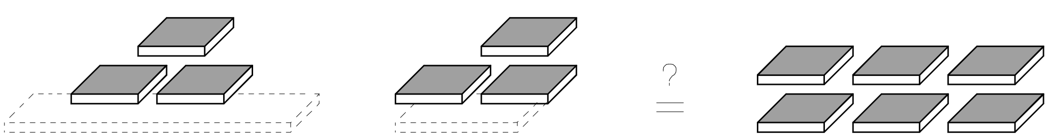
$3 = 3$



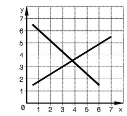
$(x) + (y) = 6$



$3 + 3 = 6$



Exercises



Find three solutions (ordered pairs) to each equation:

1. $y = x + 1$

2. $y = 2x - 3$

Use the chips to verify that the given ordered pair is a solution to the given system of equations:

3. $y = x + 1$ (4, 5)
$y = 2x - 3$

4. $2x + y = 12$ (4, 4)
$y = x$

5. $y = x + 2$ (2, 4)
$y = 6 - x$

6. $2x + y = 8$ (3,2)
$y = x - 1$

7. $y = 2x + 1$ (1, 3)
$y = {}^-x + 4$

8. $x + 2y = 3$ (3, 0)
$y = x - 3$

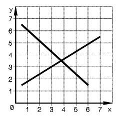
9. $y = 3x - 5$ (3, 4)
$y = x + 1$

10. $3x - y = 1$ (4, 11)
$y = 2x + 3$

Verify the solution without using the chips:

11. $2x - y = 0$ (0, 0)
$y = 3x$

12. $3x - y = 11$ $(3, {}^-2)$
$y = x - 5$



13. $2x - y = 1$ $\quad (2, 3)$

$x = y - 1$

14. $y = x + 1$ $\quad (1, 2)$

$x + y = 3$

15. $2x + 2y = 24$ $\quad (4, 8)$

$3x - y = 4$

16. $x = 8 - 4y$ $\quad ({}^-4, 3)$

$3x + 5y = 3$

Section 2
Solving by Graphing

Graphing the System

One method of solving a system of equations is to put the two equations on the same coordinate graph. Since we are dealing with lines, the lines will usually meet in one point. That point represents the ordered pair (x, y) that works for both equations at the same time. For example, consider the following system:

$$y = x + 1$$
$$y = 7 - x$$

First we make a table of 3 or more ordered pairs that satisfy each equation:

$y = x + 1$

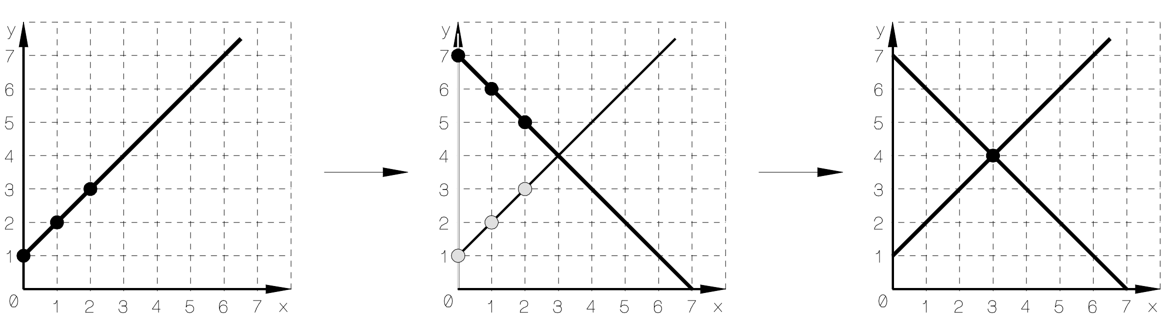
x	y
0	1
1	2
2	3

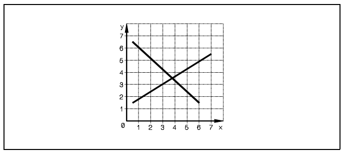
$y = 7 - x$

x	y
0	7
1	6
2	5

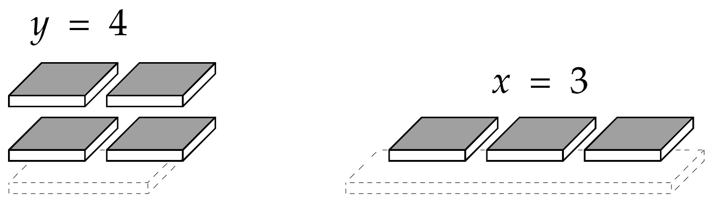
We then graph each line and find the point of intersection. This is our solution:

$y = x + 1$ $\quad$ $y = 7 - x$ $\quad$ Solution is (3, 4)

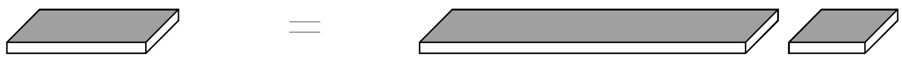




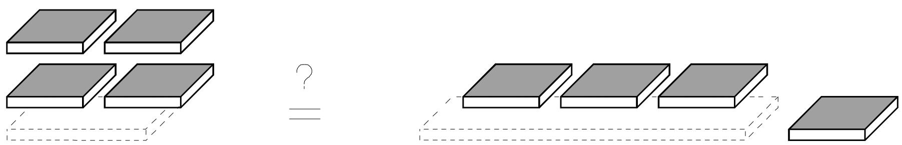
Finally, we must verify that the solution is correct. We test the solution (3, 4) by substituting $x = 3$ and $y = 4$ into each equation separately. If both equations make true statements for these values, our solution is correct:



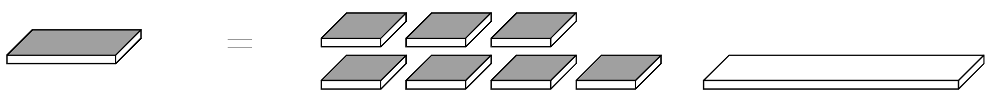
$$(y) = (x) + 1$$



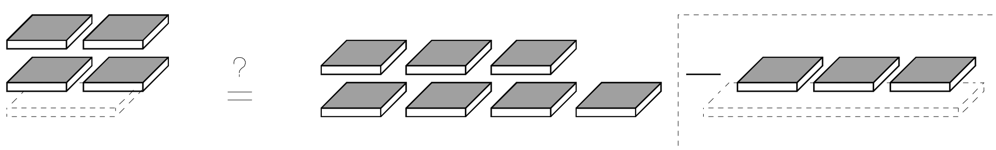
$$(4) = (3) + 1$$

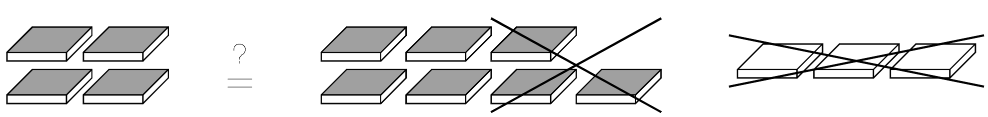


$$(y) = 7 - (x)$$

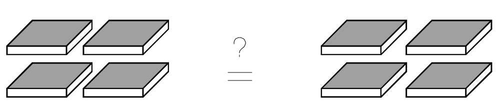


$$(4) = 7 - (3)$$

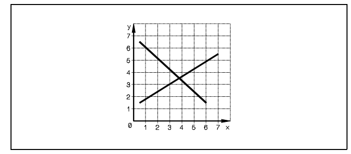




$$4 = 4$$



We will do another example to illustrate the graphing method. Consider the system show below:

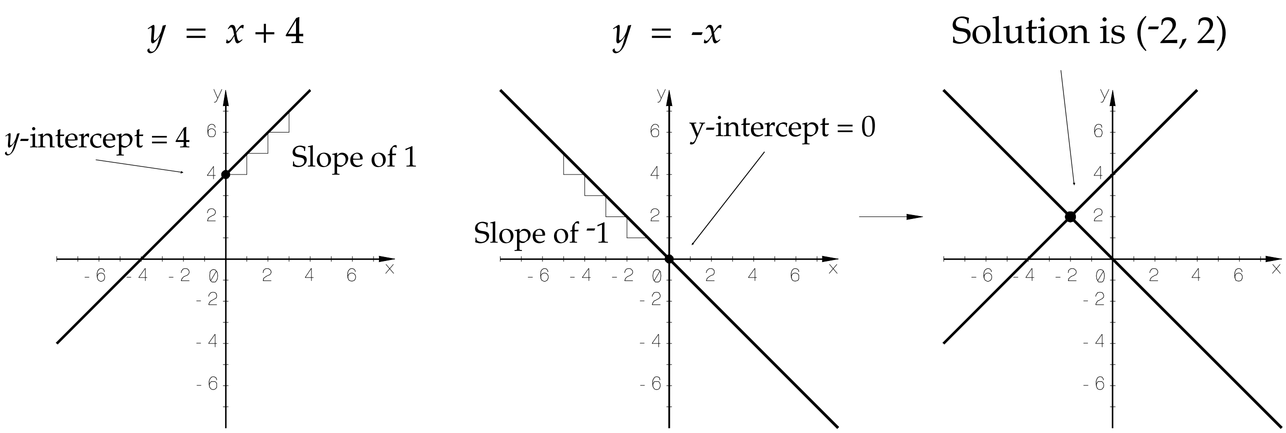


$$\begin{aligned} y - x &= 4 \\ x + y &= 0 \end{aligned}$$

Our first step is to solve each equation for y. To do this, we add and multiply on both sides of each equation until we "isolate" y. (See RULES AND GRAPHS, Section 4).

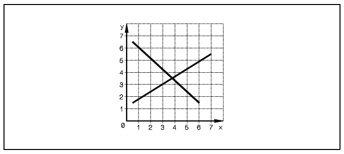
$\mathbf{y - x = 4}$	$\mathbf{x + y = 0}$
$y - x = 4$	$x + y = 0$
$+x \quad +x$	$-x \qquad -x$
$y = x + 4$	$y = -x$
$y = \mathbf{1}x + \mathbf{4}$	$y = {}^{-}\mathbf{1}x + \mathbf{0}$
Slope = 1; y-intercept = 4	Slope = $^{-}1$; y-intercept = 0

We now have two equations in the slope-intercept form. We plot both equations on the same graph:



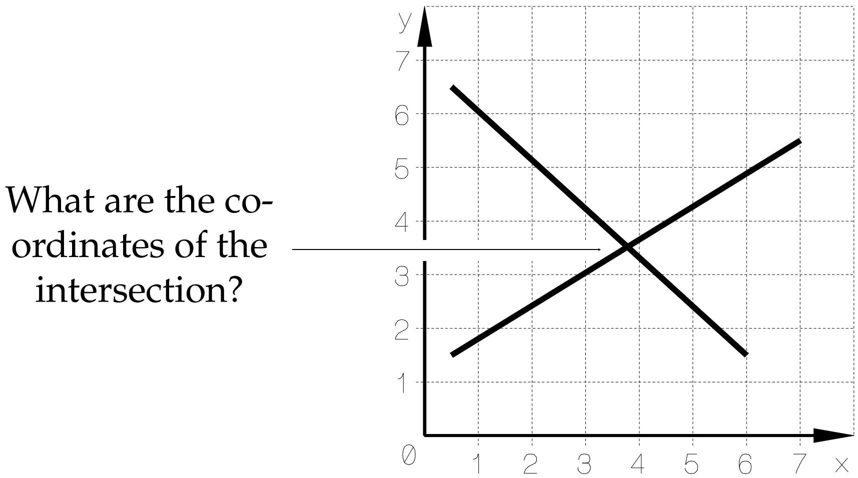
Finally we see the solution is ($^{-}2$, 2) and we verify it in *both* equations:

$\mathbf{y - x = 4}$	$\mathbf{x + y = 0}$
$(y) - (x) = 4$	$(x) + (y) = 0$
$(2) - ({}^{-}2) = 4$	$({}^{-}2) + (2) = 0$
$4 = 4$	$0 = 0$

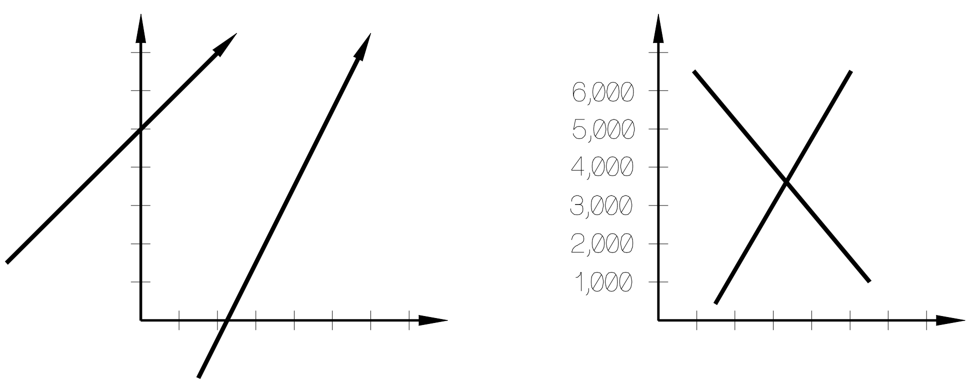


Limitations of the Graphing Method

The graphing method gives us a clear picture of the solution to a system of equations, but this method has some important limitations. First, it may take a great deal of time to graph the lines, especially if the numbers are large or the slopes are fractional amounts. Second, it is not always easy to tell where the lines actually meet; if the intersection is not on a place where the grid lines meet, then we have to guess at a fractional answer:

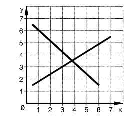


Finally, the lines may meet at a place that has such large coordinates that it is impractical to graph them at all.



In the sections that follow, we will learn two alternative techniques to supplement the graphing method for finding solutions to a system of equations. These new methods will rely on algebraic symbols instead of graphing lines.

Summary



The steps for solving a system of equations using a graph are as follows:

- **If necessary, solve each equation for *y*.**
- **Graph the equations by plotting 3 points for each line or by using the slope-intercept method.**
- **Read the solution—the point at the intersection of the lines.**
- **Check the solution by substituting the x and y values for the point of intersection into *both* of the original equations to be sure that both equations give true statements.**

Exercises

Graph each system and find the ordered pair that is the solution. Check x and y in both equations.

1. $x + y = 3$
$y = x - 5$

2. $y = x + 1$
$y = 2x - 3$

3. $x + 2 = y$
$x + y = 2$

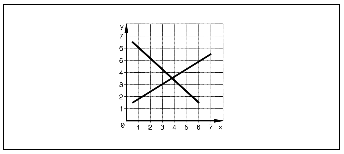
4. $y = x - 3$
$y = {}^-2x + 6$

5. $x + 2y = 5$
$x - y = 2$

6. $x + y = {}^-1$
$y = {}^-5 + x$

7. $y = x$
$2x + y = 6$

8. $y = x + 1$
$y = 2x - 1$



9. $2x - y = 4$

$2x + y = {}^{-}4$

10. $x - y = {}^{-}1$

$x + y = 3$

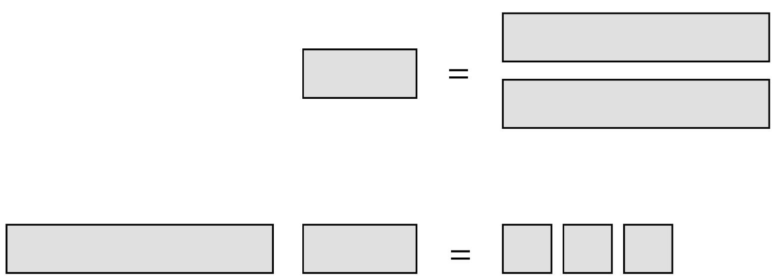
Section 3
The Substitution Method

Using the Equations Together

Another approach to solving systems of equations is called **substitution.** The substitution method gives us the exact solution, even if the values of x and y are fractional or very large. Substitution doesn't give us a picture of the equations; it just gives us the value of the solution. We will use information from both equations to determine the solution. For example, consider this system of equations:

$$\begin{aligned} y &= 2x \\ x + y &= 3 \end{aligned}$$

Using our new y bar, here is a picture of this system:

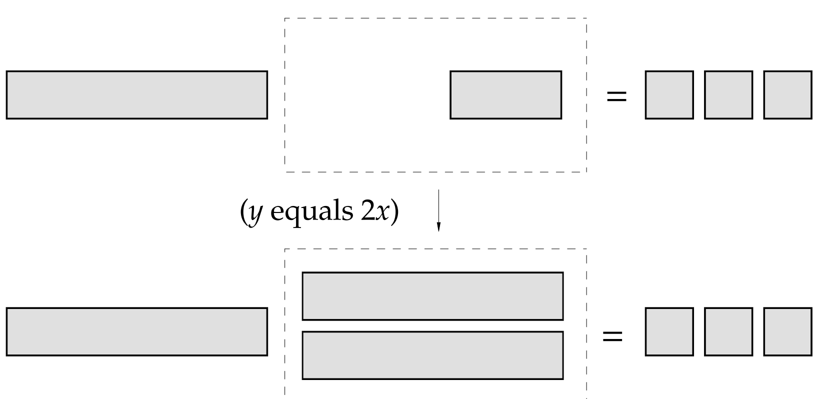


$y = 2x$

$x + y = 3$

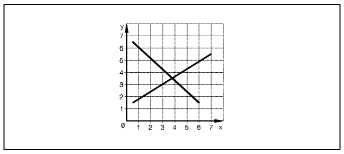
If the system has a solution, then the x and y values of that solution will work (make true statments) in both equations.

The first equation states that y is equal to $2x$, so y and $2x$ stand for the same amount and we can *substitute* $2x$ instead of y in the second equation:



$x + (y) = 3$

$x + (2x) = 3$



This substitution gives a new equation:

$$x + (2x) = 3$$

We can solve this equation because it only contains the one variable x. Here is how we complete the solution with our chips:

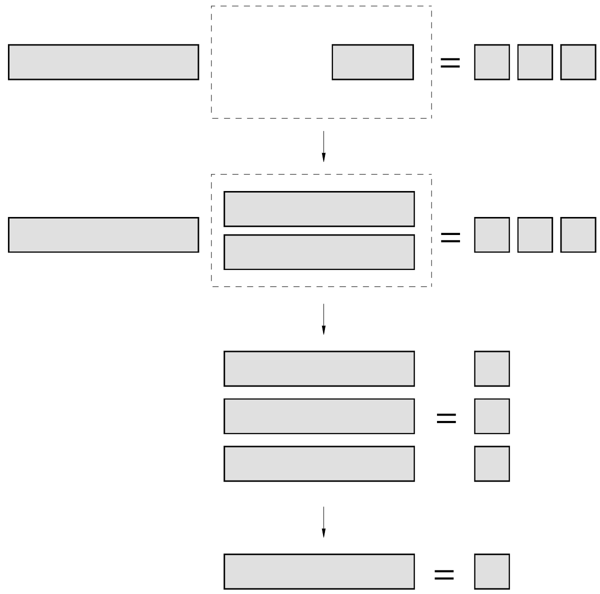
$$x + (y) = 3$$

$(y = 2x)$

$$x + (2x) = 3$$

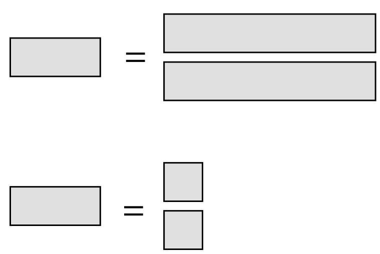
$$3x = 3$$

$$x = 1$$



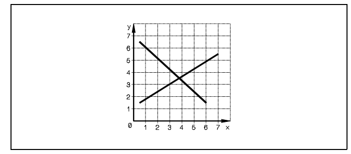
Since we know that x must be 1, our next step is to use this information to help us find y:

$$\begin{aligned} y &= 2(x) \\ y &= 2(1) \\ y &= 2 \end{aligned}$$



Here is the process with symbols alone:

$$\begin{aligned} y &= 2x \\ x + (y) &= 3 \\ \\ x + (2x) &= 3 \\ 3x &= 3 \\ \frac{1}{3}(3x) &= \frac{1}{3}(3) \\ x &= 1 \end{aligned}$$

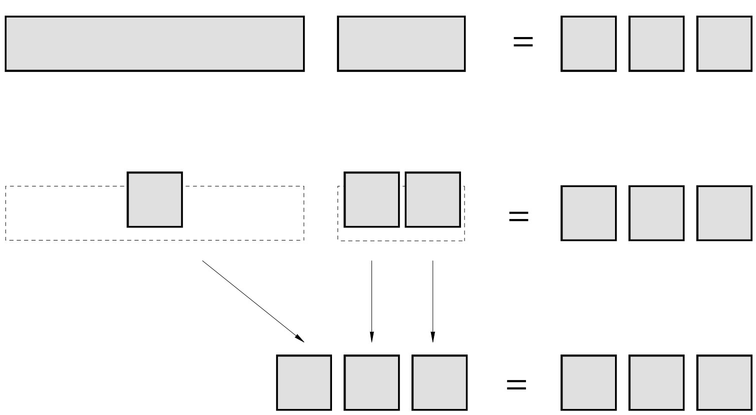


Completing the solution for y:

$$y = 2x$$
$$y = 2(1) \quad \text{(because } x = 1\text{)}$$
$$y = 2$$

Our answer is $x = 1$, $y = 2$. We can also write this as $(1, 2)$.

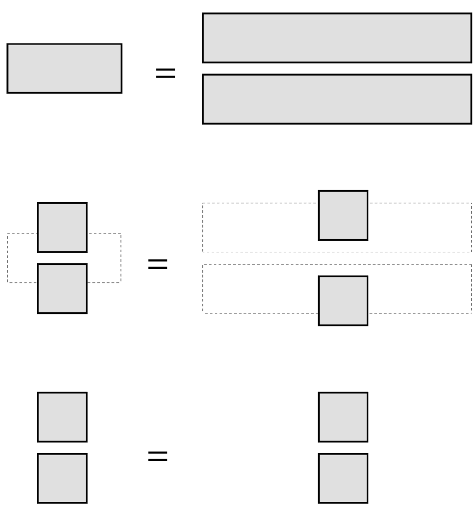
Finally, we check our solution by replacing the x and y in *both* equations with 1 and 2:



$$(x) + (y) = 3$$

$$(1) + (2) = 3$$

$$3 = 3$$



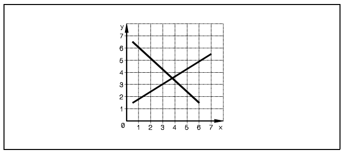
$$(y) = 2(x)$$

$$(2) = 2(1)$$

$$2 = 2$$

More Examples

We will now look at two more examples. We solve each system in the same way, by substituting the value of one variable (from one equation) into the second equation.



Here is the first system:

$$\begin{aligned} y &= 2x + 1 \\ 2x + y &= 9 \end{aligned}$$

$y = 2x + 1$

$2x + y = 9$

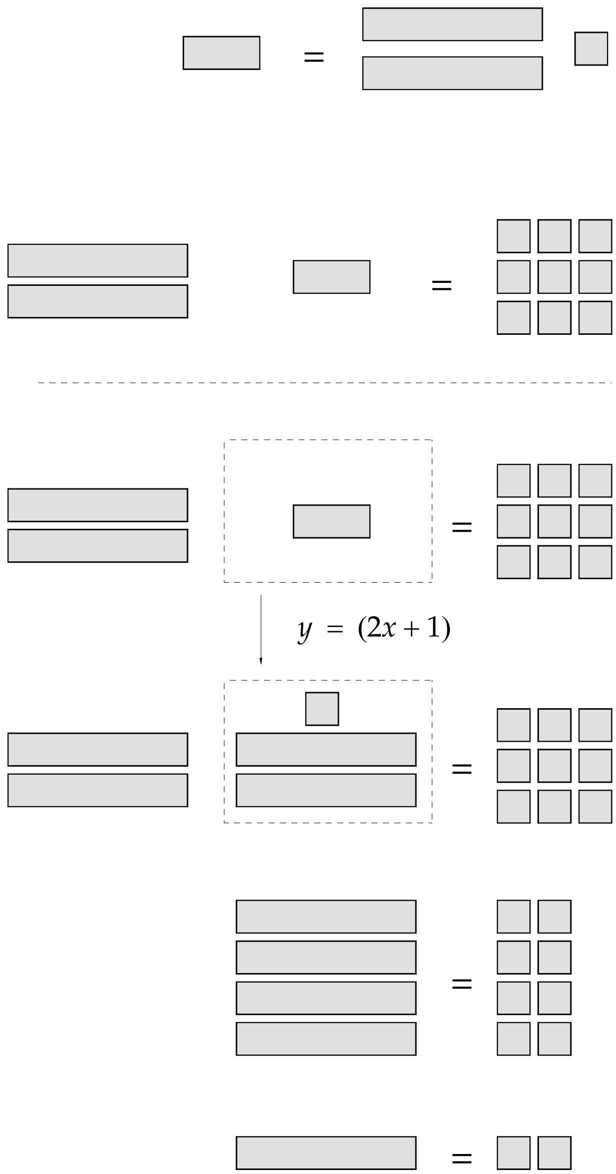
$2x + (y) = 9$

(substitute for y)

$$\begin{aligned} 2x + (2x + 1) &= 9 \\ 4x + 1 &= 9 \end{aligned}$$

$4x = 8$

$x = 2$

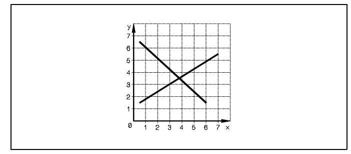


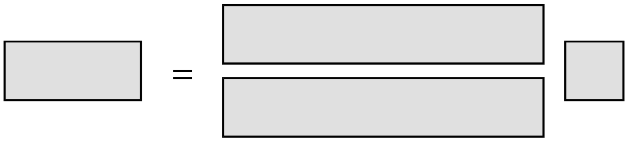
Now that we know x, we can continue on to solve for y:

$$\begin{aligned} y &= 2(x) + 1 \\ y &= 2(2) + 1 \quad \text{(because } x = 2\text{)} \\ y &= 5 \end{aligned}$$

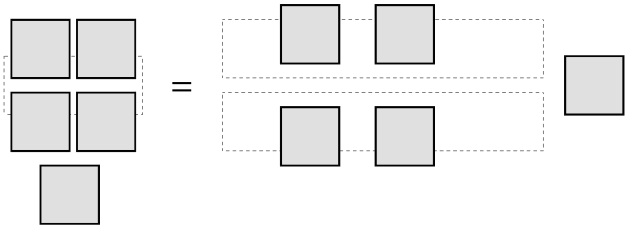
The solution is $x = 2$ and $y = 5$, or $(2, 5)$

Here is the check:

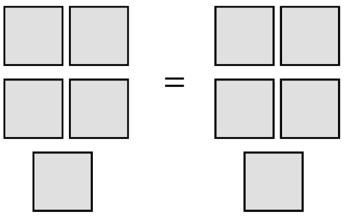




$(y) = 2(x) + 1$

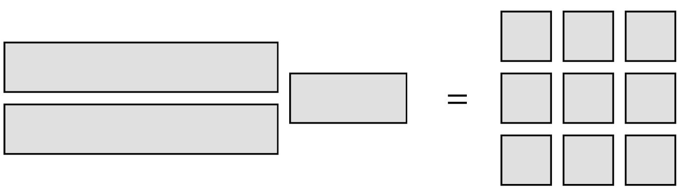


$(5) = 2(2) + 1$

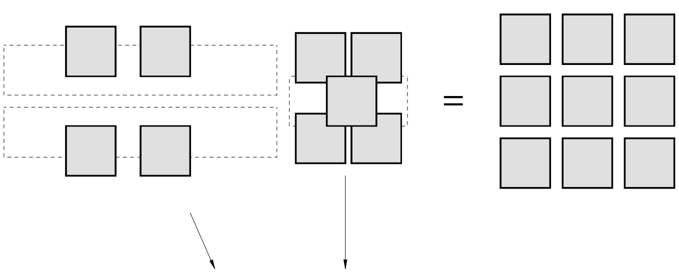


$5 = 5$

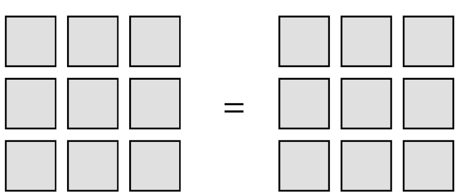
The check continues:



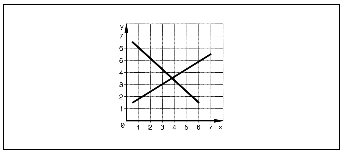
$2(x) + (y) = 9$



$2(2) + (5) = 9$



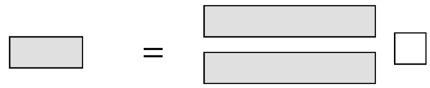
$9 = 9$



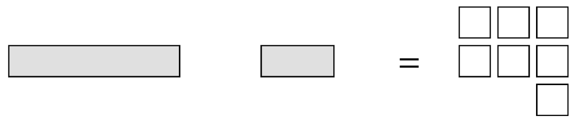
Here is a second example of a system of equations to solve:

$$y = 2x - 1$$
$$x + y = {}^{-}7$$

$y = 2x - 1$



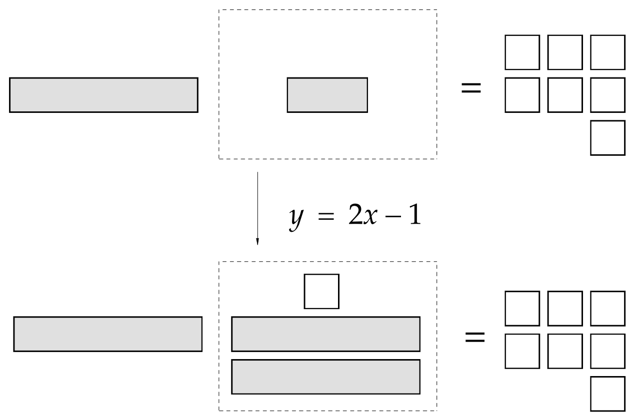
$x + y = {}^{-}7$



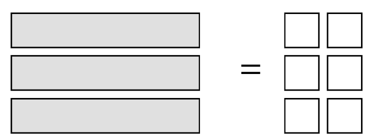
$x + (y) = {}^{-}7$

(substitute for y)

$x + (2x - 1) = {}^{-}7$
$3x - 1 = {}^{-}7$



$3x = {}^{-}6$

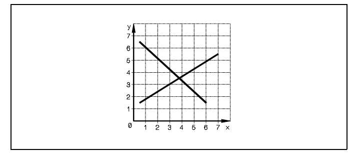


$x = {}^{-}2$

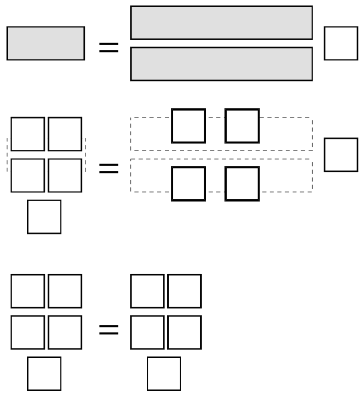
Now that we know x is $^{-}2$, we can continue on to solve for y:

$$y = 2(x) - 1$$
$$y = 2({}^{-}2) - 1 \quad \text{(because } x = {}^{-}2\text{)}$$
$$y = {}^{-}5$$

The solution is $x = {}^{-}2$ and $y = {}^{-}5$, or $({}^{-}2, {}^{-}5)$



Here is the check. For the first equation:

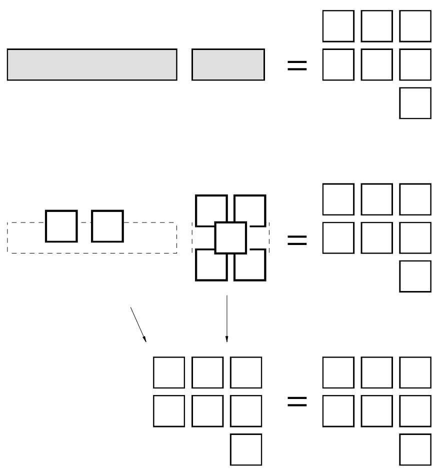


$$(y) = 2(x) - 1$$

$$(^-5) = 2(^-2) - 1$$

$$^-5 = {}^-5$$

The second equation:



$$(x) + (y) = {}^-7$$

$$(^-2) + (^-5) = {}^-7$$

$$^-7 = {}^-7$$

Working with Fractions

Systems of equations can contain fractions. When we solve for one unknown and get a fraction, we substitute the answer in the second problem in the same way as we did before:

$$y = \frac{1}{2}x \quad \rightarrow \quad y = \left(\frac{1}{2}x\right)$$

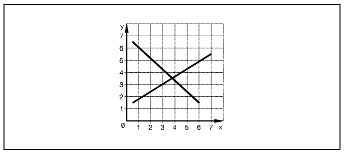
$$2x + 4y = 8 \quad \rightarrow \quad 2x + 4(y) = 8$$

$$2x + 4\left(\frac{1}{2}x\right) = 8$$

$$2x + 2x = 8$$

$$4x = 8$$

$$x = 2$$



We now return the x value (2) into the original equation and solve for y.

$$y = \frac{1}{2}x$$

$$y = \frac{1}{2}(2) \quad \text{(because } x = 2\text{)}$$

$$y = 1$$

The answer is (2, 1)

Working with Negatives

When one of the equations has a negative number of y's, we must be careful to remember that -y means the opposite of y. For example:

$$y = 3x$$

$$x - y = 2$$

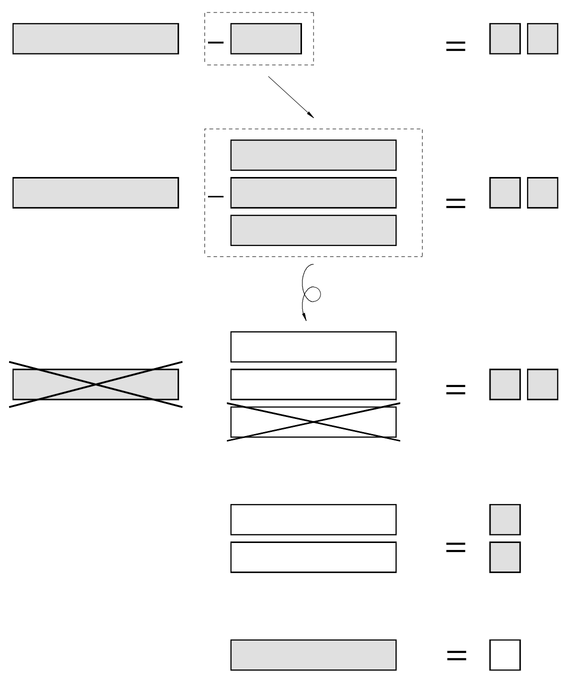
$$x - (y) = 2$$

$(y = 3x)$

$$x - (3x) = 2$$

$$^-2x = 2$$

$$x = {}^-1$$



We now have the value of x, so we can continue on in the usual way:

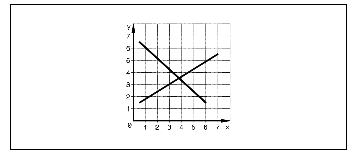
$$y = 3(x)$$

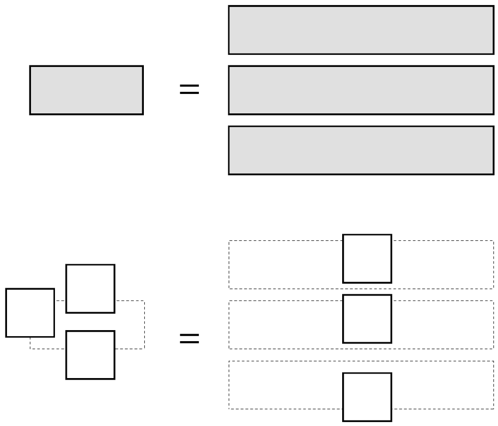
$$y = 3(^-1) \quad \text{(because } x = {}^-1\text{)}$$

$$y = {}^-3$$

The solution is ($^-1$, $^-3$)

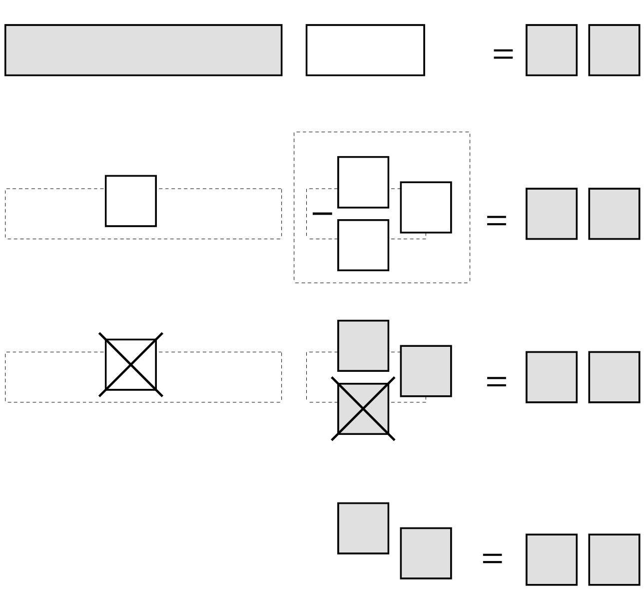
Notice that we substituted $3x$ for y, but that y was being subtracted, so the $3x$ was flipped to its opposite. When we check, we must also be careful; since y is $^{-}3$, we substitute $^{-}3$ in for y but do not ignore the negative sign:





$$(y) = 3(x)$$

$$(^{-}3) = 3(^{-}1)$$
$$^{-}3 = {}^{-}3$$



$$(x) - (y) = 2$$

$$(^{-}1) - (^{-}3) = 2$$

$$^{-}1 + 3 = 2$$

$$2 = 2$$

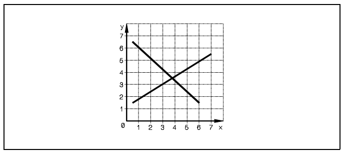
Different Forms

For simplicity, we have been using examples where one equation is already solved for y in terms of x. Systems may be in other forms where one equation is solved for x or where neither equation is in the desired form.

For example, here is a system where the second equation is solved for x:

$$2x - 3y = 7$$
$$x = 2y + 3$$

We can substitute from either equation at the beginning, so we choose the



second equation; since it is solved for x, we substitute $2y + 3$ in for x in the first equation:

$$2(x) - 3y = 7$$

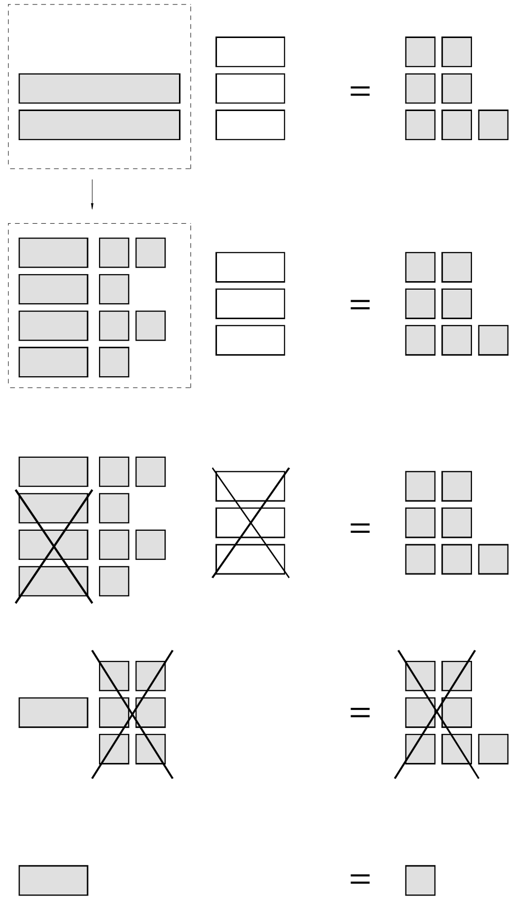
$$\downarrow \quad (x = 2y + 3)$$

$$2(2y + 3) - 3y = 7$$

$$4y + 6 - 3y = 7$$

$$y + 6 = 7$$

$$y = 1$$



Continuing on to solve for x:

$$x = 2(y) + 3$$
$$x = 2(1) + 3 \quad (\text{because } y = 1)$$
$$x = 5$$

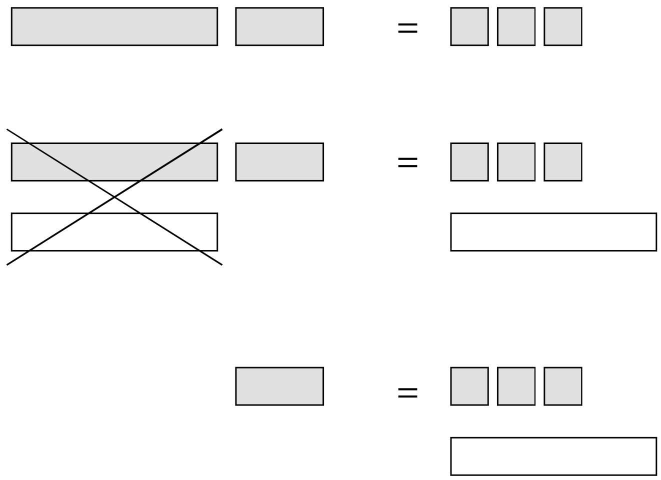
The solution is (5, 1)

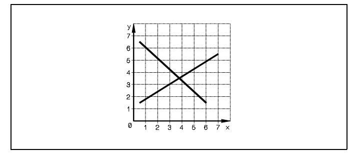
Changing the Form

If neither equation is solved for x or y, we use the technique from RULES AND GRAPHS, Section 4. We pick one equation and rearrange the terms by adding and multiplying both sides until we have the equation solved for either x or y. For example, given the equations:

$$x + y = 3$$
$$x - y = 1$$

Our first step is to solve one equation for y; choose $x + y = 3$:





$$\begin{aligned} x + y &= 3 \\ -x \quad & \quad -x \\ \hline y &= -x + 3 \end{aligned}$$

We continue on by substituting for y in the same manner as before:

$$\begin{aligned} y &= (-x + 3) \\ x - (y) &= 1 \end{aligned}$$

$$\begin{aligned} x - (-x + 3) &= 1 \\ x + x - 3 &= 1 \\ 2x - 3 &= 1 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} y &= -(x) + 3 \\ y &= {}^{-}(2) + 3 \\ y &= 1 \end{aligned}$$

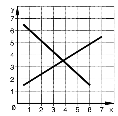
The solution is (2, 1).

Here is a more difficult example:

$$\begin{aligned} 2x + 3y &= 8 \\ 3x + 3y &= 9 \end{aligned}$$

We solve the first equation for y:

$$\begin{aligned} 2x + 3y &= 8 \\ 2x - 2x + 3y &= 8 - 2x \\ 3y &= 8 - 2x \\ y &= \frac{8}{3} - \frac{2}{3}x \end{aligned}$$



We continue by substituting the expression for y in the second equation:

$$y = \left(\frac{8}{3} - \frac{2}{3}x\right)$$
$$3x + 3(y) = 9$$

$$3x + 3\left(\frac{8}{3} - \frac{2}{3}x\right) = 9$$
$$3x + \frac{24}{3} - \frac{6}{3}x = 9$$
$$3x + 8 - 2x = 9$$
$$x + 8 = 9$$
$$x = 1$$

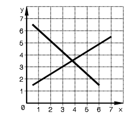
$$y = \frac{8}{3} - \frac{2}{3}x$$
$$y = \frac{8}{3} - \frac{2}{3}(1) \quad \text{(because } x = 1\text{)}$$
$$y = \frac{8}{3} - \frac{2}{3}$$
$$y = \frac{6}{3}$$
$$y = 2$$

To begin, we can choose *either* equation and solve for *either* unknown. While any choice will work eventually, it is best to "look ahead" to find the equation that looks easiest to solve. To avoid fractions, you can try to pick an equation where division isn't required to solve for either x or y.

Summary: Substitution Method

- **If necessary, solve one equation for either unknown. If an equation is already solved, use that one. We will call this the first equation.**
- **Substitute the expression for y or x in the second equation, leaving this equation with only one unknown remaining.**
- **Solve this second equation for the one unknown.**
- **Put this value back into the first equation and solve for the other unknown.**
- **Check the solution (x, y) by putting the values for x and y into both equations and confirming that these values make true statements in both equations.**

Exercises



Solve the following systems by substituting the expression for x from the first equation into the second equation:

1.
$$x = 2y + 5$$
$$2y + x = 9$$

2.
$$x = 3 - y$$
$$y - 3x = 7$$

3.
$$x = y + 5$$
$$x + 2y = 8$$

4.
$$x = 6 - y$$
$$y = {}^{-}2x + 6$$

Solve the following systems by solving the first equation for y (if necessary), and substituting the expression for y into the second equation:

5.
$$y = x + 5$$
$$x + y = 9$$

6.
$$x + y = {}^{-}5$$
$$x + 2y = {}^{-}4$$

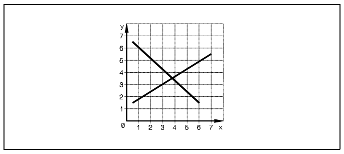
7.
$$y = 3 - 2x$$
$$3y - 2x = 1$$

8.
$$2y = x + 6$$
$$4x + y = 12$$

Solve the following systems by substitution:

9.
$$x + y = 5$$
$$3x + 2y = 13$$

10.
$$x + 2y = {}^{-}5$$
$$x - 3y = 10$$



11.
$$\begin{aligned} y &= 2x+3 \\ 2x+5y &= 3 \end{aligned}$$

12.
$$\begin{aligned} 2x+y &= 7 \\ 5x+2y &= 6 \end{aligned}$$

13.
$$\begin{aligned} x &= 4 \\ 3x-4y &= 0 \end{aligned}$$

14.
$$\begin{aligned} 2x+3y &= 2 \\ x+y &= \frac{5}{6} \end{aligned}$$

15.
$$\begin{aligned} x+y &= 0 \\ x-y &= 0 \end{aligned}$$

Section 4
The Addition Method

Adding Equations

Another method of solving systems of equations is to add the equations together. If the equations are arranged properly, this can result in a quick solution.

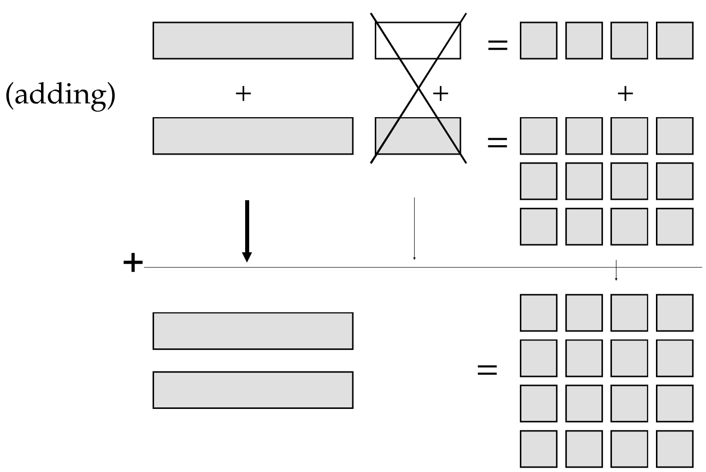
When we solve equations, we are able to add the same amount to both sides. This works because we start with two equal expressions; if we make the same changes to both sides of an equation, the expressions are still equal.

This idea will give us another method of solving a system of equations:

$$\begin{aligned} x - y &= 4 \\ x + y &= 12 \end{aligned}$$

The second equation tells us that $x + y$ is equal to 12. We start with the first equation; instead of adding 12 to both sides, we add $x + y$ to the left side and 12 to the right side. We call this process **the addition method**:

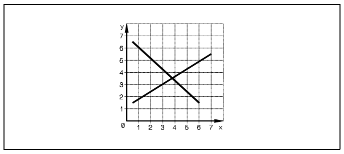
We added the two equations together and the y's canceled out, giving a new



$$\begin{array}{cccc} x & - & y & = & 4 \\ x & + & y & = & 12 \\ \hline 2x & & & = & 16 \end{array}$$

equation with x's and units only. We solve this in the usual way for x:

$$\begin{aligned} 2x &= 16 \\ \frac{1}{2}(2x) &= \frac{1}{2}(16) \\ x &= 8 \end{aligned}$$



To find the value of y, we put the value of x back into either of the original equations; we then solve for y:

$$\begin{aligned}(x) + y &= 12\\ 8 + y &= 12\\ y &= 4\end{aligned}$$

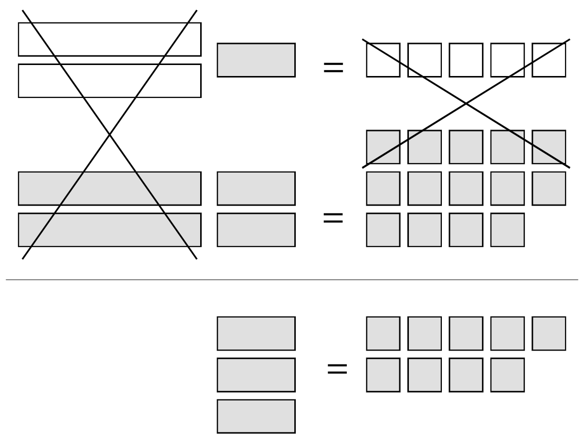
The answer is (8, 4).

Here is another example of a solving a system with the addition method:

$$\begin{aligned}{}^{-}2x + y &= {}^{-}5\\ 2x + 2y &= 14\end{aligned}$$

We "add" the two equations together. This time, the x's cancel out:

$$\begin{array}{rcrcr} -2x & + & y & = & -5\\ 2x & + & 2y & = & 12\\ \hline & & 3y & = & 9\end{array}$$



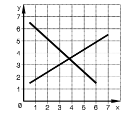
Remember that we are adding *equal amounts* to both sides of the equation, because $2x + 2y$ and 12 are equal. We finish the solution in the usual way:

$$\begin{aligned}3y &= 9\\ \frac{1}{3}(3y) &= \frac{1}{3}(9)\\ y &= 3\end{aligned}$$

$$\begin{aligned}2x + 2y &= 14\\ 2x + 2(3) &= 14\\ 2x + 6 &= 14\\ 2x &= 8\\ x &= 4\end{aligned}$$

The answer is (4, 3)

Rewriting One Equation



The examples in this section have all worked out very neatly. When we added the equations together, one of the unknowns canceled out, allowing us to solve for the other unknown. This will not always happen.

In the system shown below, adding the equations together *does not* cancel either variable. The resulting equation still has both variables and we cannot solve it for either one:

$$\begin{array}{ccccc} 2x & + & 3y & = & 4 \\ -x & + & 2y & = & 5 \\ \hline x & + & 5y & = & 9 \end{array}$$

This difficulty can be overcome if we rewrite one of our original equations *before* we add the two equations together. If we multiply both sides of the second equation by 2, it will then have a ^-2x term which will cancel the $2x$ term from the first equation:

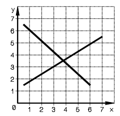
$\mathbf{2x} + 3y = 4$	$\rightarrow \quad \rightarrow$	$2x + 3y = 4$
$\mathbf{-x} + 2y = 5 \rightarrow$	$\mathbf{2}(-x + 2y) = \mathbf{2}(5) \rightarrow$	$^-2x + 4y = 10$
		$7y = 14$
		$y = 2$

We changed our original equations so their x terms were opposites (equal numbers and opposite signs). When we added the equations together, the x terms canceled out, leaving only y terms and numbers. We could then solve for y.

Here is another example:

$2x + \mathbf{3y} = {^-1}$	$\rightarrow \quad \rightarrow$	$2x + 3y = {^-1}$
$5x + \mathbf{y} = 4 \rightarrow$	$\mathbf{^-3}(5x + y) = \mathbf{^-3}(4) \rightarrow$	$^-15x - 3y = {^-12}$
		$^-13x = {^-13}$
		$x = 1$

This time we multiplied to make the y terms cancel (same number of y's and opposite signs). Since both y terms were positive, we multiplied the second equation by $^-3$. This method works whenever the x's in one equation



are a multiple of the x's in the other equation, or the y's in one equation are a multiple of the y's in the other equation.

To finish the solution:

$$\begin{aligned} 5(x) + y &= 4 \\ 5(1) + y &= 4 \\ y &= {}^{-}1 \end{aligned}$$

Rewriting Both Equations

It is not always possible to multiply both sides of one equation and then cancel out by adding. Consider the system below:

$$\begin{aligned} 2x + 3y &= {}^{-}8 \\ 3x + 4y &= {}^{-}11 \end{aligned}$$

We cannot multiply $2x$ by any number to cancel $3x$ and we cannot multiply $3y$ to cancel $4y$. Instead we have to separately multiply *both* equations to make them cancel:

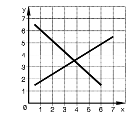
$2x + 3y = {}^{-}8 \rightarrow$	$\mathbf{3}(2x + 3y) = \mathbf{3}({}^{-}8) \rightarrow$	$6x + 9y = {}^{-}24$
$3x + 4y = {}^{-}11 \rightarrow$	$\mathbf{{}^{-}2}(3x + 4y) = \mathbf{{}^{-}2}({}^{-}11) \rightarrow$	${}^{-}6x - 8y = 22$
(Eliminate x. 6 is the common multiple of 2 and 3.)		$y = {}^{-}2$

Did you notice that we multiplied one equation by 3 and the other equation by $^{-}2$? *We must multiply both sides of each equation by the same number, but we can multiply the two different equations by two different numbers.* Our object is to get the terms of one variable to be opposite in the two equations so that they will cancel when the equations are added.

To do this, we must find the *least common multiple* of the original numbers of x's or y's. In this example, we chose x, and the least common multiple of 2 and 3 is 6. We then must choose the multipliers so that one equation has a $^{+}6x$ and the other equation has a $^{-}6x$. To finish the solution:

$$\begin{aligned} 2x + 3(y) &= {}^{-}8 \\ 2x + 3({}^{-}2) &= {}^{-}8 \\ 2x + {}^{-}6 &= {}^{-}8 \\ 2x &= {}^{-}2 \\ x &= {}^{-}1 \end{aligned}$$

The solution is $({}^{-}1, {}^{-}2)$



Here is a more complex example:

$$\begin{aligned} 4x + 5y &= 23 \\ 6x + 7y &= 33 \end{aligned}$$

The steps are as follows:

$4x + 5y = 23 \rightarrow$	$^{-}\mathbf{3}(4x + 5y) = {}^{-}\mathbf{3}(23) \rightarrow$	$-12x - 15y = {}^{-}69$
$6x + 7y = 33 \rightarrow$	$\mathbf{2}(6x + 7y) = \mathbf{2}(33) \rightarrow$	$12x + 14y = 66$
(Eliminate x. The common multiple of 4 and 6 is 12.)		$-y = -3$ $y = 3$

We finish the solution in the usual way:

$$\begin{aligned} 4x + 5(y) &= 23 \\ 4x + 5(3) &= 23 \\ 4x + 15 &= 23 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

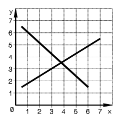
The solution is (2, 3)

The steps for adding equations are as follows:

- **Choose a variable to eliminate.**
- **Find a common multiple of the numbers of x's or y's.**
- **Multiply the original equations to give new equations where the terms for the chosen variable are opposite.**
- **Add the equations together, letting the chosen variable cancel out.**
- **Solve for the remaining variable.**
- **Plug this solution back into the original equation and solve for the remaining variable.**

How Do We Multiply?

From the examples above, we can develop a plan for multiplying the equations. Since our only tool is multiplication, we get multiples of the original numbers of x and y. We are looking for a common multiple,



preferably a **least common multiple—**familiar from the **least common denominator**.

In the last example, we looked at $4x$ and $6x$ and found the least common multiple of 4 and 6. This is 12. We then multiplied the equation containing $4x$ by $^{-}3$, so $4x$ became $^{-}12x$, and we multiplied the equation containing $6x$ by 2, so that $6x$ became $12x$.

How do we know whether we should cancel x or y? We simply pick the one that looks easiest—usually the one where we will have less multiplying to do (where the least common multiple is smaller). Here is a summary of these ideas:

- **If either x's or y's in both equations are ready to cancel, then no multiplying is required.**

- **If the x's (or y's) in one equation are a multiple of the x's (or y's) in the other equation, then multiply both sides of one equation only. Remember to use a negative number when multiplying if necessary.**

- **If the x's (or y's) in one equation are *not* a multiple of the x's (or y's) in the other equation, then separately multiply both sides of *both* equations to get a common multiple of one variable (x or y). Use a negative number if necessary.**

Changing the Form of the Equations

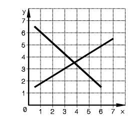
When equations are not given in the form we have been using, it is easy to rearrange the terms. This is a slightly different process than solving an equation for x or y. In this case, we are adding to both sides with a different goal—to have the equation in **standard form**, as shown below:

$$\_\,x + \_\,y = \_$$

For example, in the single equation below, we add to both sides until the x's and y's are on one side (usually the left) and the units are on the other side (usually the right):

$$\begin{aligned} \mathbf{3x + 4} &= \mathbf{y + 7} \\ 3x + 4 - 4 &= y + 7 - 4 \\ 3x &= y + 3 \\ 3x - y &= y - y + 3 \\ \mathbf{3x - y} &= \mathbf{3} \end{aligned}$$

We are not solving for x or y. Our goal is to have the x and y terms on the left side of the equation and the number term on the right side of the equation.



A second example is shown below:

$$\mathbf{9x + 3y + 4 = 7(x - 2) + 1}$$
$$9x + 3y + 4 = 7x - 14 + 1$$
$$9x + 3y + 4 = 7x - 13$$
$$9x + 3y + 4 - 4 = 7x - 13 - 4$$
$$9x + 3y = 7x - 17$$
$$9x + 3y - 7x = 7x - 7x - 17$$
$$\mathbf{2x + 3y = {}^{-}17}$$

Notice that we need to multiply out parentheses but that we do not need to divide at the end.

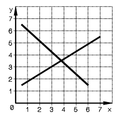
Here are the steps to get an equation into standard form:

- **Multiply out all parentheses.**
- **Combine similar terms on both sides.**
- **Decide how to get the equation into the form: _ *x* + _ *y* = _**
  Choose the left side for the unknown and the right for the units.
- **Add to cancel the *x*'s and *y*'s on the right side.**
- **Add to cancel the units on the left side.**

Summary

We now have a complete addition method for solving systems of equations. Here are the steps we have developed:

- **Rewrite each equation in the standard form by multiplying out parentheses, combining terms, and arranging the unknowns on one side.**
- **If necessary, multiply one or both equations so that either the *x*'s or the *y*'s will cancel.**
- **Add the two equations. The chosen variable will cancel out.**
- **Solve the resulting equation for the remaining variable.**
- **Substitute that result back into either original equation and solve for the other unknown.**
- **Check by substituting *x* and *y* into both equations.**



Exercises

Solve the following systems by addition. Change the form if necessary:

1.
$$\begin{aligned} 2x + y &= 8 \\ x - y &= 4 \end{aligned}$$

2.
$$\begin{aligned} -x + y &= 1 \\ x + y &= 3 \end{aligned}$$

3.
$$\begin{aligned} {}^{-}2x - 3y &= {}^{-}12 \\ 2x + y &= 8 \end{aligned}$$

4.
$$\begin{aligned} x - y &= 3 \\ y &= 6 - 2x \end{aligned}$$

5.
$$\begin{aligned} 2x + y &= 9 \\ y &= 3x - 3 \end{aligned}$$

6.
$$\begin{aligned} 3x + 2y - 7 &= 0 \\ x + 11 &= 2y \end{aligned}$$

Solve the following systems by multiplying one equation. Change the form first if necessary:

7.
$$\begin{aligned} 2x - y &= 7 \\ 3x + 2y &= 0 \end{aligned}$$

8.
$$\begin{aligned} 3x + 2y &= 8 \\ 3x + y &= 7 \end{aligned}$$

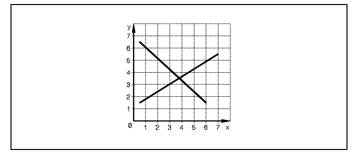
9.
$$\begin{aligned} x - 3y &= {}^{-}9 \\ 2x + 4y &= 22 \end{aligned}$$

10.
$$\begin{aligned} 3x + 2y &= 1 \\ y &= 3 - 2x \end{aligned}$$

11.
$$\begin{aligned} x - 2y &= {}^{-}5 \\ x &= y - 1 \end{aligned}$$

12. $8 - y = 5x$

$4y + 14 = 3x$



Solve the following equations by multiplying both equations. Change the form if necessary:

13. $3x + 5y = 28$

$5x - 3y = 24$

14. $7x - 5y = 8$

$^{-}5x + 4y = {}^{-}1$

15. $3x + 4y = 13$

$^{-}7x + 3y = {}^{-}18$

16. $2x + 3y + 2 = 0$

$3x = 2y + 10$

17. $3x - 8y = 7$

$^{-}5y = 45 - 10x$

18. $4x - 14 = {}^{-}3y$

$2y = 9x - 14$

Section 5
Choosing a Method

Three Methods

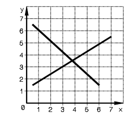
We have learned three different ways to solve a system of linear equations—**graphing**, **substitution**, and **addition**. Each method has advantages and disadvantages, but all three are useful for solving systems and illustrating the solutions.

In this book, the graphing method is included mainly for the purpose of illustration; it is important to remember that a system of linear equations can be represented by the two lines—the solution is the intersection of the lines. Graphing is obviously not very efficient as a practical method to solve linear equations exactly. In statistics and other fields, graphing is very useful when we have the *ordered pairs* instead of the equations.

Here is a comparison of some of the advantages and disadvantages of the three methods:

Method	Advantages	Disadvantages
Graphing	• Shows a picture of the solution • Not much algebraic calculation to do	• Not very accurate • Difficult with very large or very small numbers • Takes time to graph two equations
Substitution	• Only one equation to rewrite • No need to multiply equations	• Fractions can get complicated • Solving one equation for y or x can be difficult
Addition	• Adding and solving can be very fast.	• It may be difficult to find a common multiple

Substitution versus Addition



When you have a choice of methods, it will usually be a choice between the substitution and addition methods. *Either method will always give the correct solution***.** It is useful, however to develop a plan to decide which method will be easier for each given system of equations. *For each system, one method is often much quicker than the other.* Here are some important ideas to help you choose:

Substitution

A good choice if:

- One equation is solved for an unknown or is close to being solved. For example, $y = 3x + 2$.
- It will be easy to substitute into the second equation and easy to solve that equation.

A poor choice if:

- It will be difficult to solve either equation for an unknown without creating an expression with fractions. For example: $3y + 4 = 15 - 35x$.
- The other equation (where we substitute for the unknown) has a complicated expression containing the unknown. For example, it will take a great deal of work to substitute for x in the equation $y = 6(3 - 45x) + 234x$.

Addition

A good choice if:

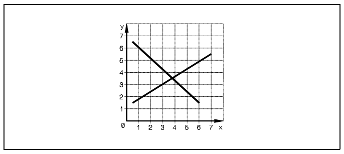
- The number of x's (or y's) in the first equation is the *same* as the number of x's (or y's) in the second equation.
- The number of x's (or y's) in the first equation is *a multiple of* the number of x's (or y's) in the second equation.
- The equations are arranged (or close to being arranged) in the format with unknowns on one side and units on the other. ($\_\_\, x + \_\_\, y = \_\_$)

A poor choice if:

- It will be difficult to find a common multiple.
- Fractions or decimals are present.
- It will take a great deal of work to rewrite the equations in the proper form.

For example, consider this system:

$$3x - 2y = 7$$
$$5x = 33 - 2y$$



The second equation can easily be rewritten as $5x + 2y = 33$, so the system would look like this:

$$\begin{aligned} 3x - 2y &= 7 \\ 5x + 2y &= 33 \end{aligned}$$

The addition method is the best choice. Substitution would be more difficult because solving either equation for x or y would result in fractions.

If one equation is easy to solve for x or y, then it is usually best to choose substitution. For example:

$$\begin{aligned} 3x + y &= 7 \\ 5x &= \frac{3}{2}y + \frac{17}{2} \end{aligned}$$

The addition method is less attractive because it would be difficult to find a common multiple for the x's or y's, but the first equation can easily be solved for y, allowing for substitution.

In summary:

- **Choose *substitution* if:**
  One equation is easily solved for x or y.
  The other equation has a simple expression for the variable which you are substituting.

- **Choose *addition* if:**
  The equations are easy to arrange in the proper form.
  The common multiple is not too large and is easily found.

- **If you are not sure, *either method will always work.***

Exercises

Solve by graphing:

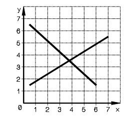
1. $$\begin{aligned} y &= 2x + 1 \\ y &= x - 3 \end{aligned}$$

2. $$\begin{aligned} x &= 2y \\ x + 2y &= 4 \end{aligned}$$

3. $$\begin{aligned} y &= 2x + 1 \\ 3x + y &= 1 \end{aligned}$$

4. $$\begin{aligned} y &= {}^{-}2x - 2 \\ y &= x + 4 \end{aligned}$$

Solve by substitution:



5. $2x + y = {}^-1$
$5x + y = {}^-13$

6. $3x - y = {}^-8$
$x = 6y + 3$

7. $2x + 5y = 12$
$x - y = {}^-1$

8. $x - 4y = {}^-5$
$3x - 2y = 5$

Solve by addition:

9. $4x + 2y = 6$
$3x + 2y = 1$

10. ${}^-4x + 5y = 10$
$4x + 3y = {}^-26$

11. ${}^-4x + 3y = {}^-1$
$2x - 3y = 0$

12. $2x - 3y = 4$
$4x + y = {}^-6$

Solve by the best method:

13. $3x - 4y = 11$
$2x + 3y = {}^-4$

14. $4x + 3y = 13$
$3x - 7y = {}^-18$

15. $3x - y = {}^-1$
$x - y = 1$

16. $y = x + 1$
$x + y = 3$

Section 6
Special Cases

Fractions and Decimals

When fractions and decimals appear in systems of equations, we treat them in the same manner as we did when solving one equation. We multiply both sides of the equation by an appropriate number so that the fractions and decimals are gone. For example, consider these equations:

$$\frac{1}{2}x + y = \frac{10}{3}$$

$$\frac{1}{18}y = x$$

We multiply the first equation times 6 (the least common multiple of 2 and 3) and the second equation times 18:

$$\begin{aligned} \frac{1}{2}x + y &= \frac{10}{3} \\ 6\left(\frac{1}{2}x + y\right) &= 6\left(\frac{10}{3}\right) \\ \frac{6}{2}x + 6y &= \frac{60}{3} \\ 3x + 6y &= 20 \quad \text{(no fractions remain)} \end{aligned}$$

$$\begin{aligned} \frac{1}{18}y &= x \\ 18\left(\frac{1}{18}y\right) &= 18(x) \\ y &= 18x \quad \text{(no fractions remain)} \end{aligned}$$

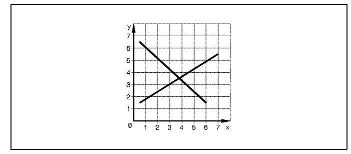
We are now ready to solve the system by substitution or addition.

If decimals are present in an equation, we multiply both sides by 10, 100, 1000, etc. For example:

$$1.2x + .32y = 6$$

$$\begin{aligned} 100(1.2x + .32y) &= 100(6) \\ 120x + 32y &= 600 \end{aligned}$$

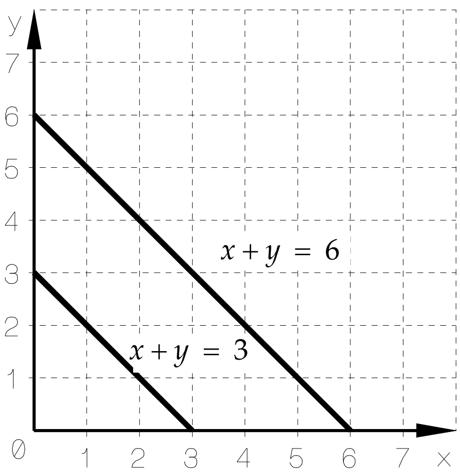
We choose a power of 10 that is large enough to give a new equation without decimals. Because decimal numbers are really fractions with denominators of 10, 100, 1000, and so on, we are really multiplying by the least common denominator.



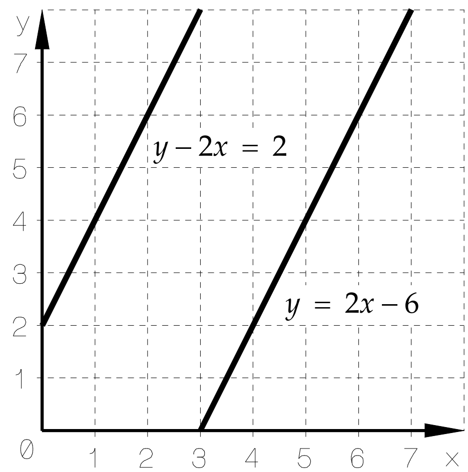
Unusual Solutions

Two lines do not always meet in one single point. The lines may be **parallel** to one another; parallel lines have no common point and no solution:

$$\begin{aligned} x + y &= 3 \\ x + y &= 6 \end{aligned}$$



$$\begin{aligned} y - 2x &= 2 \\ y &= 2x - 6 \end{aligned}$$



We call such a system **inconsistent.** *Inconsistent equations are parallel lines on a graph and have no solution.*

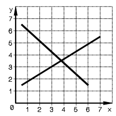
The two equations in a system may also represent the same line. The equations are actually two different forms of the same equation:

$$\begin{aligned} 2x + 3y &= 6 \\ 4x + 6y &= 12 \end{aligned}$$

If we multiply the first equation by 2, it is the same as the second equation:

$$\begin{aligned} 2x + 3y &= 6 \\ 2(2x + 3y) &= 2(6) \\ 4x + 6y &= 12 \end{aligned}$$

It is now clear that the two original equations are really the same equation; they both represent the same line. We call this type of system **dependent** because the two lines are not independent of each other. We also say that the two lines **coincide.** *Dependent equations share the same line on the graph, and all points of that line are solutions.*



Identifying Inconsistent and Dependent Systems

We have just learned how to identify the *graphs* of inconsistent and dependent equations, but how can we identify these situations when we are solving by substitution or addition? Consider the system we just looked at:

$$2x + 3y = 6$$
$$4x + 6y = 12$$

$2x + 3y = 6 \rightarrow$	$\mathbf{-2}(2x + 3y) = \mathbf{-2}(6) \rightarrow$	$-4x - 6y = -12$
$4x + 6y = 12 \rightarrow$	$\rightarrow \quad \rightarrow$	$4x + 6y = 12$
		$0 + 0 = 0$
		$\mathbf{0 = 0}$

We obtained a result that is true (zero *is* equal to zero), but useless. This means that the two equations (lines) are the same. They are dependent (the graphs of the lines coincide).

Next, let's look at two parallel lines:

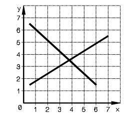
$$2x + 3y = 6$$
$$4x + 6y = 7$$

$2x + 3y = 6 \rightarrow$	$\mathbf{-2}(2x + 3y) = \mathbf{-2}(6) \rightarrow$	$-4x - 6y = -12$
$4x + 6y = 7 \rightarrow$	$\rightarrow \quad \rightarrow$	$4x + 6y = 7$
		$0 + 0 = -5$
		$\mathbf{0 = -5}$

This time, we obtained a result that is false. This means that *there is no solution —the two equations are inconsistent and the lines are parallel*. When we combined the two equations, we assumed that there was an x and a y value that were on both lines. Since this was a false assumption, we reached a false conclusion.

Note: These results will be the same whether we are using the addition or the substitution method. Meaningless results (dependent or coinciding systems) may be $3 = 3$ or $5 = 5$, not just $0 = 0$. False results (inconsistent or parallel systems) may come out to be any false statement. In both cases, the variables all cancel out.

Summary



The table below is a summary of how we can identify these two special cases:

If both variables cancel out:		
Results	**Meaning**	**Why**
True, but not helpful: $0 = 0$ $3 = 3$ ${}^{-}1 = {}^{-}1$	**Dependent (Coinciding) System** Equations are equivalent Equations represent the same line	Since we start with two lines that are the same, we do not have enough information to find an answer.
False: $3 = 5$ $7 = {}^{-}2$ $0 = 3$	**Inconsistent or Parallel system** No solution Lines are parallel	Since we start with a false idea that the lines meet, we come to a false conclusion.

Exercises

Solve the following systems by first multiplying each equation (if necessary) by the least common multiple:

1. $\frac{x}{5} + \frac{y}{7} = 1$

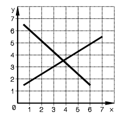
$x + y = 11$

2. $\frac{2}{3}x + \frac{2}{5}y = \frac{1}{3}$

$-\frac{2}{3}x - \frac{3}{5}y = \frac{1}{6}$

3. $\frac{3}{4}x + \frac{1}{3}y = 8$

$\frac{1}{2}x - \frac{5}{6}y = {}^{-}1$



4. $$\frac{2x}{3}+\frac{2y}{6} = \frac{10}{6}$$
$$\frac{5x}{12}+\frac{y}{4} = \frac{11}{12}$$

Solve the following systems by first multiplying each equation (if necessary) by the appropriate power of 10 (10, 100, 1000):

5. $$0.1x + 10y = 30$$
$$^{-}4.1x - 10y = 10$$

6. $$.02 + .01y = .05$$
$$.5x + .3y = 1.1$$

7. $$.5x + 1.0y = 17$$
$$.1x + .1y = 2.2$$

8. $$y = 0.1x + 3.5$$
$$y + .08x = 8$$

Solve the following systems. State the unique solution or state that the system is inconsistent (two parallel lines) or it is dependent (two lines that coincide):

9. $$3x - 6y = 6$$
$$x - 2y = 4$$

10. $$x + 3y = 3$$
$$^{-}x + y = 5$$

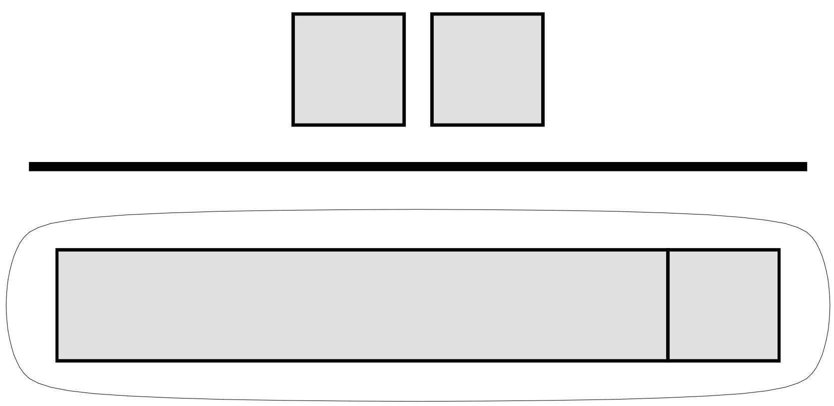
11. $$2x - 3y = 6$$
$$4x - 12 = 6y$$

12. $$x + y = {}^{-}1$$
$$y = -x + 1$$

13. $$x - y = 3$$
$$2x - 2y = 6$$

14. $$y = 2x$$
$$4x + y = 6$$

Chapter 14
Rational Expressions



Section 1
Introduction

Rational Expressions

Rational expressions are expressions written as **ratios**. When we write ratios with integers, (numbers such as 1, 2, or 25), we call them **fractions**. The ratio of 1 to 4 is written

$$1:4 \text{ or } \frac{1}{4}$$

Rational expressions are also fractions, except that they can contain both numbers and unknowns. For example:

$$\text{The ratio of } 3 \text{ to } x \text{ is } \frac{3}{x}$$

$$\text{The ratio of } x+5 \text{ to } x^2-7x+2 \text{ is } \frac{x+5}{x^2-7x+2}$$

$$\text{The ratio of } x^2+2x \text{ to } 7 \text{ is } \frac{x^2+2x}{7}$$

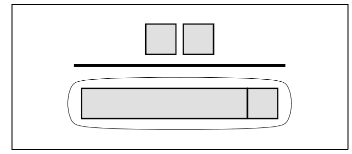
We work with rational expressions in essentially the same way as we work with fractions; in some sections of this chapter we will briefly review common fractions before expanding the discussion to more complex rational expressions.

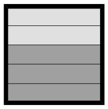
Fractions

You will recall that the fraction $\frac{3}{5}$ can be pictured in several different ways:

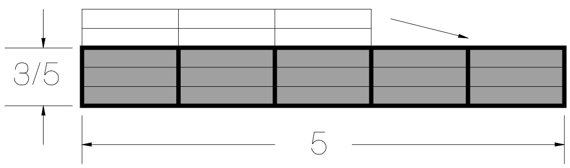
- $\frac{1}{5}$ **of 3**
- **3 parts out of 5**
- **The height of a rectangle made with an area of 3 units arranged so it is 5 units wide.**

$\frac{1}{5}$ of 3



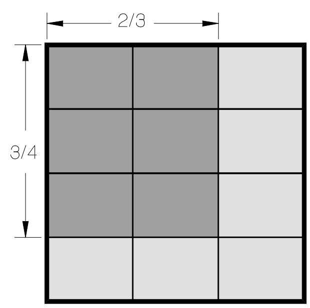


3 parts out of 5



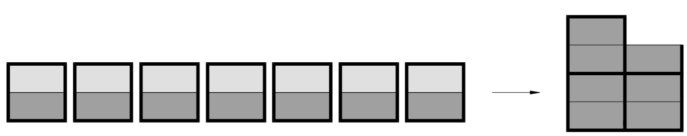
The height of a rectangle made with an area of 3 units, arranged so it is 5 units wide.

We have also represented fractions as smaller rectangles within unit rectangles, such as

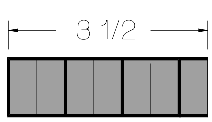


$$\frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12}$$

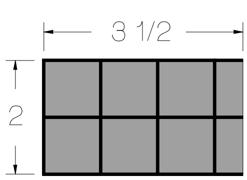
Of course, fractions need not be limited to ratios which are less than one whole unit. The fraction $7/2$ can be illustrated as:



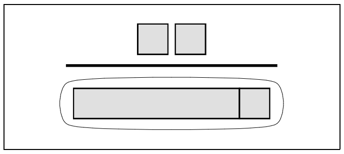
$\frac{1}{2}$ of 7



7 parts where each unit has 2 parts

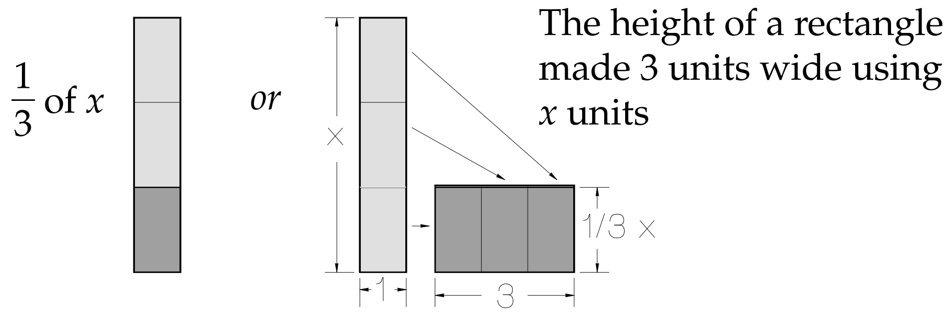


The width of a rectangle made 2 units high using 7 units



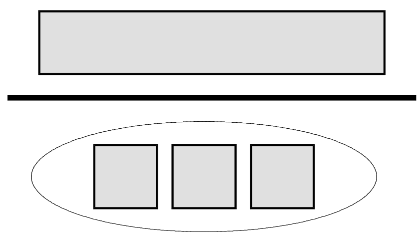
Fractions Containing Unknowns

Picturing fractions which contain unknowns is no more difficult than picturing common fractions. For example, $x/3$ can be pictured as



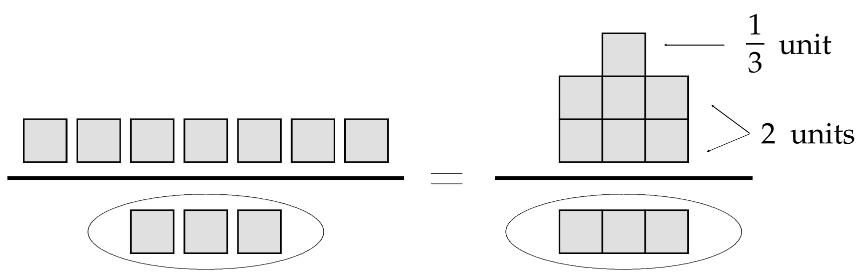
But the third picture, where we show *x* parts when each unit has 3 parts, is harder to draw. We can suggest a new method for representing this case:

x parts when a unit is 3 parts

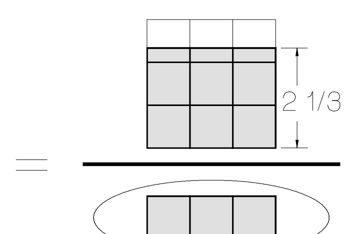


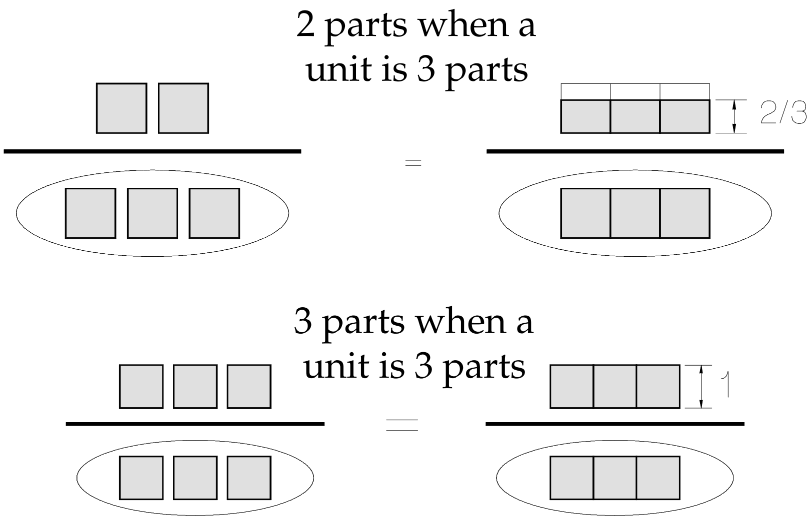
Here, the circled denominator represents the size of one unit. We can see what this representation means by substituting some particular values for the unknown (*x*), and then grouping them into the units shown in the denominator.

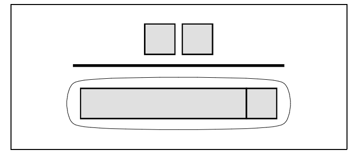
7 parts when a unit is 3 parts



or $2\frac{1}{3}$ groups of 3

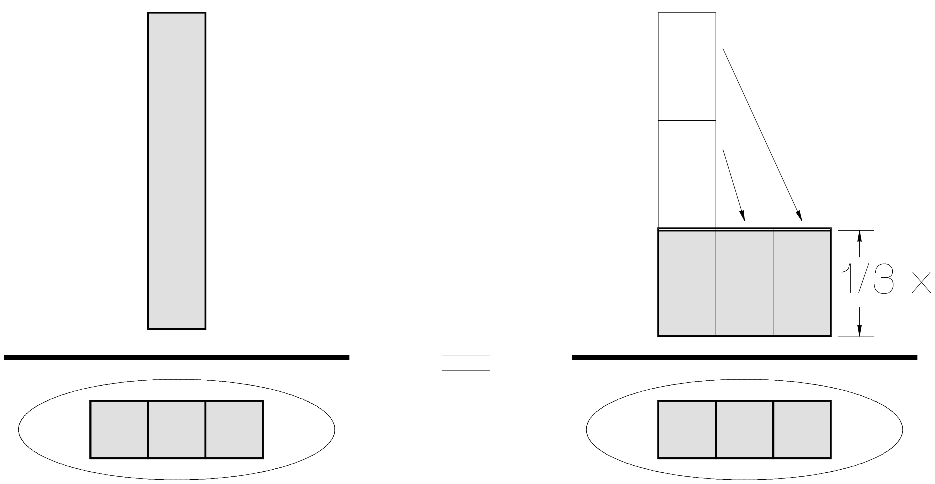






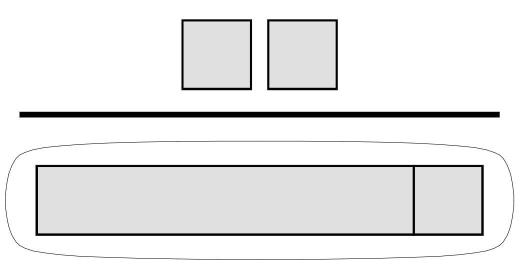
The denominator (bottom pieces) forms a rectangle which stands for one unit; we then use the top pieces to make as many of these unit rectangles as we can, with any left over part (**remainder**) still being written as a fraction.

If we don't know the value of x, we can sometimes represent the fraction using more than one picture, but the value of the result will still be a fraction containing an unknown.



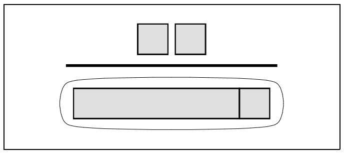
$$\frac{x}{3}$$

When we have unknowns in the denominator, our new representation will still work, but alternative pictures are sometimes harder to visualize:



$$\frac{2}{x+1}$$

This expression represents 2 chips spread out into a rectangle which is $x + 1$ units wide.



Exercises

Draw a picture to represent each of the following rational expressions:

1. $\frac{4}{5}$

2. $\frac{3}{7}$

3. $\frac{5}{4}$

4. $\frac{12}{5}$

5. $\frac{x}{5}$

6. $\frac{x}{2}$

7. $\frac{2x}{3}$

8. $\frac{3x}{2}$

9. $\frac{x+3}{4}$

10. $\frac{x+1}{3}$

11. $\frac{2}{x}$

12. $\frac{3}{x+2}$

Section 2
Simplifying Rational Expressions

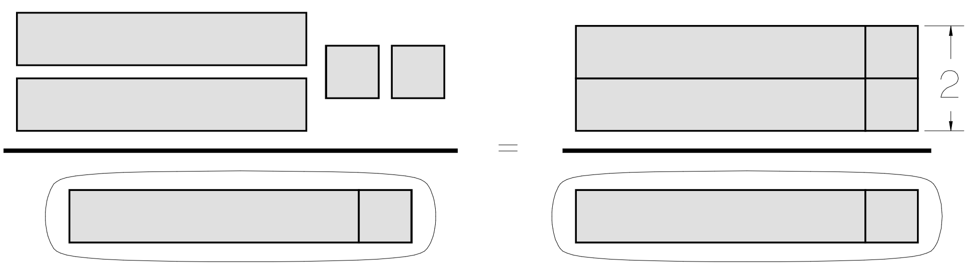
Simplifying

Consider the last expression from the previous section:

$$\frac{2}{x+1}$$

This expression cannot be simplified without knowing the value of x. However, the following expression can easily be simplified:

$$\frac{2x+2}{x+1}$$



$$\frac{2x+2}{x+1}$$

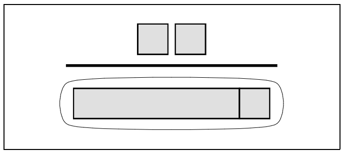
This expression simplifies to 2, because the pieces in the numerator combine to make exactly two unit rectangles (denominators).

Collapsing, Reducing, and Canceling

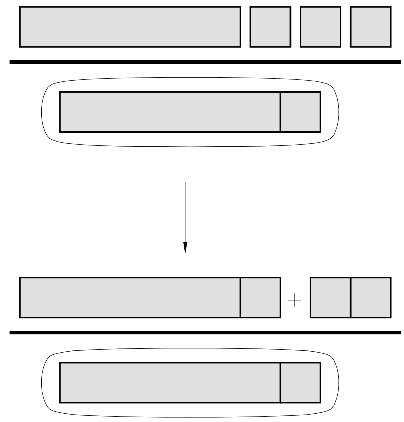
Let's look at some more rational expressions. What if we begin with

$$\frac{x+3}{x+1}$$

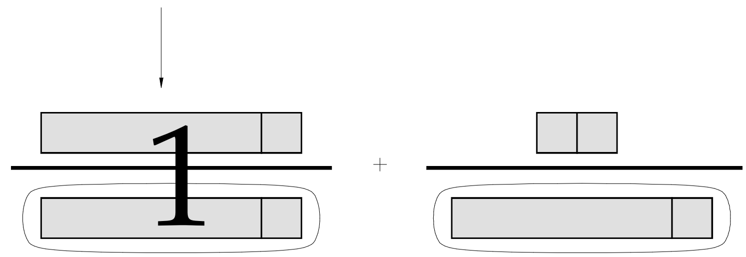
What can we do with this expression? Here we separate our original expression into the sum of two fractions: one where the numerator matches the denominator (thus reducing to one), and the second which has the left-over pieces (**remainder**) over the same denominator:



$$\frac{x+3}{x+1}$$



$$\frac{(x+1)+(2)}{x+1}$$



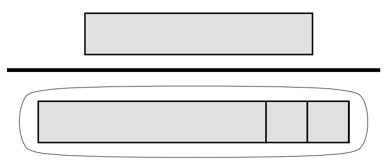
$$\frac{x+1}{x+1}+\frac{2}{x+1} = 1+\frac{2}{x+1}$$

An expression such as

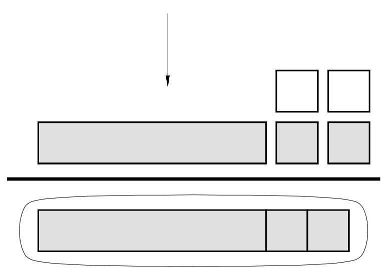
$$\frac{x}{x+2}$$

can be simplified in the following way:

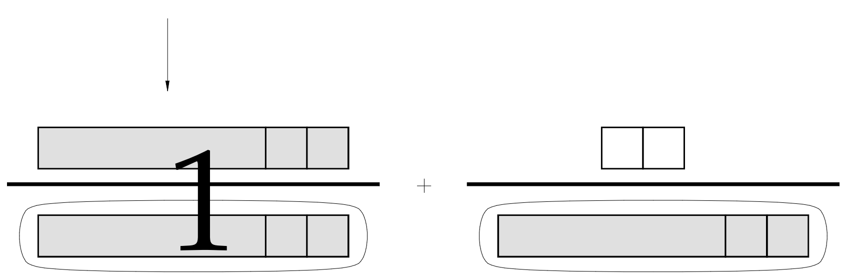
$$\frac{x}{x+2}$$



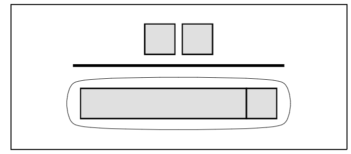
$$\frac{x+2-2}{x+2}$$

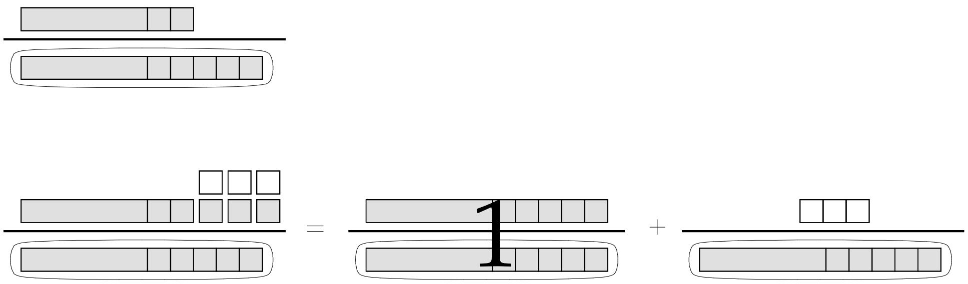


$$\frac{x}{x+2} = 1+\frac{^-2}{x+2}$$

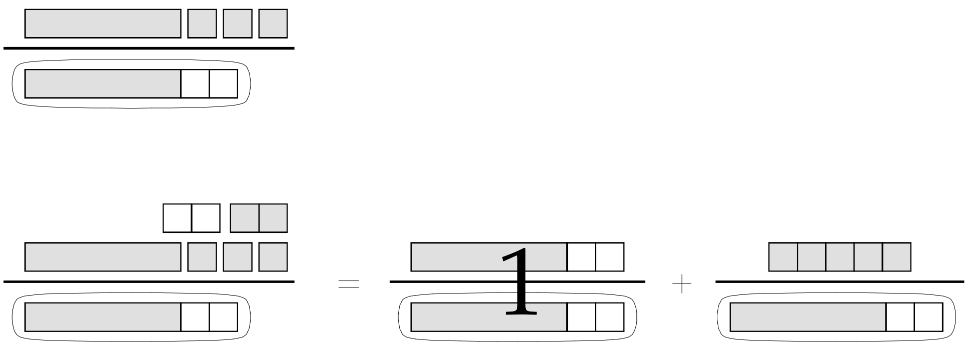


Here we have used the idea of adding equal numbers of positive and negative chips to the numerator (top) so that we can complete one unit. Some further examples of this technique follow:

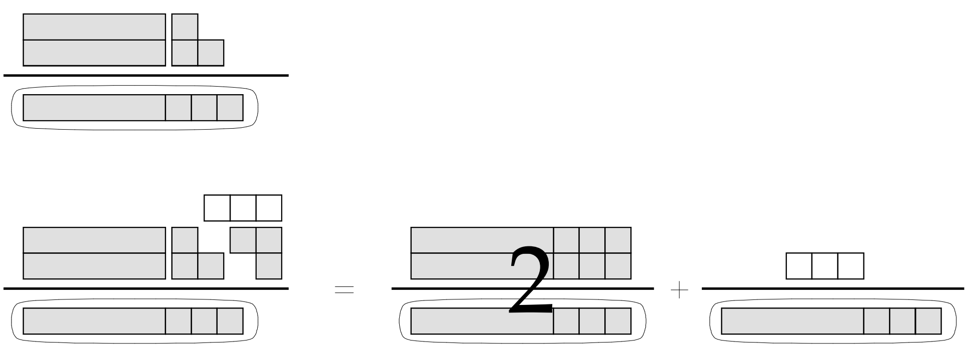




$$\frac{x+2}{x+5} = 1 + \frac{^{-}3}{x+5}$$

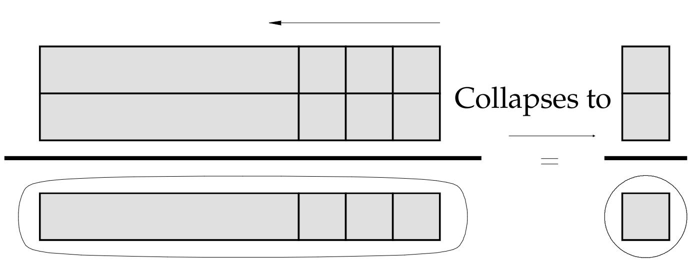


$$\frac{x+3}{x-2} = 1 + \frac{5}{x-2}$$



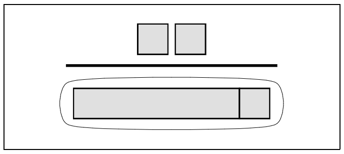
$$\frac{2x+3}{x+3} = 2 + \frac{^{-}3}{x+3}$$

In all of these examples, when the numerator (top) of the fraction matches the denominator (bottom) of the same fraction along one dimension, then the fraction reduces to being just a whole number. This is equivalent to both the top and bottom of the fraction collapsing along their direction of common length. In the symbolic language of algebra, it is the same as dividing out (canceling) like factors from the top and bottom of the fraction.



Cancels to

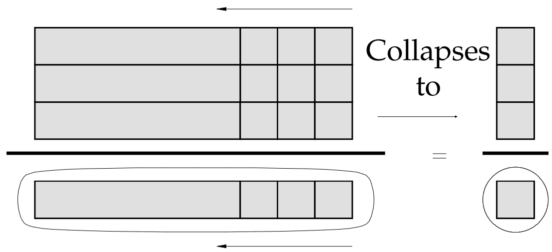
$$\frac{2x+6}{x+3} = \frac{2\cancel{(x+3)}}{1\cancel{(x+3)}} = \frac{2}{1}$$



This only works when one dimension of the top and bottom of the fraction are exactly the same.

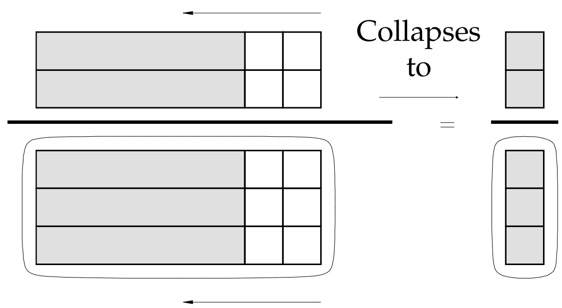
Cancels

$$\frac{3x+9}{x+3} = \frac{3\cancel{(x+3)}}{1\cancel{(x+3)}} = \frac{3}{1}$$



Cancels

$$\frac{2x-4}{3x-6} = \frac{2\cancel{(x-2)}}{3\cancel{(x-2)}} = \frac{2}{3}$$



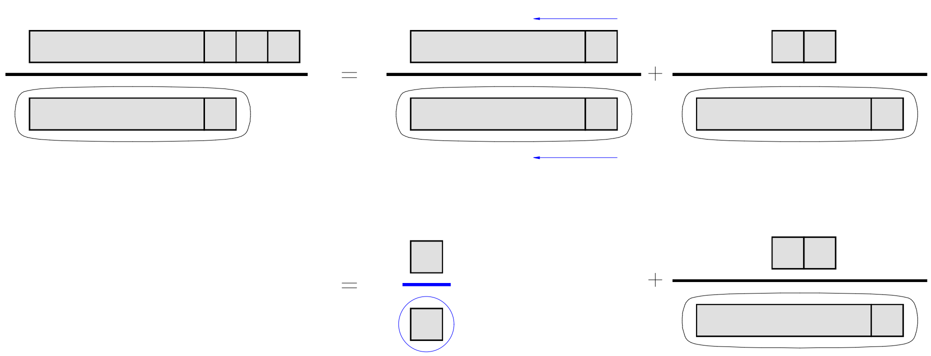
This process of collapsing, or reducing, works because fractions (rational expressions) compare the pieces on top of the fraction to the pieces on the bottom of the fraction. If the top and bottom rectangles have the same length, then the ratio (fraction) of their sizes is the same as the ratio (fraction) of their heights. Collapsing (reducing) is the same as just comparing the heights of the rectangles.

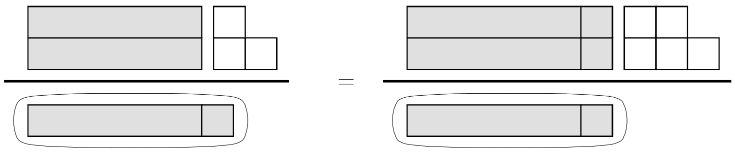
If, when adjusting the top of a fraction (by adding equal numbers of positive and negative chips), there are some pieces left over which do not match the length of the bottom of the fraction, these pieces are the **remainder**. They must be left on top of the original denominator, as shown in the next two examples:

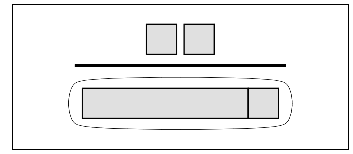
$$\frac{x+3}{x+1} = \frac{x+1+2}{x+1}$$

$$\frac{\cancel{x+1}}{\cancel{x+1}} + \frac{2}{x+1}$$

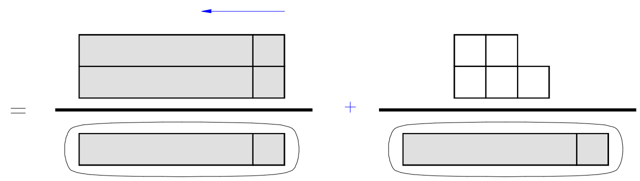
$$1 + \frac{2}{x+1}$$



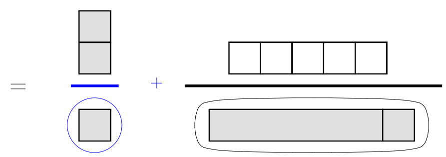




$$\frac{2x-3}{x+1} = \frac{2x+2-5}{x+1}$$



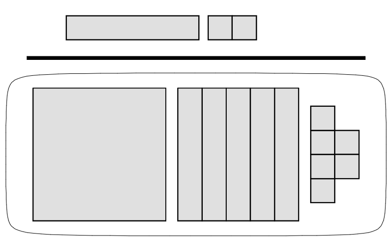
$$\frac{2(x+1)}{x+1} + \frac{{}^{-}5}{x+1}$$



$$2 + \frac{{}^{-}5}{x+1}$$

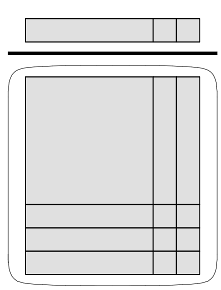
Numerators and Denominators with x^2

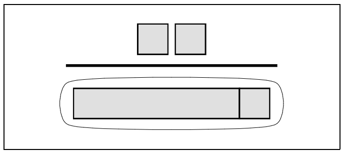
Consider the following rational expression:



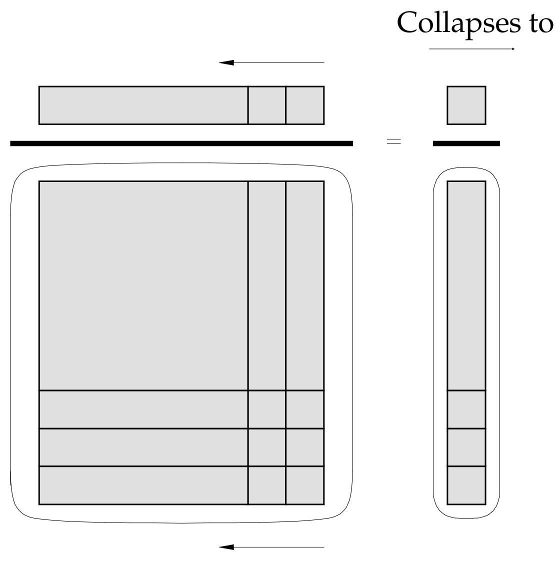
$$\frac{x+2}{x^2+5x+6}$$

Can we simplify this expression? First we must see if the denominator can be made into a rectangle (factored); this can be done in the following way:



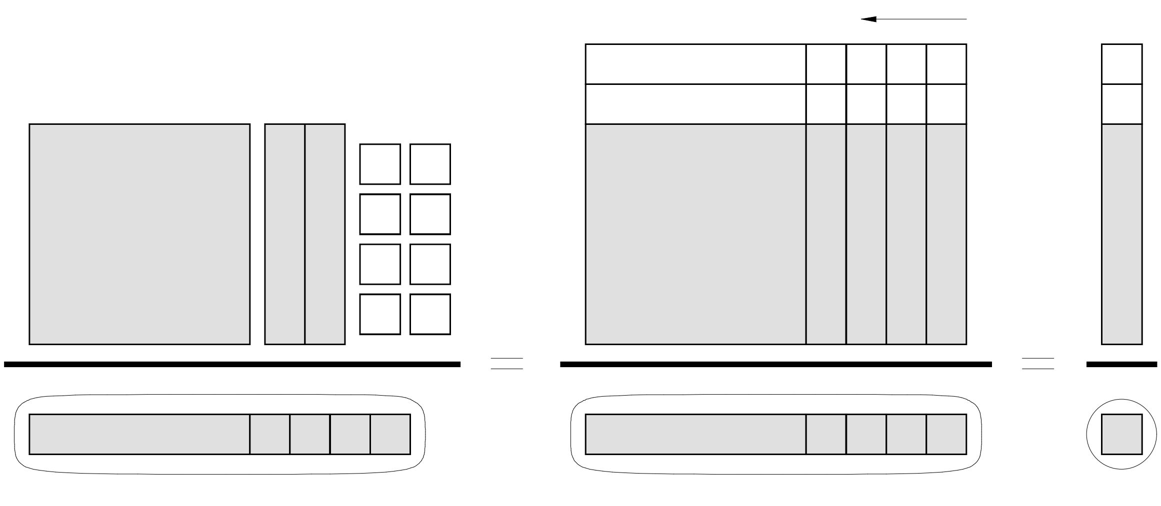


The top and bottom of the expression have the same length, so we can collapse (reduce) both the top and bottom:



$$\frac{x+2}{x^2+5x+6} = \frac{1\cancel{(x+2)}}{(x+3)\cancel{(x+2)}} = \frac{1}{x+3} \qquad (x+2 \text{ cancels})$$

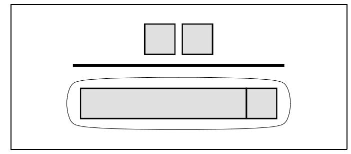
This is our reduced result. Here is another example:

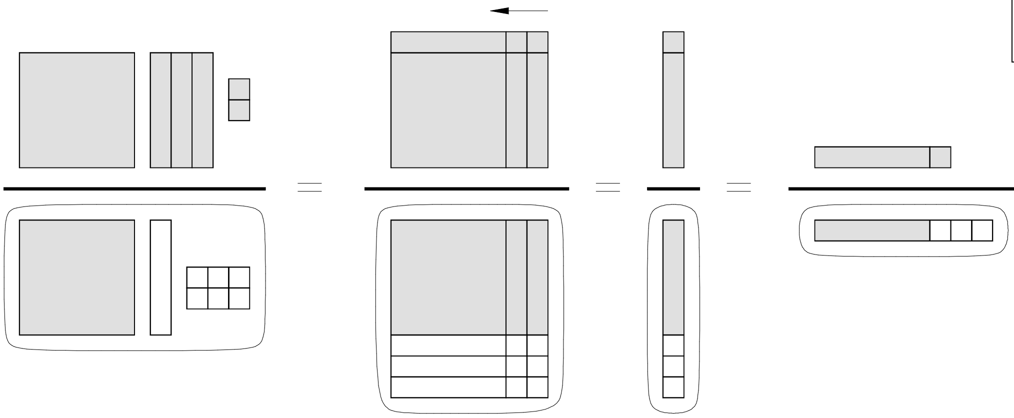


$$\frac{x^2+2x-8}{x+4} = \frac{(x-2)\cancel{(x+4)}}{1\cancel{(x+4)}} = \frac{x-2}{1} = x-2$$

When the denominator reduces to one chip, standing for one unit, we don't have to write the denominator because the ratio of any quantity to one is exactly that quantity. In other words, any quantity divided by one is unchanged.

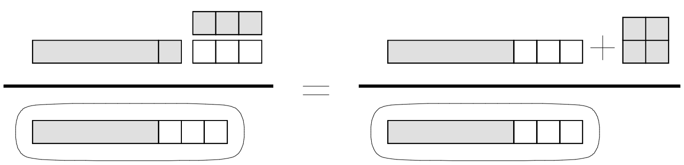
Here is a third problem of this type:

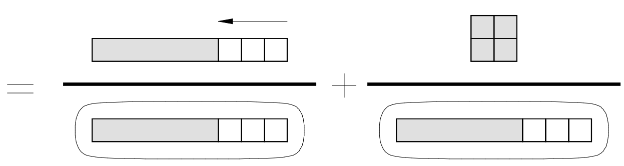


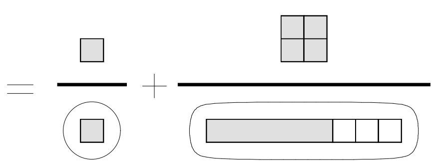


$$\frac{x^2 + 3x + 2}{x^2 - x - 6} = \frac{\cancel{(x+2)}(x+1)}{\cancel{(x+2)}(x-3)} = \frac{x+1}{x-3}$$

This can be converted further as follows:





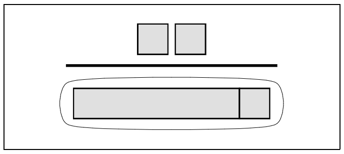


$$1 + \frac{4}{x-3}$$

When to Stop

The two expressions derived for the last problem are equivalent:

$$\frac{x+1}{x-3} = 1 + \frac{4}{x-3}$$



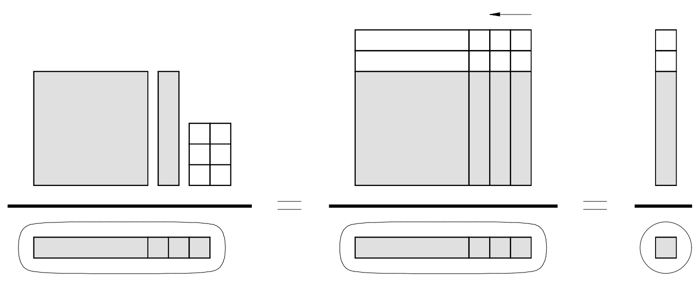
Which is the better way to write the answer?
One method is to stop reducing an expression before we are left with a remainder, at the last step when the expression can be written as a single fraction:

$$\text{Answer \#1: } \frac{x+1}{x-3}$$

But this answer is also correct:

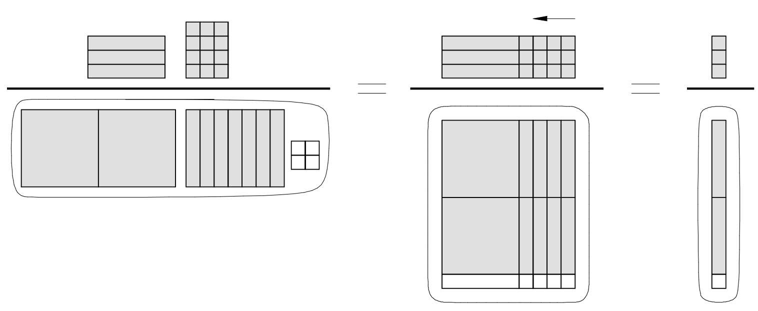
$$\text{Answer \#2: } 1+\frac{4}{x-3}$$

We often stop with answer #1 because this expression, having only one fraction, is easier to work with than answer #2, which has two different terms. To illustrate the these results, we will reduce the following examples:



$$\frac{x^2+x-6}{x+3} = \frac{(x+3)(x-2)}{1(x+3)} = \frac{x-2}{1} = x-2$$

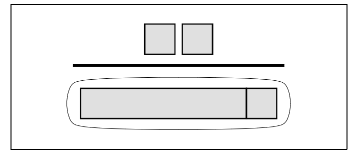
Example 1:

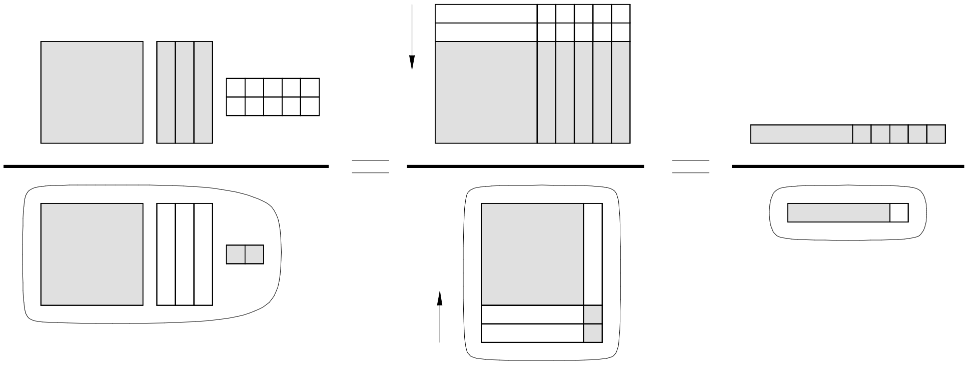


$$\frac{3x+12}{2x^2+7x-4} = \frac{3(x+4)}{(2x-1)(x+4)} = \frac{3}{2x-1}$$

Example 2:

Example 3:





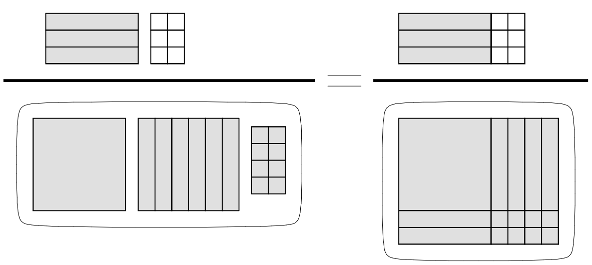
$$\frac{x^2+3x-10}{x^2-3x+2} = \frac{(x-2)(x+5)}{(x-2)(x-1)} = \frac{x+5}{x-1}$$

We will stop here, although $1+\dfrac{6}{x-1}$ is also correct.

When There are No Common Factors

If the numerator and denominator of a rational expression have no common factors, then we can factor both the numerator and the denominator as much as possible and leave them in factored form without reducing.

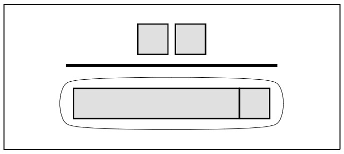
For example:



$$\frac{3x-6}{x^2+6x+8} = \frac{3(x-2)}{(x+2)(x+4)}$$

Exercises

Reduce:



1. $\dfrac{2x-6}{x-3}$

2. $\dfrac{x+5}{3x+15}$

3. $\dfrac{3x+6}{2x+4}$

4. $\dfrac{x^2-4x}{3x}$

5. $\dfrac{x+3}{3x+9}$

6. $\dfrac{2x+18}{x^2+9x}$

Reduce to an expression with a remainder (add and subtract equal amounts if necessary):

7. $\dfrac{x+5}{x+2}$

8. $\dfrac{x-3}{x-1}$

9. $\dfrac{x+2}{x+5}$

10. $\dfrac{2x+3}{2x-5}$

11. $\dfrac{3x-5}{x-3}$

12. $\dfrac{2x+6}{x+2}$

Factor. Reduce if possible:

13. $\dfrac{2x+6}{x^2+x-6}$

14. $\dfrac{x^2+3x+2}{x^2-3x-10}$

15. $\dfrac{3x-12}{x^2+5x+6}$

16. $\dfrac{x^2+6x+5}{3x+3}$

17. $\dfrac{2x^2-x-6}{x^2-7x+10}$

18. $\dfrac{3x}{x^2 - 7x}$

19. $\dfrac{x^2 + x - 12}{3x^2 + 5x - 2}$

20. $\dfrac{2x^2 - 3x}{2x^2 + 7x - 15}$

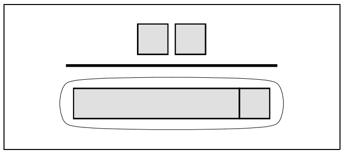
$$\begin{array}{r}
14\ {}^{15}\!/_{17} \\
17\overline{)\ 253} \\
-\underline{17} \\
83 \\
-\underline{68} \\
15
\end{array}$$

$$\frac{253}{17} = 14\,\frac{15}{17}$$

$$(17 \cdot 14\ {}^{15}\!/_{17}) = (17 \cdot 14) + (17 \cdot {}^{15}\!/_{17})$$

$$\begin{array}{r}
14 \\
\times\ 17 \\
\hline
98 \\
+\ 140 \\
\hline
238 \\
+\ 15 \\
\hline
253
\end{array}$$

$$(17 \cdot {}^{15}\!/_{17}) = 15$$

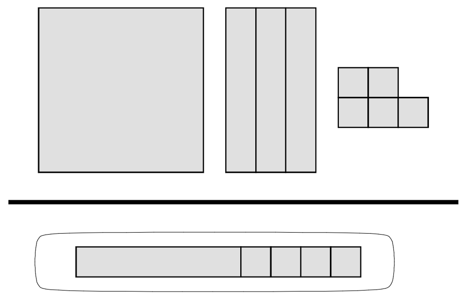


Section 3
Division Using Chips

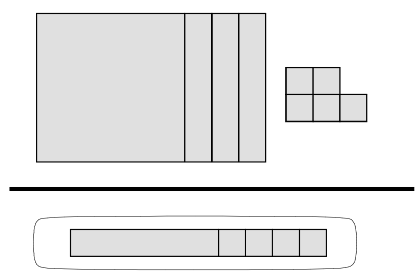
Division with Numbers

If we have a fraction containing whole numbers, with the numerator larger

$$\frac{x^2 + 3x + 5}{x + 4}$$

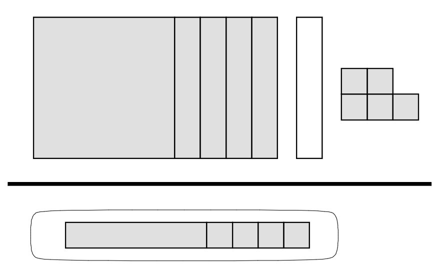


than the denominator we know that we can treat the fraction as a division problem; we divide the bottom number into the top number to get a simplified answer. For example, consider the fraction $^{253}/_{17}$. We can simplify this fraction by doing long division:

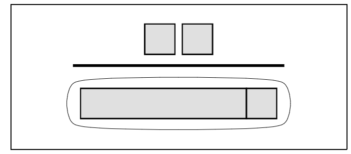


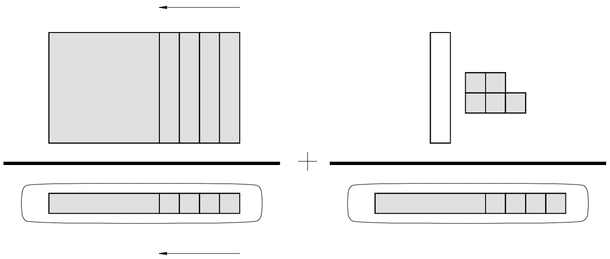
Once we have done this long division, we can check our result by multiplying the divisor (17) times the answer (14 $^{15}/_{17}$), to see if we get the original numerator (253).

$$\frac{x^2 + 4x + (-x) + 5}{x + 4}$$

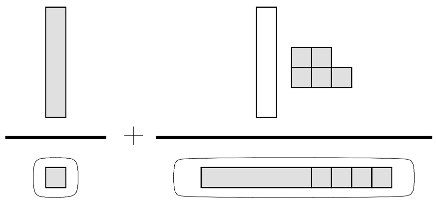


Here is the check:



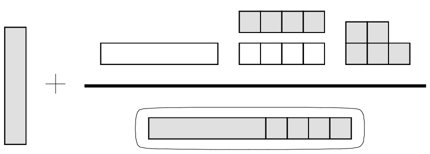


$$\frac{x(x+4)}{1(x+4)} + \frac{{}^{-}x+5}{x+4}$$



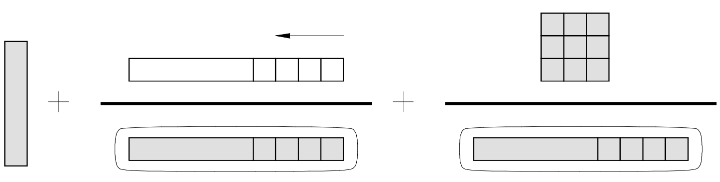
$$\frac{x}{1} + \frac{{}^{-}x+5}{x+4}$$

Division with Unknowns



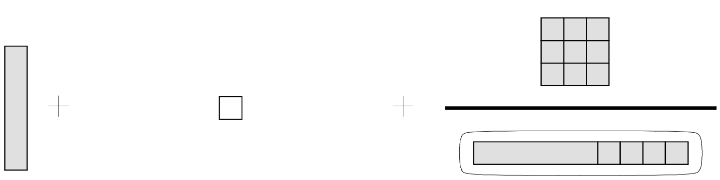
$$x + \frac{{}^{-}x - 4 + 9}{x+4}$$

There is a similar process for reducing rational expressions. If the numerator



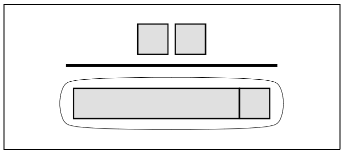
$$x + \frac{({}^{-}1)(x+4)}{1(x+4)} + \frac{9}{x+4}$$

$$x + \frac{{}^{-}1}{1} + \frac{9}{x+4}$$



$$x + {}^{-}1 + \frac{9}{x+4}$$

of a rational expression is of the same or higher order than the denominator,



we can reduce the rational expression through a process of long division. The method is similar to what we have done already in section 1 of this chapter.

For example:

$$\frac{x^2 + 3x + 5}{x + 4}$$

We need to arrange the terms of the numerator so that they are in groups which share the dimension of length with the denominator. We can do this, you will recall, by adding equal numbers of positive and negative pieces until we get terms which will work. Begin with the x^2 and the x terms of the numerator.

These first two terms can make a rectangle having the length of the denominator $(x + 4)$ if we add a $+x$ and a $-x$, and group the $+x$'s with the x^2 pieces.

This gives:

Now that the first two terms of the numerator share the same length as the denominator, we separate the expression into two fractions. The first fraction reduces (collapses) as shown:

To repeat the process, we add plus and minus units to the second fraction's numerator, making another fraction which reduces:

$$\frac{x^2 + 4x}{x + 4} + \frac{^-1x + 5}{x + 4}$$

$$\frac{x\cancel{(x + 4)}}{1\cancel{(x + 4)}} + \frac{^-1x + 5}{x + 4}$$

The numerator of the final fraction has no x-bars, so it cannot match the length of the denominator. This is the remainder; it remains a fraction and cannot be reduced. We now have two terms and a small fraction remainder.

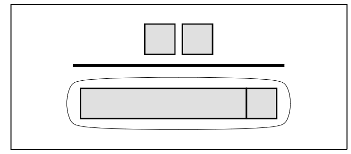
Let's follow through the steps of the long division process.

$$\frac{x^2 + 3x + 5}{x + 4}$$

We look at the first terms of the numerator and denominator. The numerator has an x^2, which is x times the first term (x) of the denominator. We build a rectangle with one dimension as the length of the denominator $(x + 4)$, and the other dimension of x (which gives us the x^2), making the rectangle

$$(x)(x + 4) = x^2 + 4x.$$

We add a $+x$ and a $-x$ to the numerator, giving us $x^2 + 4x$:



$$\frac{x^2 + 4x - 1x + 5}{x + 4}$$

The first two terms have one factor (one dimension) which matches the denominator, so we split the expression into two fractions. We then factor and reduce the first fraction:
The first terms of both the numerator and denominator of the second fraction contain an x. On the top, $(-x)$ is $({}^{-}1)$ times x; in order to have a factor of $(x + 4)$ in the numerator we will need a numerator having the terms ${}^{-}1(x + 4)$ which is $-x - 4$. To get this we add a ${}^{-}4$ and a ${}^{+}4$ to the numerator, and split off a second fraction which will reduce, leaving a remainder.

$$x + \frac{{}^{-}x + 5}{x + 4} = x + \frac{{}^{-}x - 4 + 4 + 5}{x + 4}$$

$$= x + \frac{({}^{-}1)(x + 4)}{(1)(x + 4)} + \frac{9}{x + 4}$$

$$= x + ({}^{-}1) + \frac{9}{x + 4}$$

This is our answer.
To summarize:

$(x + 4)$ goes into $(x^2 + 3x + 5)$, $x + {}^{-}1 + \frac{9}{x + 4}$ times.

(We will show how this process is the same as regular numerical long division later in this section.)

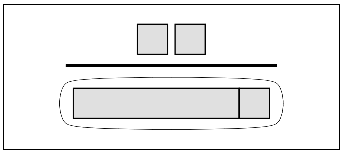
To get a feel for what we have done, let's check our results for a specific value of x, to show that our solution gives the correct answer. If we choose $x = 3$, then we can substitute this value for x into the original expression and into our answer to see if the results are the same. Our result says

$$\frac{x^2 + 3x + 5}{x + 4} = x + {}^{-}1 + \frac{9}{x + 4}$$

Substituting $x = 3$, we get the following:

$$\frac{x^2 + 3x + 5}{x + 4} = \frac{(3)^2 + 3(3) + 5}{3 + 4}$$

$$= \frac{9 + 9 + 5}{7}$$



$$= \frac{23}{7}$$

$$= \mathbf{3\frac{2}{7}} \quad \leftarrow$$

$$x + {}^{-}1 + \frac{9}{x+4} = 3 - 1 + \frac{9}{3+4}$$

$$= 2 + \frac{9}{7}$$

$$= 2 + 1\frac{2}{7}$$

$$= \mathbf{3\frac{2}{7}} \quad \leftarrow$$

We have confirmed that our long division process was correct. When x is 3,

$$\text{Is } (x+4) \text{ times } x + {}^{-}1 + \frac{9}{x+4} \text{ equal to } (x^2 + 3x + 5)?$$

$$(x+4)\left(x + {}^{-}1 + \frac{9}{x+4}\right) = (x+4)(x + {}^{-}1) + \frac{\cancel{(x+4)}}{1} \cdot \frac{9}{\cancel{(x+4)}}$$

$$= x^2 + {}^{-}x + 4x - 4 + 9$$

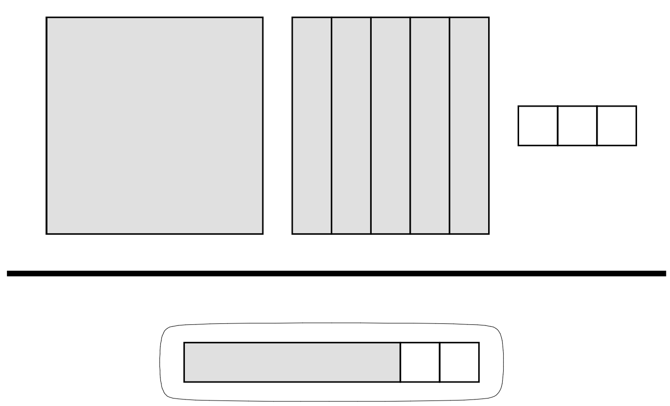
$$= x^2 + 3x - 4 + 9$$

$$= x^2 + 3x + 5$$

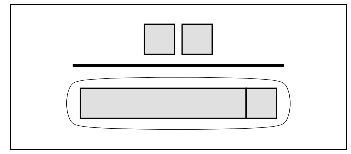
we get the same value for the original expression and for the expression after long division.

Now you try it for a different value of x. Does it work every time? How

$$\frac{x^2 + 5x - 3}{x - 2}$$

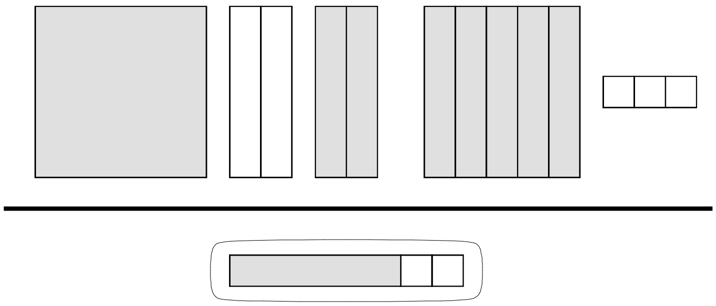


about when x is negative? Let $x = {}^{-}2$:

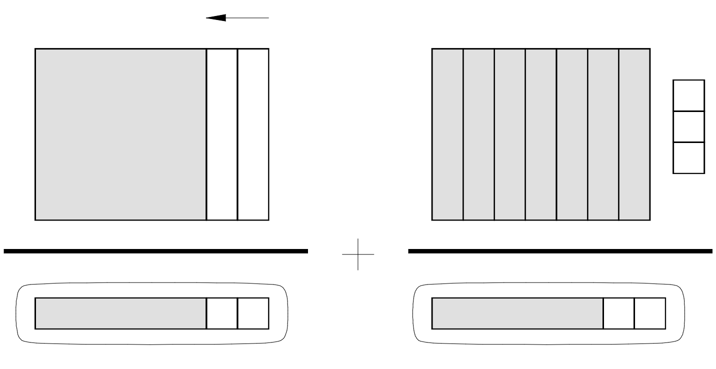


$$\frac{x^2+3x+5}{x+4} = \frac{(^-2)^2+3(^-2)+5}{(^-2)+4}$$

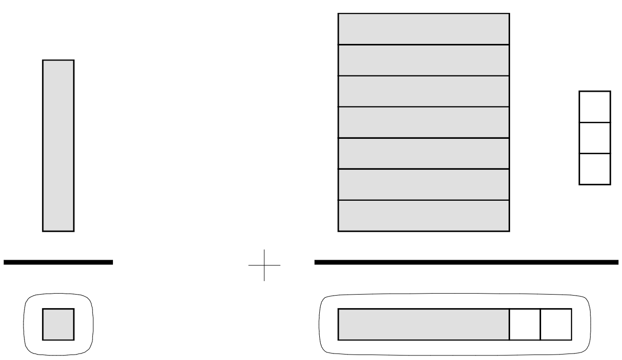
$$= \frac{4-6+5}{^+2}$$



$$\frac{x^2-2x+7x-3}{x-2}$$



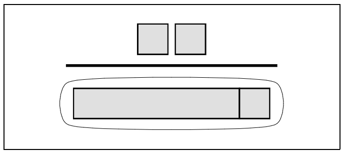
$$\frac{x^2-2x}{x-2}+\frac{7x-3}{x-2}$$



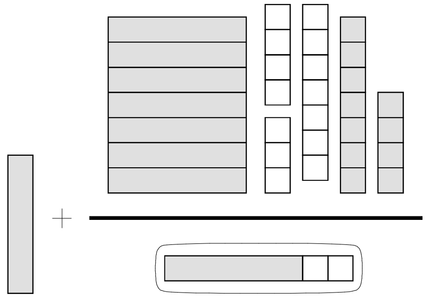
$$\frac{x\cancel{(x-2)}}{1\cancel{(x-2)}}+\frac{7x-3}{x-2}$$

$$= x+\frac{7x-3}{x-2}$$

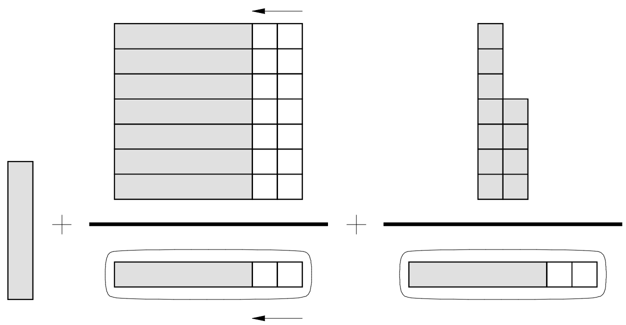
$$= \frac{3}{2} = \mathbf{1\frac{1}{2}} \quad \leftarrow$$



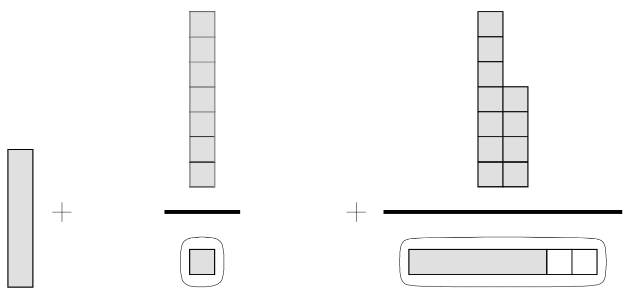
$$x + \frac{7x - 3 - 11 + 11}{x - 2}$$



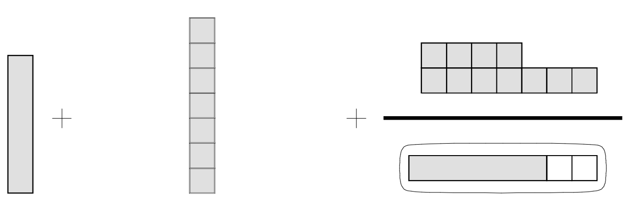
$$x + \frac{7x - 14}{x - 2} + \frac{11}{x - 2}$$

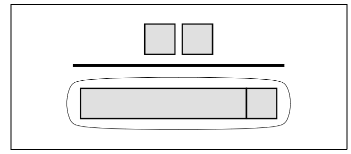


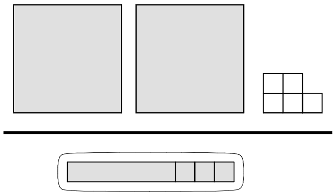
$$x + \frac{7\cancel{(x - 2)}}{\cancel{(x - 2)}} + \frac{11}{x - 2}$$



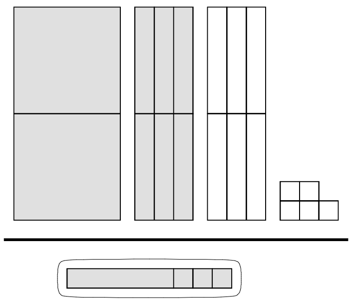
$$x + 7 + \frac{11}{x - 2}$$



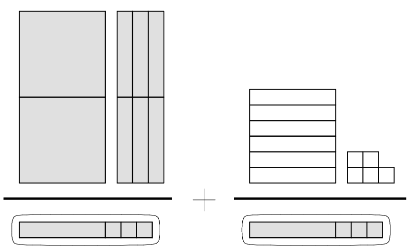




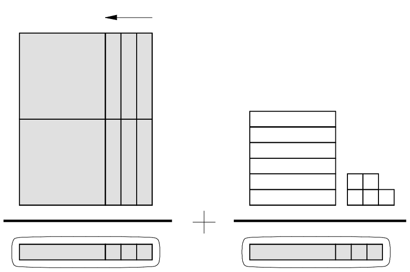
$$\frac{2x^2 - 5}{x + 3}$$



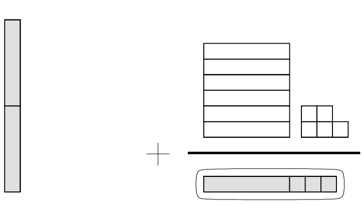
$$\frac{2x^2 + 6x - 6x - 5}{x + 3}$$



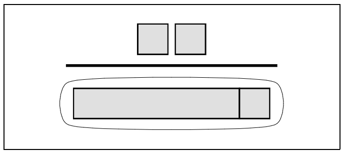
$$\frac{2x^2 + 6x}{x + 3} + \frac{{}^-6x - 5}{x + 3}$$



$$\frac{2x\cancel{(x + 3)}}{\cancel{(x + 3)}} + \frac{{}^-6x - 5}{x + 3}$$

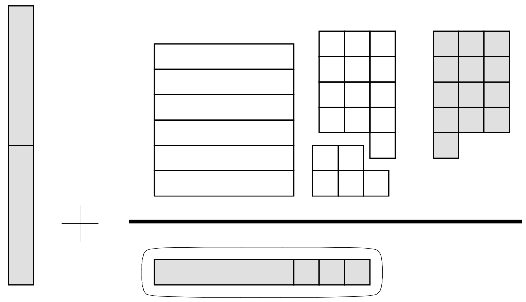


$$2x + \frac{{}^-6x - 5}{x + 3}$$

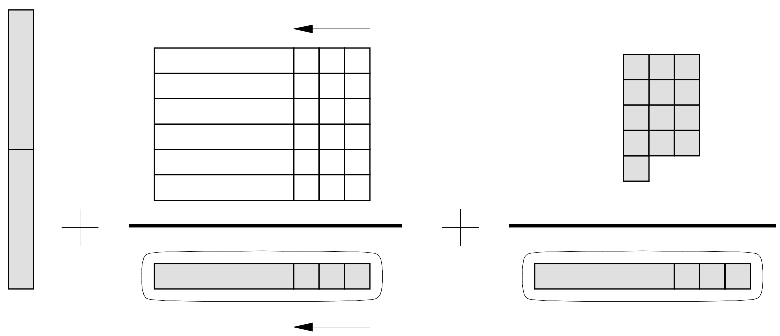


$$x - 1 + \frac{9}{x+4} = {}^{-}2 - 1 + \frac{9}{{}^{-}2+4}$$
$$= {}^{-}3 + \frac{9}{2}$$
$$= {}^{-}3 + 4\frac{1}{2}$$

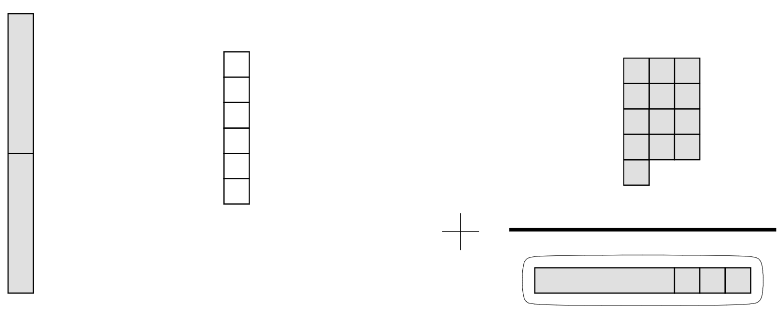
$$2x + \frac{{}^{-}6x - 5 - 13 + 13}{x+3}$$



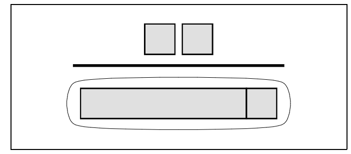
$$2x + \frac{{}^{-}6x - 18}{x+3} + \frac{13}{x+3}$$



$$2x + \frac{{}^{-}6\cancel{(x+3)}}{1\cancel{(x+3)}} + \frac{13}{x+3}$$
$$= 2x + {}^{-}6 + \frac{13}{x+3}$$



$$= \mathbf{1\frac{1}{2}} \quad \leftarrow$$



One more way to check the results of this long division is to multiply the divisor $(x + 4)$ times the answer to see if we get the original dividend:
So the solution works for any value of x.

Let's do another example. Use long division to reduce the following rational expression:

First, we need to make a rectangle in the numerator which uses the x^2, and which has length $(x - 2)$, the length of the denominator. This will require

$$x(x-2) = x^2 - 2x$$

Since the numerator doesn't have any negative x-bars, we must add $^{-}2x$ and $^{+}2x$ to the numerator. Then we break off the first two terms into a separate fraction which can reduce:
In the remaining fraction, we wish to make a rectangle $(x - 2)$ in length, while using the 7 x-bars in the numerator. For the numerator, this will require

$$7(x-2) = 7x - 14$$

Since we have $7x - 3$, we must add a $^{-}11$ and a $^{+}11$ to the numerator:
The result is

$$\frac{x^2 + 5x - 3}{x - 2} = x + 7 + \frac{11}{x - 2}$$

A third and final example will show how this method even fills in for missing terms in the numerator. Consider this expression:
We need a rectangle $2x$ by $(x + 3)$, or $2x^2 + 6x$. So we add $+6x$ and $^{-}6x$ to the numerator:
This breaks into two fractions, the first of which reduces:

The numerator of the second fraction will now need a rectangle which is $(x + 3)$ long, and $^{-}6$ high, or

$$^{-}6(x+3) = {^{-}6x} - 18$$

This requires adding $^{-}13$ and $^{+}13$ to the numerator, and then breaking it into two fractions. The first fraction will reduce, and the other fraction is the remainder.
The result is:

$$\frac{2x^2 - 5}{x + 3} = 2x + {^{-}6} + \frac{13}{x + 3}$$

Exercises

Use chips or drawings to complete the following reductions by long division:

1. $\dfrac{x^2+6x-4}{x+2}$

2. $\dfrac{x^2-5x+3}{x+1}$

3. $\dfrac{2x^2+3x-6}{x-3}$

4. $\dfrac{3x^2-5}{x+1}$

5. $\dfrac{x^2-4x}{x-2}$

Beginning with the given fraction,

$$\frac{253}{17}$$

Place the numerator under the division sign, and the denominator in front as the divisor.

$$17\overline{)\,253}$$

Now look at how many times the 17 goes into the 25 and write the first partial answer (1) above the division bar:

$$\begin{array}{r} \mathbf{1} \\ \mathbf{17}\overline{)\,\mathbf{25}3} \end{array}$$

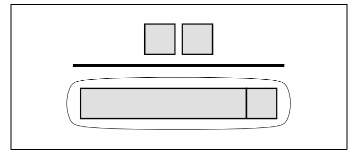
6. $\dfrac{2x^2+6x-5}{x-2}$

Next multiply the first partial answer (the 1) times the 17 and write the answer (also 17) below the 25.

$$\begin{array}{r} 1 \\ 17\overline{)\,253} \\ \underline{17} \end{array}$$

To find out what remains, subtract the bottom 17 from the 25, giving 8, and bring down the 3:

$$\begin{array}{r} 1 \\ 17\overline{)\,253} \\ \underline{-17} \\ 83 \end{array}$$



$$\begin{array}{r} 14 \\ 17\overline{)253} \\ -17 \\ \hline 83 \end{array}$$

Now look at the divisor (17) and at the 83, and decide how many times 17 can go into 83. (The answer is 4.) Place this second partial answer above the division bar as shown.

$$\begin{array}{r} 14 \\ 17\overline{)253} \\ -17 \\ \hline 83 \\ -\ 68 \\ \hline 15 \end{array}$$

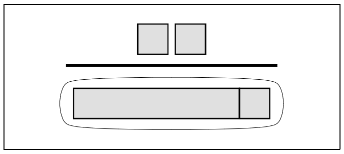
Multiply the second partial answer (the 4) times the 17, and put the result (68) under the 83. Subtract the 68 from the 83, leaving 15.

7. $\dfrac{x^2 - 5x + 1}{x + 3}$

$$\begin{array}{r} 14\tfrac{15}{17} \\ 17\overline{)253} \\ -17 \\ \hline 83 \\ -\ 68 \\ \hline 15 \end{array}$$

Finally, since 17 will not divide into 15, this is our remainder, which is written as the fraction $^{15}/_{17}$.

8. $\dfrac{2x^2 - x}{x + 5}$



$$\frac{x^2 + 3x + 5}{x + 4}$$

Begin with the given rational expression, and place the numerator under the division bar, with the denominator in front as the divisor.

$$x + 4 \overline{\big)\ x^2 + 3x + 5}$$

Now look at the first term of both expressions, (the x and the x^2) and decide how many times x will go into x^2? Write the answer (x) on top of the division bar.

$$\begin{array}{r} x \qquad\quad \\ x + 4 \overline{\big)\ x^2 + 3x + 5} \end{array}$$

Next multiply this first partial answer (the top x) times ($x + 4$), and put the result ($x^2 + 4x$) below the $x^2 + 3x$.

$$\begin{array}{l} \qquad\quad x \\ x + 4 \overline{\big)\ x^2 + 3x + 5} \\ \qquad\quad x^2 + 4x \end{array}$$

To find out what will remain, we must subtract the $x^2 + 4x$ from the $x^2 + 3x$. (The easiest way to do this is to change the signs on both bottom terms, the x^2 and the $4x$, and then add the results to the columns above them as shown.) Bring down the +5.

$$\begin{array}{l} \qquad\quad x \\ x + 4 \overline{\big)\ x^2 + 3x + 5} \\ \qquad\quad \underline{-x^2 + {}^-4x} \\ \qquad\quad\ \ 0 + {}^-1x + 5 \end{array}$$

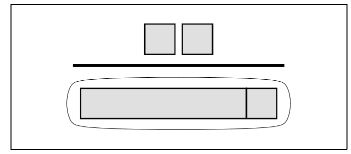
change signs
and add

Now look at the first term of the divisor ($x + 4$) and the $^-1x + 5$ and decide what must multiply the x to give ^-1x. The required multiplier is $^-1$, which is written above the division bar.

$$\begin{array}{l} \qquad\quad x - 1 \\ x + 4 \overline{\big)\ x^2 + 3x + 5} \\ \qquad\quad \underline{-x^2 + {}^-4x} \\ \qquad\qquad\quad {}^-1x + 5 \end{array}$$

Multiply this second partial answer (the $^-1$) times the divisor ($x + 4$) and write the result under the $^-1x + 5$:

$$\begin{array}{l} \qquad\quad x - 1 \\ x + 4 \overline{\big)\ x^2 + 3x + 5} \\ \qquad\quad \underline{-x^2 + {}^-4x} \\ \qquad\qquad\quad {}^-1x + 5 \\ \qquad\qquad\quad \underline{{}^-1x - 4} \end{array}$$



$$\begin{array}{r|l} & x - 1 \\ \hline x + 4 & x^2 + 3x + 5 \\ & \underline{-x^2 + {}^-4x} \\ & \quad\quad {}^-1x + 5 \\ \text{change signs and add} \rightarrow & \quad\quad \underline{x + 4} \\ & \quad\quad 0 + 9 \end{array}$$

To find out how much remains, subtract the $-x - 4$ from the ${}^-1x + 5$. (Again this is done by changing the signs on both terms of the $-x - 4$, making them $+x + 4$, and adding the result to the columns above.)

$$\begin{array}{r|l} & x - 1 + 9/(x+4) \\ \hline x + 4 & x^2 + 3x + 5 \\ & \underline{-x^2 + {}^-4x} \\ & \quad\quad {}^-1x + 5 \\ & \quad\quad \underline{x + 4} \\ & \quad\quad\quad 9/(x+4) \end{array}$$

The 9 which is left is the remainder, and it is written as the numerator of a fraction having the divisor $(x + 4)$ as the denominator. This fraction is added on to the answer.

n 4

Division

$$\frac{x^2 + 5x + 3}{x - 2}$$

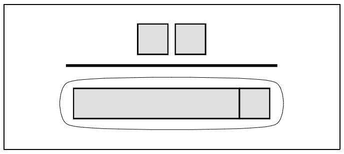
Rewrite as a division

$$\begin{array}{r|l} & x \\ \hline x - 2 & x^2 + 5x + 3 \\ & x^2 - 2x \end{array}$$

Look at first terms and put the partial answer up. Multiply through.

Short-Cut Method of Long Division

We will now look at the regular long division of numbers. Our purpose is



Change signs and add. Bring down the third term.

$$\begin{array}{r} x \qquad\qquad \\ x-2\,\big)\overline{\,x^2+5x+3\,} \\ -\,x^2+2x \qquad \\ \hline 0+7x-3 \end{array}$$

Look at the first terms and write the second partial answer. Multiply through.

$$\begin{array}{r} x+7 \qquad\quad \\ x-2\,\big)\overline{\,x^2+5x+3\,} \\ -\,x^2+2x \qquad \\ \hline +7x-3 \\ +7x-14 \end{array}$$

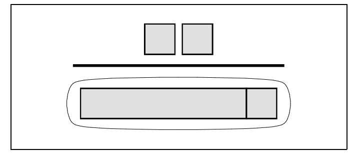
Change signs and add.

$$\begin{array}{r} x+7 \qquad\quad \\ x-2\,\big)\overline{\,x^2+5x+3\,} \\ -\,x^2+2x \qquad \\ \hline 7x-3 \\ -7x+14 \\ \hline 0+11 \end{array}$$

Form a fraction from the remainder.

$$\begin{array}{r} x+7+{}^{11}\!/_{x-2} \\ x-2\,\big)\overline{\,x^2+5x+3\,} \\ -\,x^2+2x \qquad \\ \hline 7x-3 \\ -7x+14 \\ \hline 0+{}^{11}\!/_{(x-2)} \end{array}$$

to discover how to do the same type of long division with *rational expressions*:



$$\frac{2x^2 - 5}{x + 3}$$

$$x + 3 \overline{\big) \; 2x^2 + 0x - 5}$$

Rewrite as a division problem; put a column in for x's by writing the dividend $2x^2 + 0x - 5$

$$\begin{array}{r}
2x \\
x + 3 \overline{\big) \; 2x^2 + 0x - 5} \\
\underline{2x^2 + 6x}
\end{array}$$

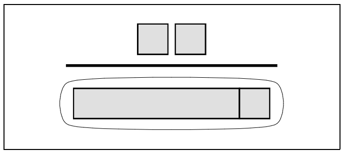
Look at the first terms and write a first partial answer. Multiply through.

$$\begin{array}{r}
2x \\
x + 3 \overline{\big) \; 2x^2 + 0x - 5} \\
\underline{{}^{-}2x^2 + {}^{-}6x} \\
0 + {}^{-}6x - 5
\end{array}$$

Change the signs and add. Bring down the third term.

$$\begin{array}{r}
2x - 6 \\
x + 3 \overline{\big) \; 2x^2 + 0x - 5} \\
\underline{{}^{-}2x^2 + {}^{-}6x} \\
{}^{-}6x - 5 \\
\underline{{}^{-}6x - 18}
\end{array}$$

Look at the first terms and write a second partial answer. Multiply through.



$$
\begin{array}{r|l}
 & 2x \quad - \quad 6 \quad + 13/(x+3) \\
\hline
x + 3 & 2x^2 \quad + \quad 0x \quad - \quad 5 \\
 & {}^-2x^2 \quad + \quad {}^-6x \\
\hline
 & \qquad {}^-6x \quad - \quad 5 \\
 & \qquad {}^+6x \quad + \quad 18 \\
\hline
 & \qquad 0 \quad + \; 13/(x+3)
\end{array}
$$

Change signs and add. Form the remainder into a fraction.

Dividing Rational Expressions

The long division method for rational expressions is nearly identical with that for numbers as shown above. We will illustrate using our first example from the last section. As you work through the problems, refer back to the pictures of the method we developed earlier for dividing rational expressions so you can see where the steps come from.

Example 1:

$$\frac{x^2 + 3x + 5}{x + 4}$$

The final result is:

$$\frac{x^2 + 3x + 5}{x + 4} = x - 1 + \frac{9}{x + 4}$$

Now we will work through this method for the other two examples.

Example 2:

The final answer:

$$\frac{x^2 + 5x - 3}{x - 2} = x + 7 + \frac{11}{x - 2}$$

Example 3:

The final answer:

$$\frac{2x^2 - 5}{x + 3} = 2x - 6 + \frac{13}{x + 3}$$

Exercises

Reduce the following by the long division method:

1. $\dfrac{x^2+3x-5}{x-2}$

2. $\dfrac{2x^2-3x-1}{x+3}$

3. $\dfrac{x^2+7x+2}{x+1}$

4. $\dfrac{3x^2+5}{x-4}$

5. $\dfrac{x^2-2x}{x+2}$

6. $\dfrac{5x^2+7x-11}{x-5}$

7. $\dfrac{x^2-2x+5}{x-3}$

8. $\dfrac{2x^2-3x}{x+5}$

9. $\dfrac{x^2-3x-2}{x+1}$

10. $\dfrac{3x^2+2x+1}{x+4}$

Section 5
Multiplication

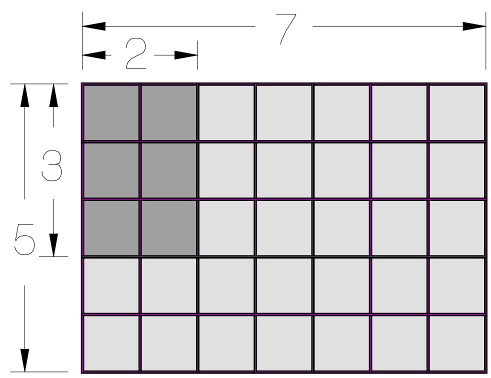
Multiplying Fractions

To understand how multiplication works with rational expressions, we need to briefly review multiplication of common fractions. You will remember (from Chapter 4) that when we multiply two fractions

$$\frac{3}{5} \cdot \frac{2}{7}$$

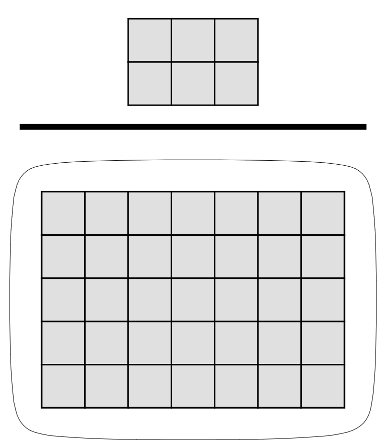
the denominators multiply $(5 \cdot 7)$ to give the total number of pieces needed to make one whole unit; and the numerators multiply $2 \cdot 3$ to tell us how many of these pieces we have.

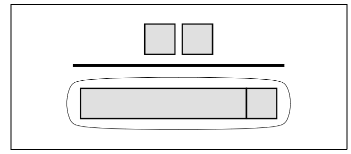
$$\frac{3}{5} \cdot \frac{2}{7} = \frac{6}{35}$$



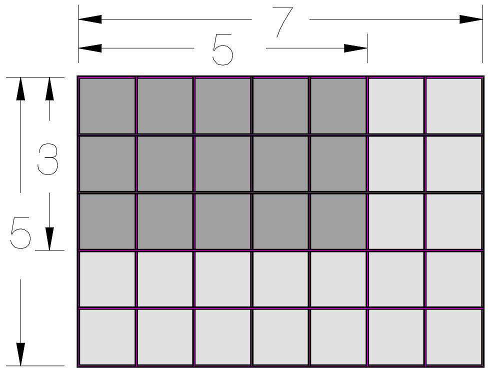
In this example the result is $^6/_{35}$, or six thirty-fifth's of one whole unit.

Using our new form of representation this result can also be shown as:



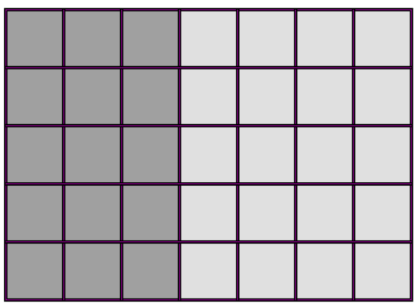


If we had chosen to multiply the fractions $\frac{3}{5} \cdot \frac{5}{7}$ our result would have looked like this:



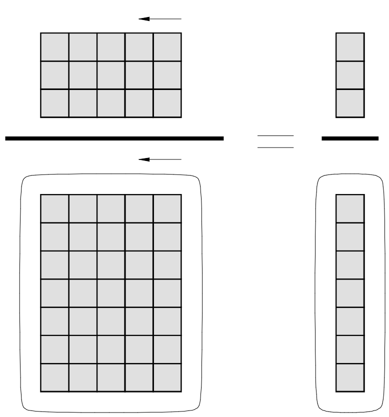
$$\frac{3}{5} \cdot \frac{5}{7} = \frac{15}{35}$$

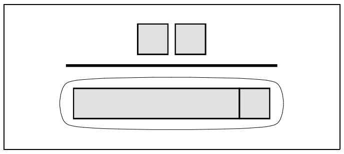
We can see the result is 15 chips out of 35, but the arrangement of the chips can be changed to show how this result can be reduced. 15 out of 35 can also be arranged like this:



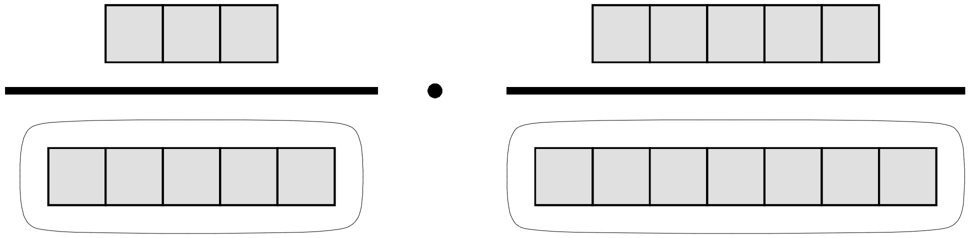
We see 3 shaded columns out of a total of 7 columns, for a reduced result of $\frac{3}{7}$.

In the new representation, this reduction is easily seen, since the rectangles of the numerator and the denominator are the same length in one dimension, which means that they can reduce (collapse), also giving $\frac{3}{7}$.



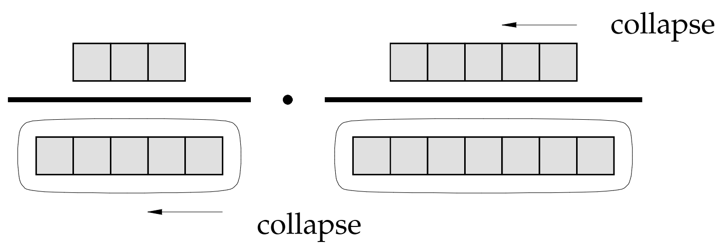


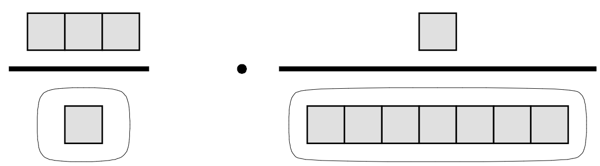
We could have reduced even before we multiplied:

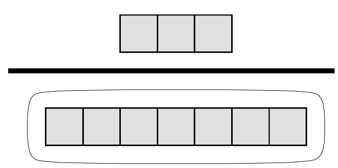


$\frac{3}{5}$ $\frac{5}{7}$

Since the numerator of each fraction in the multiplication will be one dimension of the final numerator rectangle, and the denominator of each fraction will be one dimension of the final denominator rectangle, the 5 in the second fraction's numerator will reduce (cancel) with the 5 in the first fraction's denominator.





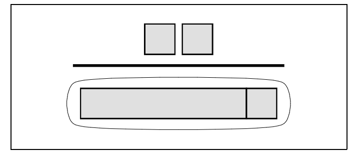


Numerically, we would write this result as

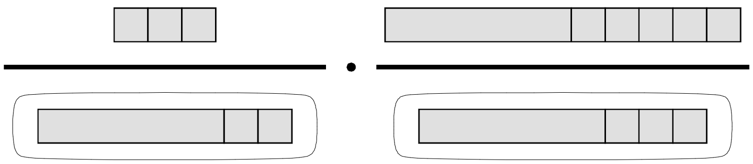
$$\frac{3}{5} \cdot \frac{5}{7} = \frac{3}{\cancel{5}} \cdot \frac{\cancel{5}}{7} = \frac{3}{1} \cdot \frac{1}{7} = \frac{3}{5}$$

(You might wish to review the multiplication and reduction of fractions from Chapter 4 before going on.)

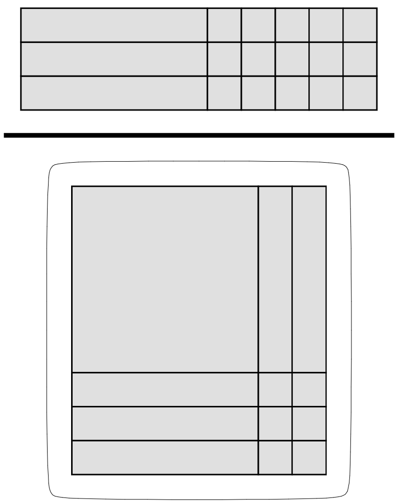
Multiplying Rational Expressions



We are now ready to consider the multiplication of two rational expressions having unknowns. Let's begin with an uncomplicated example:



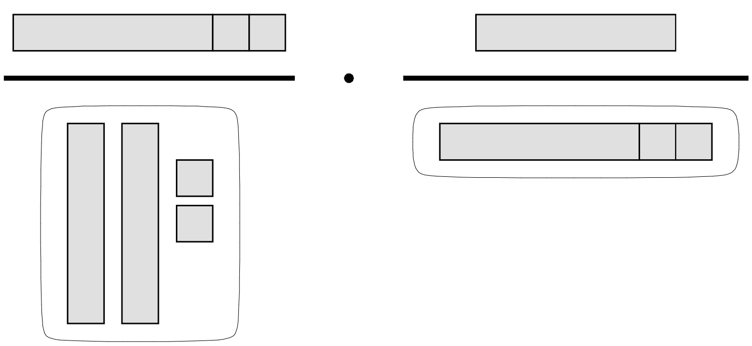
$$\frac{3}{x+2} \cdot \frac{x+5}{x+3}$$



$$\frac{(3)\cdot(x+5)}{(x+2)\cdot(x+3)} = \frac{3x+15}{x^2+5x+6}$$

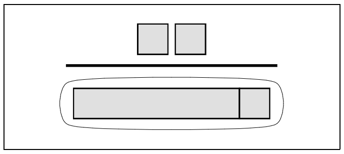
Since none of the dimensions of the numerator and denominator rectangles are the same, this result cannot reduce. *The result written in symbols can be correctly shown in either factored or unfactored form.*

Now let's multiply two other expressions:



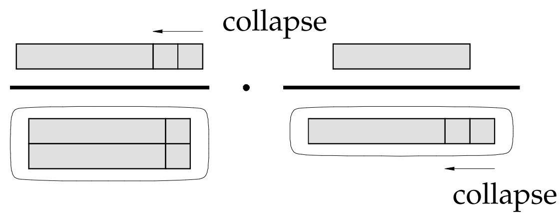
$$\frac{x+2}{2x+2} \cdot \frac{x}{x+2}$$

Remember that we can't cancel pieces; we must make rectangles from the pieces on the top and bottom of each fraction, and if the rectangles have a *dimension* (*factor*) in common on the top and bottom, then that *dimension* can collapse (canceling that *factor*).

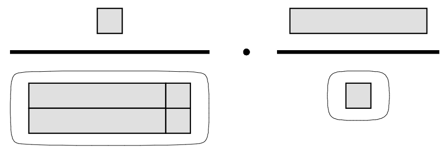


We continue:

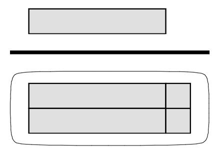
$$\frac{\cancel{(x+2)}}{2(x+1)} \cdot \frac{x}{\cancel{(x+2)}}$$



$$\frac{1}{2(x+1)} \cdot \frac{x}{1}$$



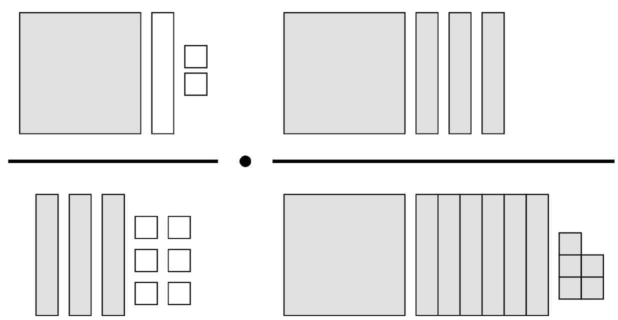
$$\frac{x}{2(x+1)}$$



In this example we first factor (make rectangles on) the top and bottom of each fraction, and then we reduce (cancel) factors common to the numerators and denominators before multiplying to get the final result.

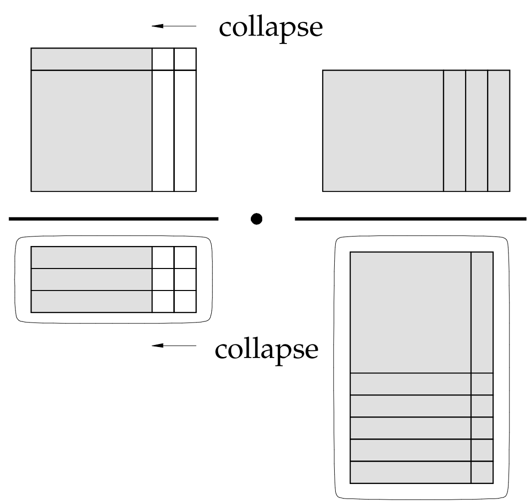
Here's one more example:

$$\frac{x^2 - x - 2}{3x - 6} \cdot \frac{x^2 + 3x}{x^2 + 6x + 5}$$

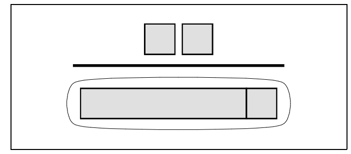


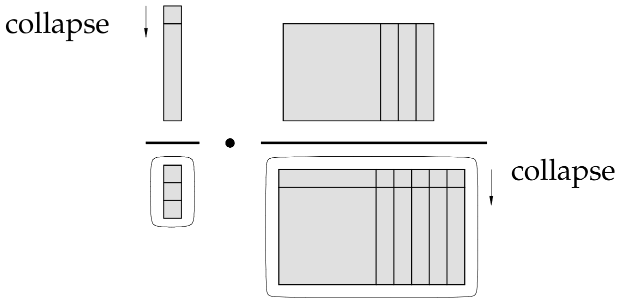
$$\frac{(x+1)\cancel{(x-2)}}{3\cancel{(x-2)}} \cdot \frac{x(x+3)}{(x+1)(x+5)}$$

$$\frac{(x+1)}{3} \cdot \frac{x(x+3)}{(x+1)(x+5)}$$

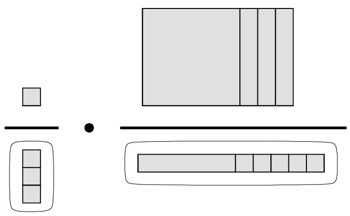


Continuing on:

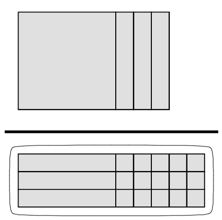




$$\frac{\cancel{(x+1)}}{3} \cdot \frac{x(x+3)}{\cancel{(x+1)}(x+5)}$$



$$\frac{1}{3} \cdot \frac{x(x+3)}{(x+5)}$$



$$\frac{x(x+3)}{3(x+5)}$$

Multiplication of rational expressions is a process of factoring the given numerators and denominators, and reducing by canceling *like factors (dimensions)* from top and bottom. Finally, the remaining factors are multiplied across the top and across the bottom. The result can be left in factored form.

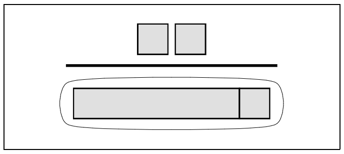
Exercises

Multiply the following rational expressions, canceling when possible and leaving answers in factored form:

1. $\frac{2}{x-2} \cdot \frac{x+2}{x+5}$

2. $\frac{x+1}{2x+3} \cdot \frac{x+3}{2x+2}$

3. $\frac{3x-6}{2x-4} \cdot \frac{2x+4}{x-3}$



4. $\dfrac{5x}{x+3} \cdot \dfrac{x-4}{x^2-4x}$

5. $\dfrac{x^2+6x}{x^2-7x+12} \cdot \dfrac{x-3}{3x+18}$

6. $\dfrac{x+7}{x^2+2x-35} \cdot \dfrac{x-5}{x^2-4x+4}$

7. $\dfrac{x+2}{x^2-x-6} \cdot \dfrac{x^2-4x+3}{x^2-5x+6}$

8. $\dfrac{x^2+5x+4}{x^2+3x-4} \cdot \dfrac{x^2-2x+1}{x^2+7x+6}$

9. $\dfrac{x^2-x-12}{3x+9} \cdot \dfrac{x^2-8x+16}{x^2-16}$

10. $\dfrac{x+2}{x-2} \cdot \dfrac{x^2+x-2}{x^2-4}$

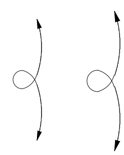
Section 6
Division

Dividing With Rational Expressions

Before attempting to divide using rational expressions, you may want to review the section on dividing with fractions in Chapter 4.

The process of division with rational expressions works in the same way as with ordinary fractions. We begin with the idea that dividing one rational expression by another is accomplished by inverting (turning over) the dividing fraction and then multiplying. When performing the multiplication we can reduce (cancel) as in the last section. For example, divide these rational expressions:

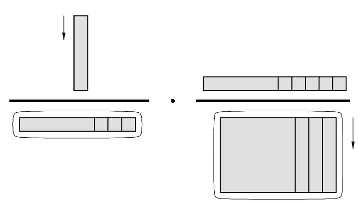
$$\frac{x}{x+3} \div \frac{x^2+3x}{x+5}$$



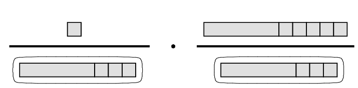
First invert the divisor and change the sign to multiply.

$$\frac{x}{x+3} \cdot \frac{x+5}{x^2+3x}$$

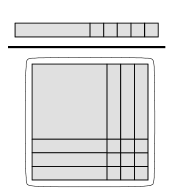
Now multiply as before:



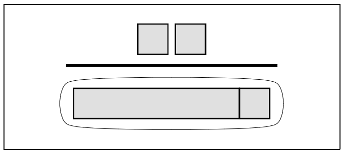
$$\frac{(x)}{x+3} \cdot \frac{x+5}{(x)(x+3)}$$



$$\frac{1}{x+3} \cdot \frac{x+5}{(x+3)}$$



$$\frac{x+5}{(x+3)(x+3)} \text{ or } \frac{x+5}{x^2+6x+9}$$



It is very important to remember in these division problems that *you cannot cancel (reduce) the given fractions until you have inverted the divisor and are ready to multiply*. Remember that

$$\frac{6}{1} \text{ divided by } \frac{1}{3} \text{ is 18, } \textit{not } 2$$

Exercises

Divide or multiply as indicated:

1. $\dfrac{x+2}{x^2-3x} \div \dfrac{2x+4}{x^2}$

2. $\dfrac{x^2-2x}{3x-9} \div \dfrac{x^2-4x+4}{x^2-5x+6}$

3. $\dfrac{x^2+7x+6}{3x+6} \cdot \dfrac{x+6}{x^2+2x+1}$

4. $\dfrac{x^2+7x+6}{3x+6} \div \dfrac{x+6}{x^2+2x+1}$

5. $\dfrac{2x+3}{2x^2+5x+3} \div \dfrac{3x^2+6x}{x^2-x-2}$

6. $\dfrac{2x-6}{x^2+x-6} \cdot \dfrac{x^2-4x+4}{x^2-3x}$

7. $\dfrac{x-2}{x+4} \div \dfrac{x^2-4}{x+1}$

8. $\dfrac{4x^2-1}{2x^2+x} \cdot \dfrac{2x+3}{2x-1}$

9. $\dfrac{x+3}{x-3} \div \dfrac{x^2-9}{x^2-6x+9}$

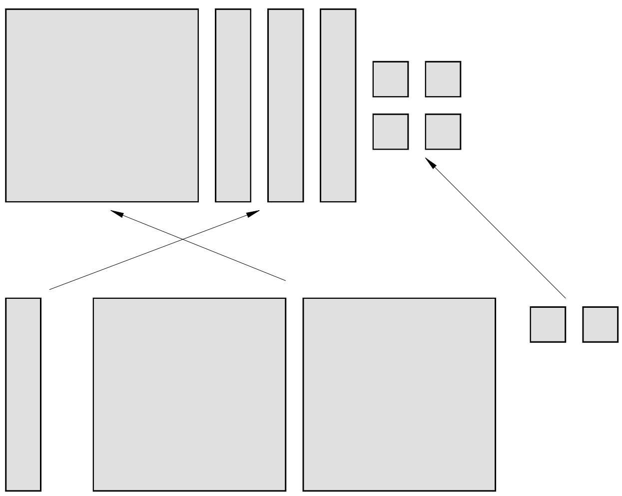
10. $\dfrac{x^2-25}{x^2-10x+25} \cdot \dfrac{x^2-2x-15}{3x-12}$

Section **7**

Addition

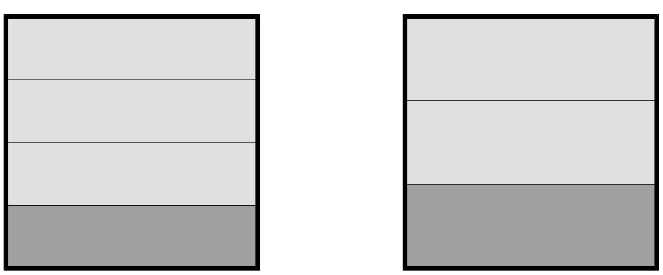
Adding and Subtracting Rational Expressions

To understand how to add rational expressions, we must briefly review the process of adding algebraic expressions and common fractions. When we combine two groups into one group (what we call adding) *we can only combine the pieces that are the same size*. For example, combine these pieces:

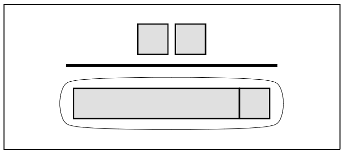


The x^2 pieces go with other x^2 pieces, x's with other x's and units with other units (which we call **combining like terms**). Pieces of different sizes must be kept in separate groups connected by a + or – sign.

When adding fractions we must go through a process to be sure that all the pieces we wish to combine are the same size. When adding two fractions, the fractions may begin as *different* size pieces. For example if we wish to add ¼ of a unit and ⅓ of a unit we begin with two pieces which are different sizes.



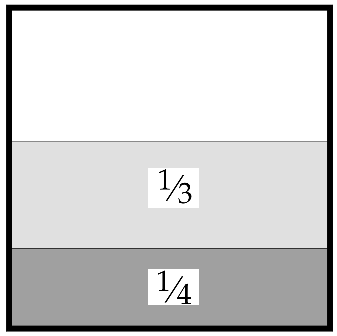
$\frac{1}{4}$ $\frac{1}{3}$



We can put them side by side, but how do we write the result other than

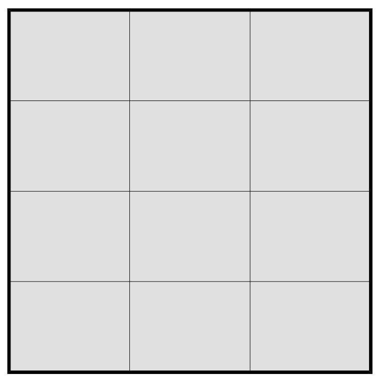
$$\frac{1}{4}+\frac{1}{3}?$$

Pieces of different sizes cannot combine.

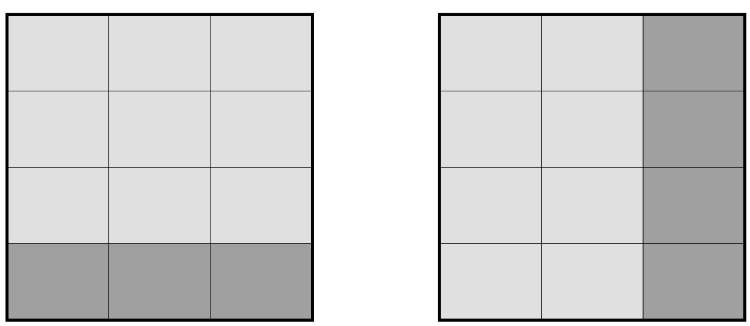


The process we go through to make both ¼ and ⅓ out of pieces which are the same size is called **finding the common denominator**. Here is how it is done:

Take the unit square and divide it one direction into fourths and divide it the other direction into thirds.

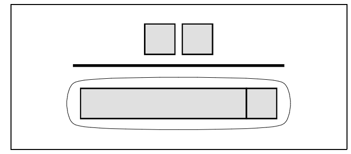


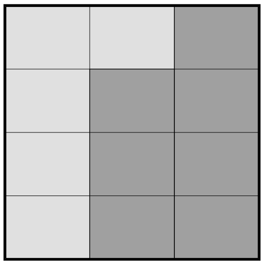
Now the unit square is divided into 12 parts, and the ¼ and ⅓ look like this:



$\frac{1}{4}$ $\frac{1}{3}$

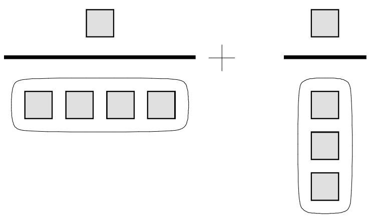
We can see that ¼ equals 3⁄12 and ⅓ equals 4⁄12. Rearranging the shaded rectangles will allow us to show that the total is 7⁄12:



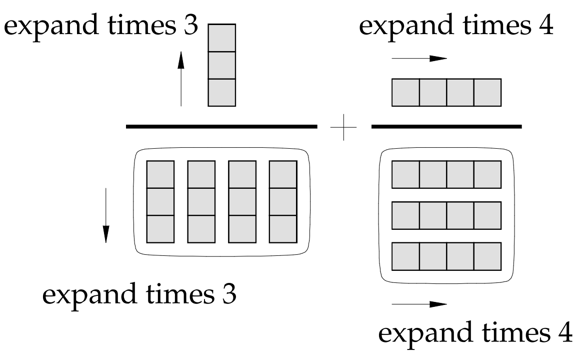


$$\frac{1}{4} = \frac{3}{12} \quad \text{and} \quad \frac{1}{3} = \frac{4}{12}$$

If we use the new method of representing rational expressions, this would be pictured as follows:

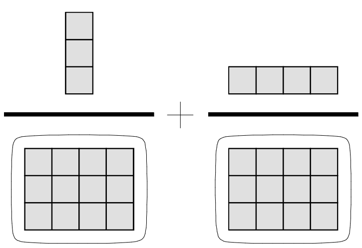


$$\frac{1}{4} + \frac{1}{3}$$



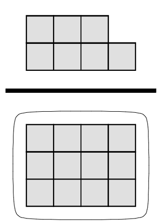
Now we expand the top and bottom of both fractions so that their denominators become rectangles of the same size. (This is the reverse process of collapsing or reducing.)

$$\frac{3 \cdot 1}{3 \cdot 4} + \frac{4 \cdot 1}{4 \cdot 3}$$

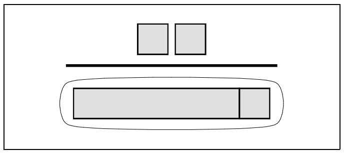


$$\frac{3}{12} + \frac{4}{12}$$

When the units (denominators) are the same, the pieces can be combined (added together).



$$\frac{7}{12}$$



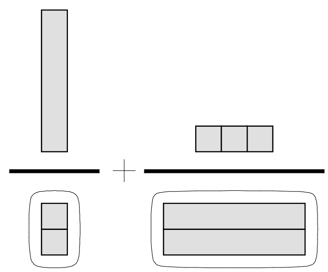
Here we see that, rather than collapsing rectangles which have like dimensions, we have expanded rectangles to give them like dimensions. We are making the units the same size (same denominator) so we can combine (add) the numerators together. If we begin with two denominators which share no factors (like 4 and 3), then the final rectangle required for the denominator will have each of the original denominators as one dimension ($4 \cdot 3 = 12$).

In order to add fractions together, they must both have the same denominator (the same size pieces). The same requirement is true for adding positive and negative fractions, or subtracting fractions.

In order to add fractions together, they must both have the same denominator (the same size pieces).

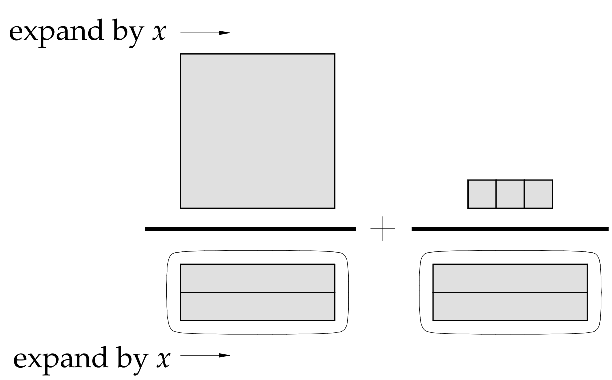
Since rational expressions are simply fractions having unknown terms, this same requirement of having common denominators also holds when we wish to add or subtract rational expressions. To meet this requirement, we often must expand the dimensions of the rectangles used in one or both fractions to create denominator rectangles which are the same for both fractions being added. For example, let's add the following:

$$\frac{x}{2} + \frac{3}{2x}$$

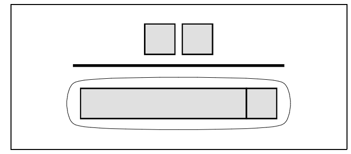


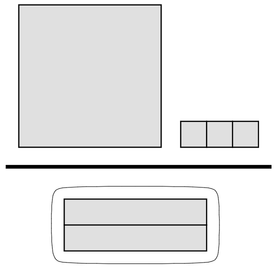
The first denominator rectangle has dimensions 2 by 1, while the other denominator rectangle has dimensions 2 by x. To make the dimensions of the two denominators the same, we must expand the first denominator (and also the first numerator) to have a width of x.

$$\frac{x \cdot x}{x \cdot 2} + \frac{3}{2x} = \frac{x^2}{2x} + \frac{3}{2x}$$



Now that the denominators are the same, we can put both numerators over the same denominator.



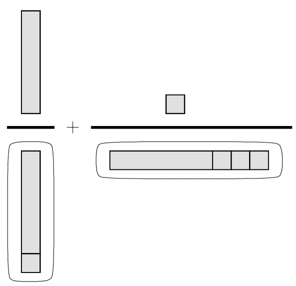


$$\frac{x^2+3}{2x}$$

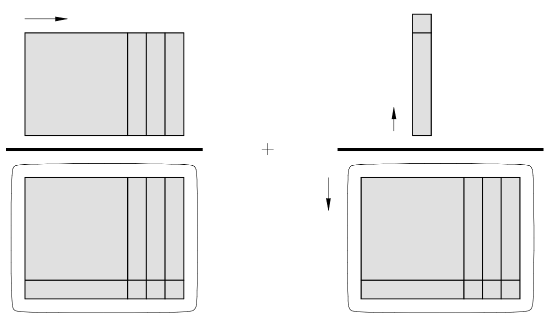
Since the numerator of this expression won't factor, this is our final result.

You will notice that adding two rational expressions means that they get combined over the same denominator (into one fraction), but it doesn't necessarily mean that the numerator simplifies greatly in form.

Here's another example:

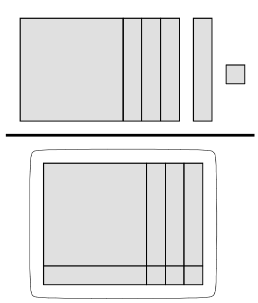


$$\frac{x}{x+1}+\frac{1}{x+3}$$



Expand to give like denominators:

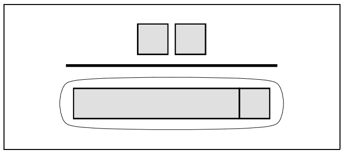
$$\frac{x^2+3x}{(x+1)(x+3)}+\frac{x+1}{(x+1)(x+3)}$$



Combine into one fraction:

$$\frac{x^2+4x+1}{(x+1)(x+3)}$$

Since the numerator cannot be factored, this fraction cannot be reduced.

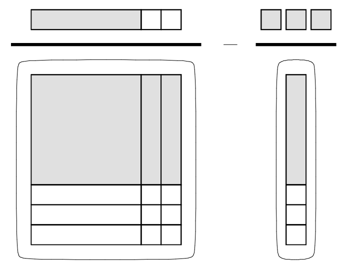


We will give one example of a **subtraction** of rational expressions. Just like with common fractions, a negative sign in front of a rational expression will flip the chips (change the sign) of the numerator, but leave the denominator (the size of a unit) unchanged. For example;

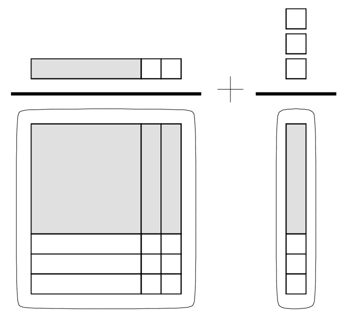
$$-\left(\frac{3}{7}\right) = \frac{{}^{-}3}{7}$$

Here is an example of subtracting rational expressions:

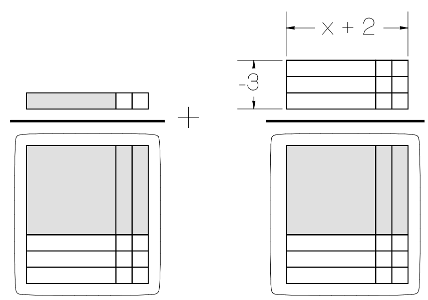
$$\frac{x-2}{x^2-x-6} - \frac{3}{x-3}$$



$$\frac{x-2}{(x+2)(x-3)} + \frac{{}^{-}3}{x-3}$$

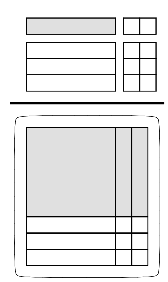


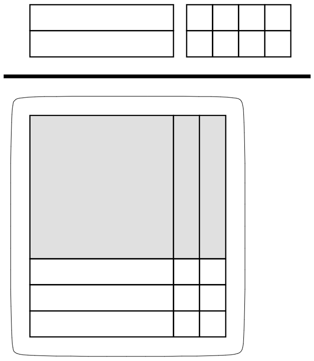
$$\frac{x-2}{(x+2)(x-3)} + \frac{{}^{-}3(x+2)}{(x-3)(x+2)}$$

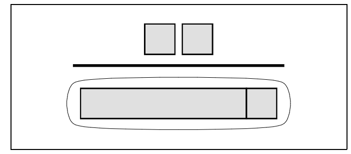


$$\frac{x-2}{(x+2)(x-3)} + \frac{{}^{-}3x-6}{(x-3)(x+2)}$$

$$\frac{x-2+{}^{-}3x-6}{(x+2)(x-3)}$$







$$\frac{-2x-8}{(x+2)(x-3)} = \frac{-2(x+4)}{(x+2)(x-3)}$$

Exercises

Add or subtract as indicated; reduce your answers:

1. $\frac{2}{3}+\frac{1}{4}$

2. $\frac{3}{5}-\frac{1}{2}$

3. $\frac{3}{7}-\frac{1}{3}$

4. $\frac{5}{6}+\frac{1}{2}$

5. $\frac{3}{x}+\frac{2}{3}$

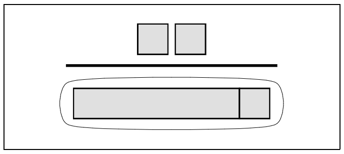
6. $\frac{2}{x+1}-\frac{3}{x}$

7. $\frac{3x}{x-2}+\frac{1}{x+1}$

8. $\frac{x}{8}-\frac{3}{5}$

9. $\frac{3x}{x+3}-\frac{5}{6}$

10. $\frac{x-1}{x-5}+\frac{7}{x+2}$



11. $\dfrac{2}{2x+3}-\dfrac{x+1}{x-1}$

12. $\dfrac{2x-4}{x^2+6x}+\dfrac{3}{x+6}$

13. $\dfrac{3x-2}{x^2-4x+4}-\dfrac{2}{x-2}$

14. $\dfrac{x+1}{x^2-x-6}+\dfrac{x}{x-3}$

15. $\dfrac{3x+1}{x^2+3x-10}+\dfrac{4}{x+5}$

16. $\dfrac{x+3}{x^2+x-6}-\dfrac{x-1}{x-2}$

Section 8
Summary

Summary of the Steps

A summary of the processes involved in multiplication, division, addition and subtraction of rational expressions is as follows:

Multiplication

- Factor the numerator and denominator of the expressions to be multiplied.
- Any factors which appear in both a numerator and a denominator cancel out.
- After canceling, multiply the remaining numerators together and the remaining denominators together to give the numerator and the denominator respectively of the resulting fraction.

Division

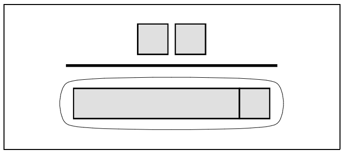
- Invert the divisor (the expression being divided by)
- Proceed to multiply the resulting expressions.

Addition

- Expand one or both fractions (top and bottom) to give both fractions the same (common) denominator.
- Add the numerators together and leave the result over this common denominator as a single fraction.
- Factor the numerator and denominator and check to see if they have any dimensions in common which could reduce (collapse) out.

Subtraction

- In the fraction being subtracted, take the opposite (negative) of the numerator, and proceed by adding the fractions together.



Exercises

Perform the indicated operations; reduce your answers:

1. $\frac{3}{x}+\frac{x}{8}$

2. $\frac{x}{x+2} \cdot \frac{2x+1}{3x}$

3. $\frac{x^2+5x+6}{3x+15} \div \frac{x+2}{x+5}$

4. $\frac{x+1}{x^2-4x}-\frac{2}{x}$

5. $\frac{x+2}{x^2-4}+\frac{x}{x+1}$

6. $\frac{x-2}{x^2+4x+3} \div \frac{x+1}{x^2-x-2}$

7. $\frac{x-2}{x^2+4x+3} \cdot \frac{x+1}{x^2-x-2}$

8. $\frac{x}{x-2}-\frac{2}{x}$

9. $\frac{x^2+6x+9}{x^2-x-2} \cdot \frac{x^2+3x+2}{x^2+5x+6}$

10. $\frac{x-2}{x+3}-\frac{3}{x-1}$

11. $\frac{x^2+3x}{x^2-4x+4} \div \frac{x^2+4x+3}{x^2-3x+2}$

12. $\frac{x+3}{x}+\frac{2x}{x-5}$

Answers to Exercises

Chapter 1: Introduction

Learning by Discovery

1. $x = 6$
2. $x = 5$
3. $x = 4$
4. $(x + 2)$ by $(x + 6)$
5. $(x + 3)$ by $(x + 5)$
6. $(2x + 1)$ by $(x + 2)$
7. 18
8. 8
9. 6
10. 16
11. 9
12. 3

Chapter 2: Positive and Negative Numbers

Section 1: Positive and Negative Numbers

3. $^{+}11$
4. $^{-}3$
5. $^{-}9$
6. $^{+}10$
12. $^{+}17$
13. 0
14. 0

Section 2: Addition of Signed Numbers

1. 0
2. $^{+}14$
3. $^{-}6$
4. $^{-}6$
5. $^{-}2$
6. $^{+}12$
7. $^{-}7$
8. $^{-}7$
9. $^{-}13$
10. $^{-}13$
11. $^{-}1$
12. $^{+}1$
13. $^{-}9$
14. $^{+}9$
15. $^{-}1$
16. $^{+}1$
17. $^{-}1$
18. $^{-}4$
19. $^{+}2$
20. $^{-}5$
21. $^{+}2$
22. $^{-}8$
23. $^{+}2$
24. $^{+}8$
25. $^{+}4$
26. $^{-}4$
27. $^{-}8$
28. $^{+}9$
29. $^{-}1$
30. $^{+}1$

Section 3: Subtraction of Signed Numbers

1. $^{+}8$
2. $^{-}8$
3. $^{-}2$
4. $^{-}3$
5. $^{-}2$
6. $^{+}8$
7. $^{-}8$
8. $^{+}2$
9. $^{+}17$
10. $^{+}4$
11. $^{+}6$
12. $^{+}2$
13. $^{+}17$
14. $^{-}7$
15. $^{-}16$
16. $^{+}2$
17. $^{-}8$
18. 0
19. $^{+}8$
20. $^{-}10$
21. $^{-}4$
22. $^{+}4$
23. $^{+}10$
24. $^{+}3$
25. $^{-}3$
26. $^{-}3$
27. $^{+}3$
28. $^{-}7$
29. $^{+}7$
30. 12

Section 4: Addition and Subtraction

2. $^{-}5$
3. $^{+}2$
4. $^{-}5$
5. $^{-}1$
6. $^{+}16$
7. 0
14. $^{+}5$
15. $^{-}3$
16. $^{+}4$
17. $^{+}4$

18. $^{+}2$
19. $^{-}2$
20. $^{-}8$
21. $^{+}8$
22. $^{-}8$
23. $^{-}4$
24. $^{-}10$
25. $^{+}4$
26. $^{+}10$
27. $^{-}14$

Section 5: Multiplication

3. $^{-}10$
4. $^{+}15$
5. $^{+}12$
6. $^{-}3$
7. $^{+}4$
8. $^{-}4$
9. $^{+}28$
10. $^{+}28$
11. $^{-}28$
12. $^{+}1$
13. $^{-}1$
14. $^{+}1$
15. $^{+}17$
16. $^{-}17$
17. $^{+}0$
18. $^{+}30$
19. $^{-}6$
20. $^{+}15$
21. $^{-}12$
22. $^{-}14$
23. $^{-}14$
24. $^{+}14$
25. $^{-}18$
26. $^{-}18$
27. $^{+}18$
28. $^{+}12$
29. $^{+}9$
30. $^{-}25$

Section 6: Division

1. $^{-}6$
2. $^{-}6$
3. $^{+}6$
4. $^{-}2$
5. $^{+}4$
6. $^{+}1$
7. $^{-}1$
8. $^{-}1$
9. $^{-}1$
10. $^{+}1$
11. 0
12. 0
13. $^{-}2$
14. $^{+}8$
15. $^{-}6$
16. $^{+}2$
17. $^{-}4$
18. $^{-}4$
19. $^{+}4$
20. $^{+}1$
21. $^{-}4$
22. $^{+}3$
23. $^{-}2$
24. $^{+}4$
25. $^{-}5$
26. $^{+}5$
27. $\frac{^{-}8}{3}$ *or* $^{-}2\frac{2}{3}$
28. $\frac{8}{3}$ *or* $2\frac{2}{3}$
29. $\frac{3}{2}$ *or* $1\frac{1}{2}$
30. $\frac{^{-}12}{5}$ *or* $^{-}2\frac{2}{5}$

Section 7: The Number Line

1. $^{+}6$
2. **$^{-}4$**
3. **$^{-}1$**
4. $^{+}4$
5. $^{-}1$
6. **$^{-}5$**
7. 0
8. $^{-}2$
9. $^{+}4$
10. $^{-}8$

Chapter 3: Symbols and the Order of Operations

Section 1: Rules of Language

4. 7
5. $^{-}7$
6. $^{-}15$
7. $^{-}3$
8. -10
9. 2
10. $^{-}12$
11. $^{-}35$
12. 35

Section 2: Order of Operations

1. 3
2. 12
3. 23
4. 11
5. 16
6. $^{-}30$
7. 47
8. $^{-}2$
9. $^{-}2$
10. $^{-}13$
11. 5
12. 0
13. $^{-}1$
14. 11
15. $^{-}6$
16. 16
17. $^{-}5$
18. 7
19. 17
20. $^{-}12$
21. 2
22. 26
23. $^{-}58$
24. 13
25. 9
26. $^{-}26$

Section 3: Parentheses

1. 9
2. 5
3. $^{-}28$
4. 14
5. $^{-}16$
6. $^{-}3$
7. 24
8. $^{-}31$
9. 16
10. $^{-}16$

Section 4: Fractions and Division

1. 2
2. 16
3. 4
4. 3
5. $^{-}3$
6. 10
7. $^{-}4$
8. 20
9. $^{-}17$
10. $^{-}34$

Review Exercises:

1. $^{-}15$
2. $^{-}19$
3. $^{-}11$
4. 26
5. 17
6. $^{-}73$
7. 23
8. $^{-}30$
9. $^{-}32$
10. $^{-}63$
11. $^{-}13$
12. $^{-}10$
13. $^{-}11$
14. 2
15. 1

Section 5: Absolute Value

1. 7
2. 12
3. 5
4. 9
5. 3
6. 7
7. 19

8. 2
9. $^-9$
10. $^-9$
11. 36
12. $^-16$
13. $^-10$
14. 36
15. 6
16. 7
17. 3
18. $^-5$

Chapter 4: Multiplication and Division of Fractions

Section 1: Multiplication of Fractions

1. $\frac{1}{8}$
2. $\frac{2}{9}$
3. $\frac{6}{15}$ *or* $\frac{2}{5}$
4. $\frac{12}{20}$ *or* $\frac{3}{5}$
5. $\frac{16}{25}$
6. $\frac{10}{24}$ *or* $\frac{5}{12}$
7. $\frac{15}{36}$ *or* $\frac{5}{12}$
8. $\frac{3}{12}$ *or* $\frac{1}{4}$
9. $\frac{15}{32}$
10. $\frac{20}{35}$ *or* $\frac{4}{7}$

Section 2: Division of Fractions

1. 4
2. 3
3. 6
4. 9
5. $\frac{9}{2}$ *or* $4\frac{1}{2}$
6. 10
7. $\frac{3}{2}$ *or* $1\frac{1}{2}$

Section 3: Compound Fractions

4. $\frac{56}{48}$ *or* $\frac{7}{6}$ *or* $1\frac{1}{6}$
5. $\frac{48}{56}$ *or* $\frac{6}{7}$
6. $\frac{56}{3}$ or $18\frac{2}{3}$
7. 8
8. $\frac{8}{9}$
9. 1
10. $\frac{1}{2}$

Chapter 5: Properties

Section 1: Properties of Addition and Multiplication

1. Commutative property of multiplication.
2. Commutative property of addition.
3. Associative property of addition.
4. Associative property of multiplication.
5. Associative property of addition.
6. Commutative property of addition, used twice.
7. Commutative property of multiplication.

Section 2: Distributive Property

1. $(3 \cdot 5) + (3 \cdot 6) = 15 + 18 = 33$
   Check: $3 \cdot 11 = 33$
2. $(2 \cdot 1) + (2 \cdot 4) = 2 + 8 = 10$
   Check: $2 \cdot 5 = 10$
3. $(4 \cdot 3) - (1 \cdot 3) = 12 - 3 = 9$
   Check: $3 \cdot 3 = 9$
4. $(4 \cdot 5) - (4 \cdot 2) = 20 - 8 = 12$
   Check: $4 \cdot 3 = 12$
5. $(1 \cdot 3) + (1 \cdot 4) + (2 \cdot 3) + (2 \cdot 4)$
   $= 3 + 4 + 6 + 8 = 21$
   Check: $3 \cdot 7 = 21$
6. $(3 \cdot 1) + (3 \cdot 1) + (2 \cdot 1) + (2 \cdot 1)$
   $= 3 + 3 + 2 + 2 = 10$
   Check: $5 \cdot 2 = 10$
7. $(8 \div 2) + (10 \div 2) = 4 + 5 = 9$
   Check: $18 \div 2 = 9$
8. $(6 \div 3) + (12 \div 3) = 2 + 4 = 6$
   Check: $18 \div 3 = 6$
9. $(10 + 3) \cdot (10 + 5)$
   $= (10 \cdot 10) + (10 \cdot 5) + (3 \cdot 10) + (3 \cdot 5)$
   $= 100 + 50 + 30 + 15 = 195$
   Check: $13 \cdot 15 = 195$
10. $(10 + 4) \cdot (10 + 6)$
    $= (10 \cdot 10) + (10 \cdot 6) + (4 \cdot 10) + (4 \cdot 6)$
    $= 100 + 60 + 40 + 24 = 224$
    Check: $14 \cdot 16 = 224$
11. $(10 + 10 + 1) \cdot (10 + 6)$
    $= (10 \cdot 10) + (10 \cdot 6) + (10 \cdot 10) + (10 \cdot 6)$
    $+ (1 \cdot 10) + (1 \cdot 6)$
    $= 100 + 60 + 100 + 60 + 10 + 6 = 336$
    Check: $21 \cdot 16 = 336$

Section 3: Identities and Inverses

1. 3
2. 21
3. 14
4. 8
5. 1
6. $^-49$
7. 12
8. 0
9. 0
10. 17

Section 4: Properties of Zero

1. 0
2. Not defined
3. 0
4. Not defined
5. Not defined
6. Not defined

Chapter 6: Expressions

Section 1: Simple Expressions

1. 9, 6, 4
2. $^{-}1$, 2, 4
3. 14, 5, $^{-}1$
4. 0, 0, 0
5. 4, 7, 9
6. 4, $^{-}5$, $^{-}11$
7. 12, $^{-}3$, $^{-}13$
8. 7, 4, 2
9. 0, 0, 0
10. 5, 5, 5
11. 3, 0, -2
12. $^{-}9$, $^{-}6$, $^{-}4$
13. 0, 3, 5
14. 0, 3, 5
15. 0, $^{-}3$, $^{-}5$

Section 2: Multiples of *x*

1. $^{-}12$, 0, $^{-}15$, $^{-}18$
2. $^{-}15$, $^{-}27$, $^{-}12$, $^{-}9$
3. 4, 0, 5, 6
4. $^{-}3$, $^{-}11$, $^{-}1$, 1
5. 1, 1, 1, 1
6. 4, 0, 5, 6
7. $^{-}1$, $^{-}13$, 2, 5
8. 15, 11, 16, 17
9. 5, 1, 6, 7
10. 5, 17, 2, $^{-}1$
11. $^{-}5$, 7, $^{-}8$, $^{-}11$
12. 0, 8, $^{-}2$, $^{-}4$
13. 4, 28, $^{-}2$, $^{-}8$
14. 3, 7, 2, 1
15. 10, 22, 7, 4
16. $^{-}6$, $^{-}14$, $^{-}4$, $^{-}2$
17. 8, 28, 3, $^{-}2$
18. 0, $^{-}8$, 2, 4
19. 0, 20, $^{-}5$, $^{-}10$
20. $^{-}4$, $^{-}32$, 3, 10

Section 3: Combining Similar Terms

1. $3x$, $-x$, 5
2. 0
3. $4x$, 1, 1
4. 1, $^{-}2$, x
5. $13x + 1$
6. $10x + 5$
7. $11x + 4$
8. $-x + 1$ *or* $1 - x$
9. $2x - 4$ *or* $^{-}4 + 2x$
10. 21, $^{-}4$
11. 29, $^{-}1$
12. 48, $^{-}7$
13. $^{-}3$, 2
14. 4, $^{-}6$

Section 4: Expressions and Parentheses

1. $11x - 13$ *or* $11x + {}^{-}13$
2. $-x - 2$ *or* $-x + {}^{-}2$
3. $13 - 2x$ *or* $^{-}2x + 13$
4. $2x + 13$
5. 20
6. $4x - 8$, $^{-}4$
7. $11 - 4x$ *or* $^{-}4x + 11$, 15
8. x, 256
9. x, $^{-}17$

Section 5: Expressions Containing Fractions

1. $\frac{3}{4}x$ or $\frac{3x}{4}$
2. $-\frac{x}{4}$ *or* $\frac{-x}{4}$
3. $\frac{1}{24}x$ or $\frac{x}{24}$
4. $\frac{1}{6}x$ or $\frac{x}{6}$
5. $x + 2$
6. $x + \frac{1}{4}$
7. $3x$
8. x
9. x
10. $20x$
11. $\frac{13x}{15}$
12. $\frac{13x}{6}$
13. $\frac{3x}{2}$

14. $\frac{13x}{6}$

15. $\frac{x}{9}$

16. $\frac{2x}{15}$

17. $2x-\frac{5}{3}$

18. $6x+3$

Section 6: Properties of Expressions

1. $6x + 30$
2. $^-1 - 2x$
3. $22 + x$
4. 0
5. x
6. $^-5x - 13$
7. $6x - 20$
8. 0
9. $3x$
10. $2x + 2$
11. $7x$
12. $3x + 2$
13. $3x$
14. $3x - 12$
15. $x + 1$
16. $x - 1$
17. $x+\frac{1}{6}$
18. ^-5x
19. $35x$
20. $35x$
21. $35x$
22. ^-35x
23. 0
24. 0
25. 0
26. $5x-70$
27. $^-12x+24$
28. $13x-23$
29. $21x-84$
30. $^-8x+38$

Chapter 7: Equations

Section 1: Introduction to Equations

1. Expression
2. Expression
3. Equation
4. Equation
5. Equation
6. Expression
7. Equation
8. Expression
9. Expression
10. Expression
11. Equation
12. Equation
13. Expression
14. Equation
15. Equation

Section 2: The Equation Game

1. $x = 5$
2. $x = 8$
3. $x = 8$
4. $x = 7$
6. $x = 7$
7. $x = 9$
8. $x = 5$
9. $x = 4$
10. $x = 6$
11. $x = 5$
12. $x = 12$
13. $x = 7$
14. $x = 9$
15. $x = 13$
16. $x = 21$
17. $x = 7$
18. $x = 8$
19. $x = 8$
20. $x = 9$

Section 3: Equations Using Unknowns

1. $x = 9$
2. $x = {}^{-}1$
3. $y = {}^{-}3$
4. $n = 3$
5. $y = 0$
6. $x = 12$
7. $x = 1$
8. $y = {}^{-}16$
9. $y = {}^{-}1$
10. $x = 23$
11. $x = {}^{-}15$
12. $y = 2$
13. $y = 8$
14. $x = 15$
15. $x = 0$
16. $x = 17$
17. $x = 7$
18. $y = {}^{-}1$
19. $y = {}^{-}5$
20. $x = {}^{-}7$
21. $y = {}^{-}3$
22. $n = {}^{-}1$
23. $n = 8$
24. $x = 0$
25. $x = {}^{-}14$

Section 4: Equations with Multiples of Unknowns

1. $x = 4$
2. $x = {}^{-}7$
3. $x = 2$
4. $x = \frac{9}{5}$ *or* $1\frac{4}{5}$
5. $y = 2$
6. $n = \frac{5}{6}$
7. $b = 0$
8. $x = 2$
9. $x = 11$
10. $x = \frac{10}{3}$ *or* $3\frac{1}{3}$
11. $y = 5$
12. $x = 0$
13. $x = {}^{-}2$
14. $x = {}^{-}2$
15. $y = 8$
16. $n = \frac{1}{7}$

17. $x = \frac{4}{3}$ *or* $1\frac{1}{3}$

18. $x = \frac{3}{5}$

19. $x = \frac{1}{2}$

Section 5: Unknowns in More than One Term

1. $x = {}^{-}11$
2. $x = 3$
3. $y = {}^{-}3$
4. $n = 1$
5. $y = 0$
6. $x = 2$
7. $z = 4$
8. $x = {}^{-}3$
9. $x = 7$
10. $x = {}^{-}2$
11. $y = {}^{-}2$
12. $x = {}^{-}5$
13. $x = 3$
14. $x = 0$
15. $x = 5$
16. $x = {}^{-}4$
17. $x = 0$

Section 6: Equations with Parentheses

1. $x = 6$
2. $x = 0$
3. $x = 4$
4. $x = 4$
5. $x = 0$
6. $x = {}^{-}3$
7. $x = {}^{-}2$
8. $y = 0$
9. $y = 8$
10. $x = {}^{-}1$
11. $x = \frac{1}{2}$
12. $x = 3$
13. $x = 1$
14. $x = 0$
15. $x = \frac{2}{3}$
16. $x = 8$
17. $x = 3$
18. $x = 15$
19. $x = {}^{-}2$
20. $x = 11$

Section 7: Equations with Fractions or Decimals

1. $x = 24$
2. $x = {}^{-}8$
3. $x = {}^{-}2$
4. $x = 1$
5. $x = \frac{21}{4}$ *or* $5\frac{1}{4}$
6. $x = 12$
7. $x = 6$
8. $x = {}^{-}2$
9. $x = 15$
10. $x = 12$
11. $x = \frac{12}{7}$ *or* $1\frac{5}{7}$
12. $x = {}^{-}3.6$
13. $x = 1.8$
14. $x = 4$
15. $x = 1$
16. $x = 100$
17. $x = 6$
18. $x = 1$
19. $x = 0$
20. $x = 20$

Section 8: Special Solutions

1. x can be any number
2. No solution
3. $x = 3$
4. No solution
5. x can be any number
6. No solution
7. $x = 0$
8. x can be any number
9. x can be any number
10. No solution

Chapter 8: Powers and Roots

Section 1: Introduction to Exponents

1. $5 \cdot 5 \cdot 5 \cdot 5 = 625$
2. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 1{,}024$
3. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$
4. $10 \cdot 10 \cdot 10 = 1{,}000$
5. 4^6
6. 7^3
7. 32^4
8. 0^2
9. $3^3 5^4$
10. 1^{10}
11. $^-8$
12. 16
13. $^-13$
14. 1

Section 2: Squares and Second Powers

1. 49
2. 1
3. $\frac{4}{9}$
4. $\frac{9}{25}$
5. $\frac{16}{9}$ *or* $1\frac{7}{9}$
6. 225
7. $4 \cdot 49 = 196$
8. $14 \cdot 14 = 196$
9. $1 \cdot 4 \cdot 9 \cdot 16 = 576$
10. $52 \cdot 1 = 52$
11. 11^2
12. 15^2
13. 100^2
14. 9^2
15. 12^2
16. 52
17. $^-46$
18. 48
19. 29
20. 72

Section 3: Cubes

1. 8
2. 1
3. 343
4. 9,261
5. $\frac{343}{27} = 12\frac{19}{27}$
6. $4 \cdot 8 = 32$
7. 0
8. 0
9. $\frac{343}{27} \cdot 27 = 343$
10. $9 \cdot 9 = 81$

Section 4: Higher Powers

1. $243 + 16 = 259$
2. $243 \cdot 81 = 19{,}683$
3. $4 \cdot 4 = 16$
4. $6 \cdot 6 \cdot 6 = 216$
5. $8 \cdot 8 \cdot 25 = 1600$
6. $9 \cdot 9 = 81$
7. $25 \cdot 25 = 625$
8. 2,401
9. $^-1$
10. 1
11. 256
12. 108
13. 1024
14. 128
15. 1024
16. $^-1024$
17. 131
18. 126
19. $^-124$
20. $^-125$

Section 5: Other Exponents

1. 999
2. 1
3. $\frac{1}{216}$
4. 1

5. $\frac{1}{5}$
6. 5
7. 1
8. $\frac{1}{64}$
9. $\frac{1}{625}$
10. 1
11. 1
12. 1
13. Not defined
14. 273
15. 1
16. $\frac{1}{81}$
17. $\frac{1}{9}$
18. $2\frac{1}{2}$ *or* $\frac{5}{2}$
19. $3\frac{1}{9}$ *or* $\frac{28}{9}$
20. $16\frac{1}{16}$ *or* $\frac{257}{16}$

Section 6: Properties of Powers

1. a^{10}
2. x^{14}
3. a^{18}
4. 1
5. $10 \cdot 4 = 40$
6. $\frac{2}{15}$
7. $\frac{27}{125}$
8. $\frac{x}{y^7}$
9. $\frac{72}{25}$
10. x^{-2} *or* $\frac{1}{x^2}$
11. False
12. True: $(a^3)^3 = (a^3)(a^3)(a^3) = a^9$
13. True: $\sqrt{25 \cdot 4} = \sqrt{25} \cdot \sqrt{4} = 5 \cdot 2$
14. True: $\sqrt{\frac{36}{100}} = \frac{\sqrt{36}}{\sqrt{100}} = \frac{6}{10} = 0.6$
15. False
16. True: $2^5 \cdot 2^{-2} = \frac{2^5}{2^2} = 2^3 = 8$
17. False
18. True: $4^0 = 1, \quad 3^3 = 27$
19. False
20. True: $(15)^4 = (3 \cdot 5)^4 = 3^4 \cdot 5^4$

Section 7: Simplifying Expressions

1. $x \cdot y^{-5}$ *or* $\frac{x}{y^5}$
2. $9x^4y^{-1}$ *or* $9\frac{x^4}{y}$
3. a
4. $\frac{8m}{n^{10}}$ *or* $8mn^{-10}$
5. $\frac{5}{4 \cdot 3} = \frac{5}{12}$
6. $\frac{y^2z^2}{x^2}$ *or* $x^{-2}y^2z^2$
7. 10,000
8. $\frac{1}{3^6} = 3^{-6} = \frac{1}{729}$
9. $\frac{-8}{x^9} = -8x^{-9}$
10. $\frac{9}{4}$
11. x^{10}
12. $\frac{1}{x^{10}}$ *or* x^{-10}
13. $\frac{1}{y^{10}}$ *or* y^{-10}
14. x^3
15. $\frac{3xy}{4}$
16. $\frac{1}{y}$ *or* y^{-1}
17. $a^6b^{10}c^{14}$
18. $a^3b^5c^7$
19. $a^9b^8c^5$
20. $\frac{a^{10}c^{10}}{b^4}$ *or* $a^{10}b^{-4}c^{10}$
21. $\frac{x^{24}}{y^4}$ *or* $x^{24}y^{-4}$
22. 1

Section 8: Roots and Radicals

1. 8
2. 5
3. 10
4. 1
5. 100
6. 1
7. 0
8. $^{-}10$
9. $^{-}5$
10. Undefined
11. $5 + 6 = 11$
12. $5 \cdot 6 = 30$
13. $2 \cdot 4 = 8$
14. 8

Section 9: Irrational Numbers

1. irrational
2. integer
3. irrational
4. irrational
5. integer
6. $4\frac{1}{8}$
7. 3
8. $5\frac{1}{10}$
9. $6\frac{1}{6}$
10. $2\frac{1}{2}$
11. $3\frac{1}{2}$

Section 10: Properties of Roots

1. $10 \cdot 4 = 40$
2. $2 \cdot 6 = 12$
3. $3 \cdot 5 = 15$
4. $4 \cdot 7 = 28$
5. $\frac{6}{5}$
6. $\frac{10}{3}$
7. $\frac{7}{9}$
8. $\frac{8}{5}$
9. $2 \cdot 10 = 20$
10. $5 \cdot 10 = 50$
11. $7 \cdot 10 = 70$
12. $9 \cdot 10 = 90$

Chapter 9: Polynomials

Section 1: Using Unknowns

1. $7x$
2. $7x$, $^-2$
3. $4x^2$
4. $3x^2$, $^-6$
5. 6, $^-2x^2$
6. $2x^2$, ^-3x, 12
7. $^-2x^2$, ^-5x, $^-1$
8. $^-0x^2$
9. 5, $^-3x^2$
10. $2x$, 3
11. x^2,^-5x, 6
12. $2x$, $-x^2$, 4
13. $4x$, $3x^2$
14. $2x^2$, $^-7$
15. $3x^2$, ^-5x, 2

Section 2: Adding and Subtracting Polynomials

1. x
2. $^-9x + 5$
3. $^-4x^2 + x + 5$ or $5 - 4x^2 + x$
4. $3x^2 + 4x$
5. $^-4x^2$
6. $2x^2 - x + 2$
7. $x^2 - 3x - 3$
8. $^-3x^2 - x - 2$
9. $x^2 + 5x - 5$
10. $4x + 12$
11. $8x - 8$
12. $3x^2 + 2x + 3$
13. $^-2$
14. $x^2 + 5x - 3$
15. $2x^2 - 3x - 6$
16. $2x^2 - 8x + 3$
17. $4x^2 + 2x - 7$
18. $4x^2 - 3x$
19. $^-2x + 4$
20. $-x^2 + 4x + 3$
21. $^-4x^2 - 2x$
22. $8x - 5$
23. $5x - 6$
24. $-x^2 + 7x + 3$
25. $x^2 - 5x + 1$
26. $2x^2 - 2x + 3$

Section 3: Multiplying Polynomials

1. $2x - 8$
2. $6x + 3$
3. $^-3x + 3$
4. $^-2x + 6$
5. $2x + 2$
6. $6x - 2$
7. $3x - 9$
8. $^-4x + 10$
9. $10x - 15$
10. $^-3x + 15$
11. $^-4x + 2$
12. $10x + 15$
13. $^-15x + 10$
14. $10 - 6x$
15. $^-12 + 4x$
16. $x^2 + 5x + 4$
17. $x^2 + x - 12$
18. $x^2 - 6x + 5$
19. $x^2 + 2x - 15$
20. $x^2 - 6x$
21. $2x^2 - 7x - 4$
22. $^-3x^2 + 2x$
23. $2x^2 - 7x + 6$
24. $x^2 - 2x - 15$
25. $x^2 - 8x + 12$
26. $2x^2 + 5x - 3$
27. $2x^2 + x - 6$
28. $^-3x + 2x^2$
29. $2x^2 - 3x - 2$
30. $4x^2 + 4x - 3$

Section 4: Special Products

1. $x^2 - 4x + 4$
2. $9x^2 + 6x + 1$
3. 49
4. $4x^2 - 28x + 49$
5. $9x^2$
6. $9x^2 + 12x + 4$

7. $x^2 + 8x + 16$

8. $4x^2 - 4x + 1$

9. $x^2 - 18x + 81$

10. $25x^2 + 30x + 9$

11. *No* ($4x^2 + 4x - 15$)

12. $x^2 - 4$

13. *No* ($9x^2 + 24x + 16$)

14. *No* ($15x^2 + 16x - 15$)

15. $9x^2 - 25$

16. $x^2 - 49$

17. *No* ($4x^2 + 4x - 3$)

18. *No* ($2x^2 + x - 1$)

19. $4x^2 - 1$

20. *No* ($6x^2 + 5x - 6$)

21. $25x^2 - 36$

22. $49x^2 - 1$

Chapter 10: Factoring Polynomials

Section 1: Introduction to Rectangles and Factoring

1. $x(x + 4)$
2. $x(x + 5)$
3. $(x + 3)(x + 3)$
4. $(x + 4)(x + 1)$
5. $(x + 5)(x + 3)$
6. $(x + 6)(x + 2)$
7. $(x + 4)(x + 3)$
8. $(x + 7)(x + 2)$
9. $(x + 4)(x + 4)$
10. $(x + 5)(x + 4)$

Section 2: Positive Units, Negative Bars

1. $(x - 3)(x - 1)$
2. $(x - 4)(x - 2)$
3. $(x - 6)(x - 2)$
4. $(x - 4)(x - 3)$
5. $(x - 5)(x - 2)$
6. $(x - 8)(x - 2)$

Section 3: Rectangles Having Negative Units

1. $(x + 6)(x - 1)$
2. $(x - 4)(x + 2)$
3. $(x - 8)(x + 1)$
4. $(x - 12)(x + 1)$
5. $(x - 6)(x + 1)$
6. $(x + 4)(x - 3)$
7. $(x + 9)(x - 1)$
8. $(x - 5)(x + 3)$
9. $(x + 5)(x - 3)$
10. $(x - 8)(x + 2)$

Section 4: Factoring Trinomials with More than One x^2

1. $(2x + 1)(2x + 1)$
2. $(3x + 1)(x + 2)$
3. $(2x + 1)(x + 3)$
4. $(3x + 1)(x + 3)$
5. $(2x + 1)(x + 2)$
6. $(2x + 1)(x + 1)$
7. $(2x + 3)(3x + 1)$
8. $(2x + 1)(3x + 2)$
9. $(2x + 1)(3x + 4)$
10. $(2x + 1)(2x + 3)$
11. $(3x + 4)(4x + 5)$

Section 5: Factoring Using the Grid

1. $(3x + 5)(x + 1)$
2. $(2x + 3)(x + 4)$
3. $(3x + 2)(x + 6)$
4. $(3x + 4)(x + 2)$
5. $(3x + 2)(x + 4)$
6. $(3x + 1)(x + 8)$
7. $(2x + 3)(x + 5)$
8. $(x - 3)(x + 2)$
9. $(x + 6)(x - 3)$
10. $(x - 1)(2x + 5)$
11. $(2x - 3)(x - 2)$
12. $(2x + 3)(2x - 5)$
13. $(2x - 3)(x + 5)$
14. $(2x + 3)(3x - 5)$
15. $(3x - 2)(2x + 5)$
16. $(2x - 3)(x - 5)$
17. $(x + 1)(3x - 5)$
18. $(2x + 3)(x - 2)$
19. $(2x - 1)(3x + 2)$

Section 6: A Shortcut Method

1. $(2x+ 3)(x - 5)$
2. $(x + 1)(2x - 5)$
3. $(x - 1)(2x + 5)$
4. $(2x - 3)(x - 2)$
5. $(2x + 3)(2x - 5)$
6. $(2x - 3)(x + 5)$
7. $(2x + 3)(3x - 5)$
8. $(3x - 2)(2x + 5)$
9. $(2x - 3)(x - 5)$
10. $(4x + 3)(3x + 4)$
11. $(10x + 2)(2x - 3)$ *or* $2(5x + 1)(2x - 3)$
12. $(5x + 1)(3x + 1)$
13. $(5x + 3)(5x + 3)$ *or* $(5x + 3)^2$
14. $(4x + 3)(3x - 4)$
15. $(x - 1)(3x + 5)$
16. $(2x + 1)(2x + 3)$
17. $(2x - 3)(x + 2)$

Section 7: Recognizing Special Products

1. Yes. $(x+3)^2$
2. No.
3. No.
4. Yes. $(2x+5)^2$
5. No.
6. Yes. $(2x-1)^2$
7. No.
8. No.
9. No.
10. Yes. $(4x-3)^2$
11. No.
12. Yes. $(2x-7)^2$
13. DTPS: $(2x+1)(2x-1)$
14. Neither
15. PS: $(x+3)^2$
16. DTPS: $(x+3)(x-3)$
17. Neither
18. Neither
19. PS: $(2x-3)^2$
20. Neither
21. PS: $(4x+1)^2$
22. DTPS: $(5x+2)(5x-2)$
23. Neither
24. PS: $(x-5)^2$
25. Neither
26. DTPS: $(2x-5)(2x+5)$
27. Neither
28. Neither

Section 8: Expressions which Cannot be Factored

1. $3(x+2)(x+3)$
2. Not factorable
3. $2(x-3)(x+3)$
4. $3(x+3)^2$
5. Not factorable
6. $(x+5)(x-1)$
7. $(3x+5)(x-1)$
8. Not factorable
9. Not factorable
10. $2(x+4)^2$
11. $5(x+2)(x-2)$
12. $x(4x-9)$
13. $3(x^2+4)$
14. $x(x+1)^2$
15. $(x+5)(x+1)$
16. $(x+6)(x-1)$
17. Not factorable
18. $2x(9x-4)$

Chapter 11: Quadratic Equations

Section 1: Introduction

1. Expression, Linear.
2. Equation, Quadratic.
3. Expression, Quadratic.
4. Equation, Cubic.

Section 2: The Zero Product Rule

1. $x = 4, 2$
2. $x = 4, 4$
3. $x = 6, 2$
4. $x = 4, 3$
5. $x = 3, 1\frac{1}{2}$
6. $x = 1, \frac{5}{3}$
7. $x = 5, \frac{1}{3}$
8. $x = 4, \frac{3}{2}$
9. $x = 3, \frac{4}{3}$
10. $x = 7, 3$

Section 3: Standard Form

1. $x^2 - 5x + 6 = 0$
2. $x^2 - 7x + 12 = 0$
3. $2x^2 - 11x + 12 = 0$
4. $x^2 - 6x + 8 = 0$
5. $2x^2 + 5x - 7 = 0$
6. $x^2 - 7x - 18 = 0$
7. $3x^2 + 8x + 5 = 0$
8. $x^2 - 10x + 21 = 0$
9. $3x^2 - 16x + 5 = 0$
10. $2x^2 - 9x + 9 = 0$
11. $2x^2 - 2x - 5 = 0$
12. $x^2 - 6x - 8 = 0$
13. $2x^2 + x + 1 = 0$
14. $3x^2 - 3x - 13 = 0$
15. $2x^2 - 4x - 17 = 0$

Section 4: Factoring Quadratic Equations

1. $x = 2, 3$
2. $x = 1, -\frac{7}{2}$
3. $x = 4, 2$
4. $x = {}^-1, -\frac{5}{3}$
5. $x = 4, {}^-2$
6. $x = \frac{3}{2}$
7. $x = {}^-1, {}^-8$
8. $x = {}^-2, -\frac{3}{2}$
9. $x = 3, {}^-5$
10. $x = -\frac{1}{2}, 2$
11. $x = 1, {}^-5$
12. $x = \frac{2}{3}, -\frac{5}{2}$
13. $x = {}^-3, {}^-9$
14. $x = \frac{5}{2}, 3$
15. $x = 9, 7$

Section 5: Completing the Square

1. $x = {}^-7, 1$
2. $x = 5, {}^-1$
3. $x = 1, {}^-9$
4. $x = \frac{10 \pm \sqrt{164}}{2}$ *or* $5 \pm \sqrt{41}$
5. $x = {}^-2, {}^-10$
6. $x = \frac{-10 \pm \sqrt{28}}{2}$ *or* $-5 \pm \sqrt{7}$
7. $x = {}^-3 \pm 3\sqrt{2}$
8. $x = {}^-1, {}^-11$
9. $x = 4 \pm \sqrt{19}$
10. $x = 5 \pm 2\sqrt{2}$
11. $x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$
12. $x = \frac{5}{2} \pm \frac{\sqrt{17}}{2}$

Section 6: Equations with More than One x^2

1. $^-12, 2$
2. $\frac{-6 \pm \sqrt{52}}{2} = -3 \pm \sqrt{13}$
3. $2, ^-10$
4. No solution
5. $\frac{8 \pm \sqrt{112}}{4} = 2 \pm \sqrt{7}$
6. 3, 1
7. $\frac{12 \pm \sqrt{84}}{6} = \frac{6 \pm \sqrt{21}}{3}$
8. $\frac{15 \pm \sqrt{153}}{6} = \frac{5 \pm \sqrt{17}}{2}$
9. $\frac{-7 \pm \sqrt{113}}{4}$
10. $\frac{9 \pm \sqrt{161}}{4}$

Section 7: Imaginary Solutions

1. $x = 3 \pm 4i$
2. $x = 5 \pm 3i$
3. $x = -2 \pm i$
4. $x = -4 \pm 2i$
5. $x = -3 \pm i\sqrt{6}$
6. $x = 1 \pm i\sqrt{1}$
7. $x = -6 \pm i$
8. $x = 4 \pm i\sqrt{7}$
9. $x = 2 \pm i\sqrt{7}$
10. $x = -3 \pm i\sqrt{3}$
11. $x = 5 \pm 2i\sqrt{2}$
12. $x = -4 \pm i\sqrt{5}$

Section 8: The Quadratic Formula

1. $x = \frac{5 \pm \sqrt{37}}{2}$
2. $x = \frac{7 \pm \sqrt{63}}{2} = \frac{7 \pm 3\sqrt{7}}{2}$
3. $x = \frac{-3 \pm \sqrt{3}}{2}$
4. $x = \frac{4 \pm \sqrt{14}}{2}$
5. $x = \frac{3 \pm i\sqrt{11}}{2}$
6. $x = \frac{1 \pm i\sqrt{23}}{2}$
7. $x = 2, -\frac{1}{3}$
8. $x = \frac{6 \pm \sqrt{52}}{10} = \frac{3 \pm \sqrt{13}}{5}$
9. $x = 1, \frac{5}{3}$
10. $x = 2, 1\frac{1}{2}$
11. $x = -3 \pm 2\sqrt{3}$
12. $x = -1, -\frac{3}{2}$
13. $x = 2 \pm 2i$
14. $x = \frac{1}{3}, -1$
15. $x = 3 \pm 2i$
16. $x = \frac{-2 \pm \sqrt{3}}{2}$
17. $x = \frac{1 \pm \sqrt{11}}{2}$
18. $x = \frac{-1 \pm \sqrt{57}}{4}$
19. $x = 4 \pm 3i$
20. $x = 2 \pm i$

Chapter 12: Rules and Graphs

Section 1: Related Numbers

1. Answers: ¯1, 2, 5, ¯4, ¯7, ¯10
2. Answers: 5, 7, 9, 3, 1, ¯1
3. Answers: 4, 3, 2, 5, 6, 7
4. Answers: 1, ¯1, ¯3, 3, 5, 7

(For exercises 5-12, your answers may be different)

5. x: 1, 2, 3, 0, ¯1, ¯2
   ¯3x: ¯3, ¯6, ¯9, 0, 3, 6
6. x: 1, 2, 3, 0, ¯1, ¯2
   5x – 3: 2, 7, 12, ¯3, ¯8, ¯13
7. x: 1, 2, 3, 0, ¯1, ¯2
   4x – 5: ¯1, 3, 7, ¯5, ¯9, ¯13
8. x: 1, 2, 3, 0, ¯1, ¯2
   ¯3x + 2: ¯1, -4, ¯7, 2, 5, 8
9. x: 1, 2, 3, 0, ¯1, ¯2
   ¯5 – x: ¯6, ¯7, ¯8, ¯5, ¯4, ¯3
10. x: 1, 2, 3, 0, ¯1, ¯2
    7 – 2x: 5, 3, 1, 7, 9, 11
11. x: 1, 2, 3, 0, ¯1, ¯2
    3x + 1: 4, 7, 10, 1, ¯2, ¯5
12. x: 1, 2, 3, 0, ¯1, ¯2
    ¯4x: ¯4, ¯8, ¯12, 0, 4, 8

Section 2: Charts, Machines, and a Second Variable

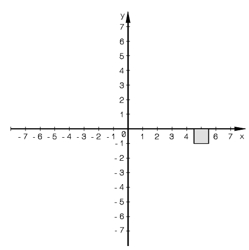
1. y's: 15, 30, 9, 3, ¯3
2. y's: ¯1, ¯5, ¯3, 11
3. y's: 0, ¯2, 3, ¯17
4. (2, ¯15), (2, ¯12), (2, ¯22), (2, 0)
5. (2, 7), (2, 16), (2, ¯14), (2, 52)
6. (2, 15), (2, 12), (2, 22), (2, 0)

(For exercises 7-15, your answers may be different)

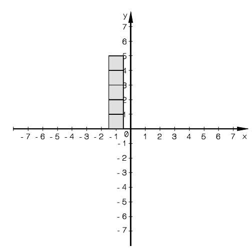
7. (0, ¯17), (1, 0), (¯1, ¯34), (3, 34)
8. (0, ¯17), (1, ¯16), (¯1, ¯16), (3, ¯8)
9. (0, 17), (1, 16), (¯1, 18), (3, 14)
10. (0, ¯3), (1, 0), (¯1, ¯6), (3, 6)
11. (0, ¯4), (1, ¯1), (¯1, ¯7), (3, 5)
12. (0, ¯5), (1, ¯1), (¯1, ¯9), (3, 7)
13. (0, ¯1), (1, ¯2), (¯1, 0), (3, ¯4)
14. (0, 1), (1, 0), (¯1, 2), (3, ¯2)
15. (0, 7), (1, 7), (¯1, 7), (3, 7)
16. x = 2
17. x = 21
18. x = 17
19. x = ¯2
20. x = ¯2
21. x = 5
22. x = 5
23. x = 5
24. x = 0
25. x = ¯1

Section 3: Graphs and Coordinates

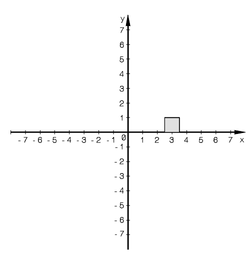
1.



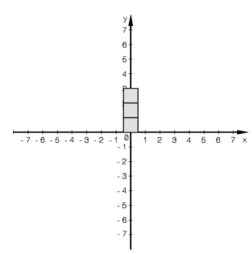
2.



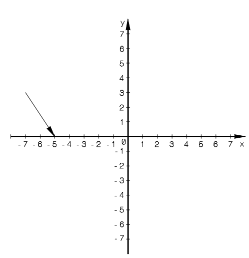
3.



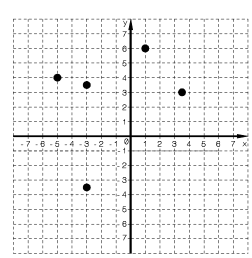
4.



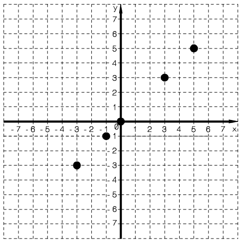
5.



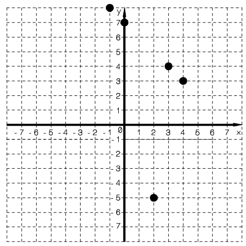
6-10.



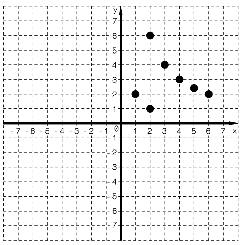
11.



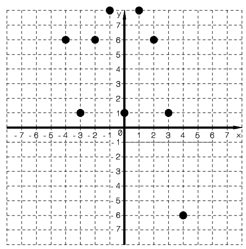
12.



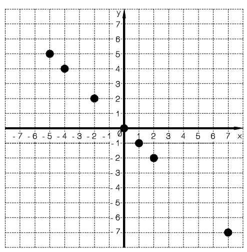
13.



14.



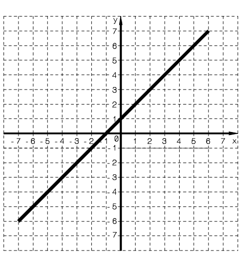
15.



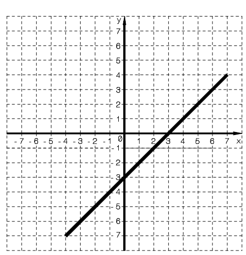
Section 4: Graphs of Lines

1. Linear
2. Linear
3. Linear
4. Linear
5. Not linear (has x^2)
6. Linear ($0x^2 = 0$, rule is actually $y = 2x + 1$)

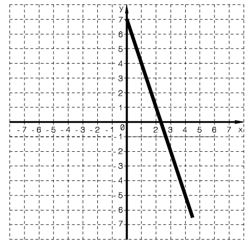
7.



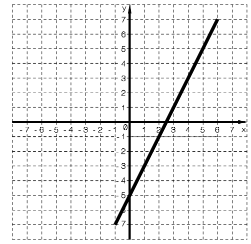
8.



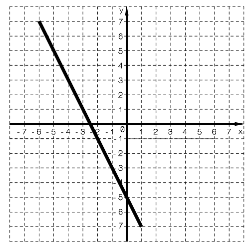
9.



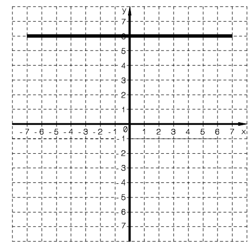
10.



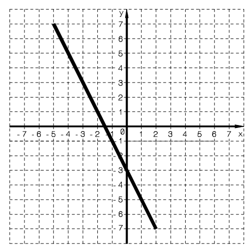
11.



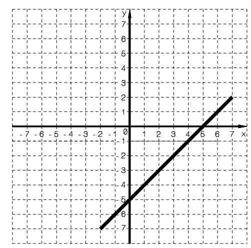
12.



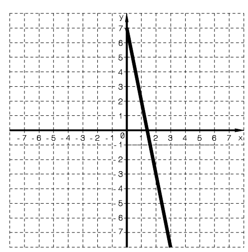
13. $y = {}^{-}2x - 3$



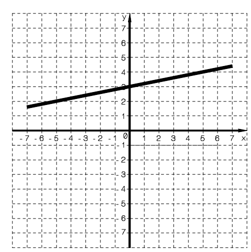
14. $y = x - 5$



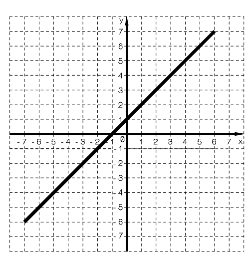
15. $y = {}^{-}5x + 7$



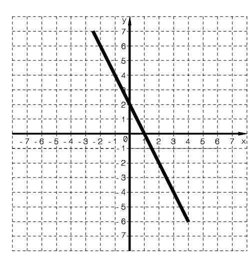
16. $y = \dfrac{1}{5}x + 3$



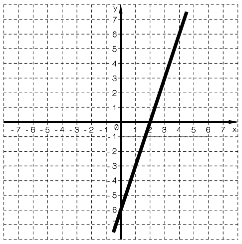
17. $y = x + 1$



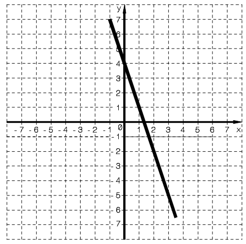
18. $y = {}^{-}2x + 2$



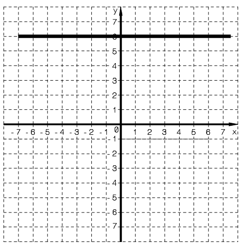
19. $y = 3x - 6$



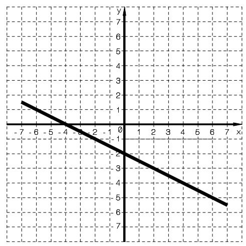
20. $y = {}^-3x + 4$



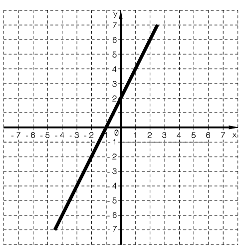
21. $y = 6$



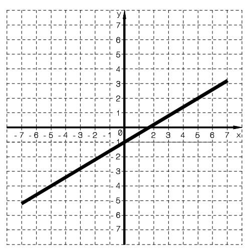
22. $y = -\frac{1}{2}x - 2$



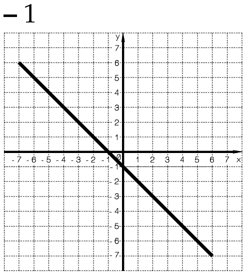
23. $y = 2x - 2$



24. $y = \frac{3}{5}x - 1$



25. $y = {}^-x - 1$



Section 5: Slopes and Intercepts

1. $-\frac{1}{3}$

2. $\frac{5}{3}$

3. ${}^-1$

4. $\frac{7}{6}$ *or* $1\frac{1}{6}$

5. 1

6. 0

7. ${}^-1$

8. $-\frac{3}{5}$

9. Slope ${}^-1$, x intercept ${}^-7$, y intercept ${}^-7$

10. Slope ${}^-3$, x intercept ${}^-4$, y intercept ${}^-12$

11. Slope 2, x intercept $-\frac{3}{2}$, y intercept 3

12. x intercept 6, y intercept ${}^-6$

13. x intercept ${}^-1$, y intercept ${}^-2$

14. x intercept 6, y intercept 2

15. ${}^-3$

16. $\frac{1}{2}$

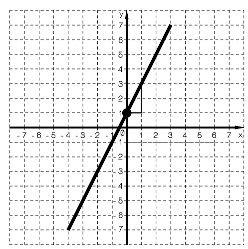
17. 2

18. ${}^-1$

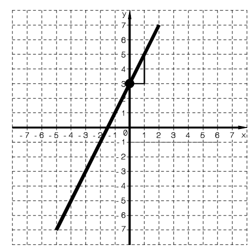
19. $\frac{10}{3}$ *or* $3\frac{1}{3}$

Section 6: Graphing with Slopes and Intercepts

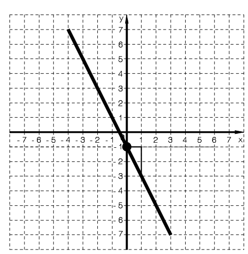
1.



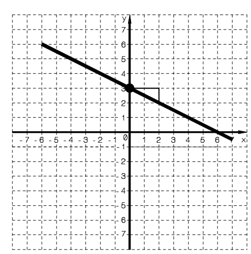
2.



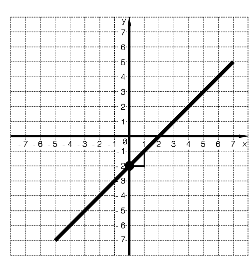
3.



4.



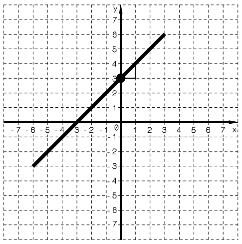
5.



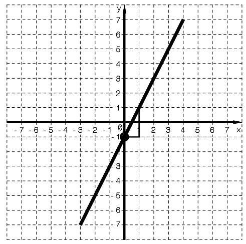
6. $y = \frac{1}{2}x + 3$

7. $y = -x + 7$

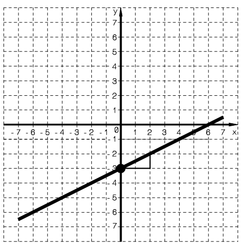
8. $y = x + 3$



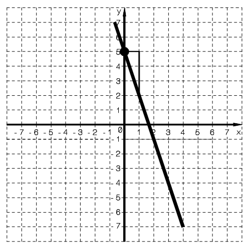
9. $y = 2x - 1$



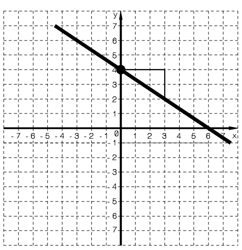
10. $y = \frac{1}{2}x - 3$



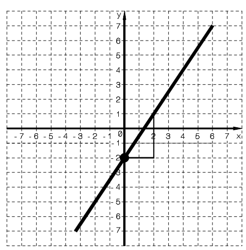
11. $y = 3x + 5$



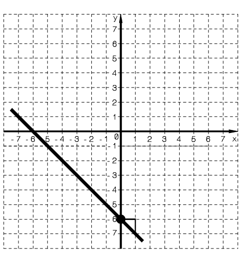
12. $y = -\frac{2}{3}x + 4$



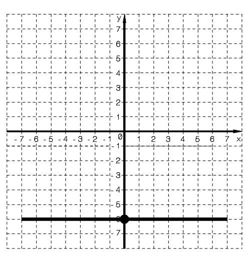
13. $y = \frac{3}{2}x - 2$



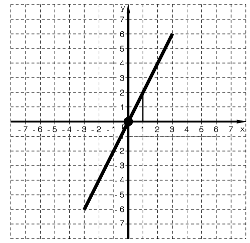
14. $y = -x - 6$



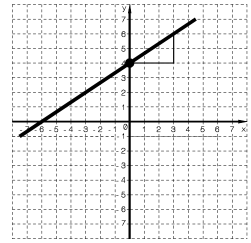
15. $y = {}^{-}6$



16. $y = 2x$



17. $y = \frac{2}{3}x + 4$



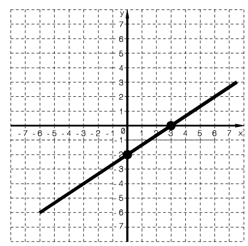
18. Slope is 0

19. Slope is not defined

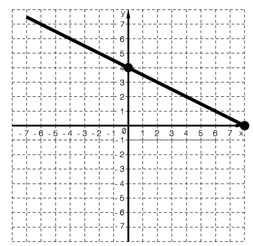
20. Slope is not defined

Section 7: Graphing With Two Intercepts

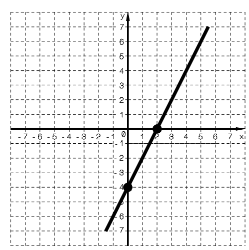
1.



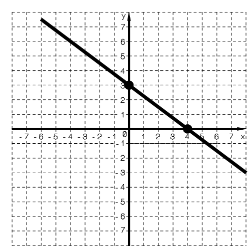
2.



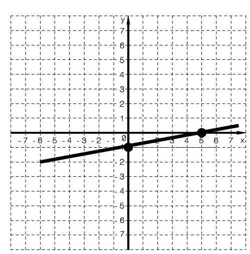
3.



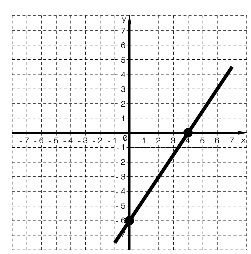
4.



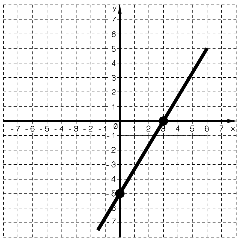
5.



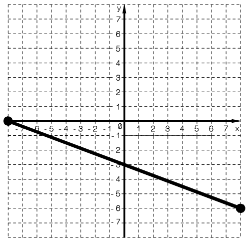
6.



7.



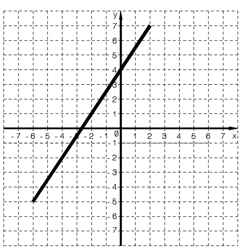
8.



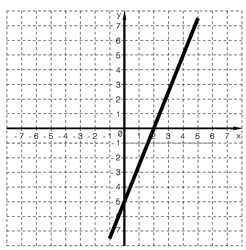
9.

10.

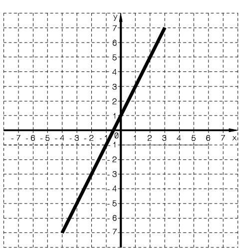
11. $y = \frac{3}{2}x + 4$



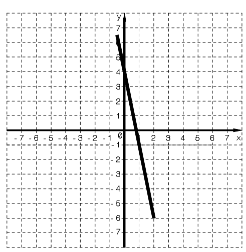
12. $y = 2x - 5$



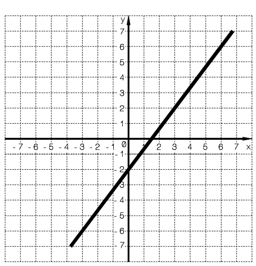
13. $y = 2x + 1$



14. $y = {}^{-}5x + 4$

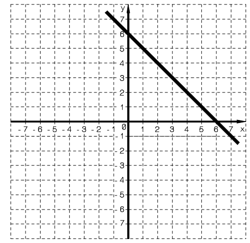


15. $y = \frac{4}{3}x - 2$

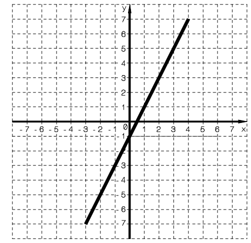


Section 8: Summary

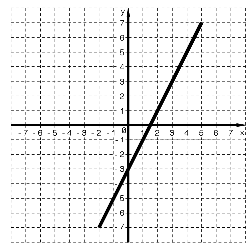
1.



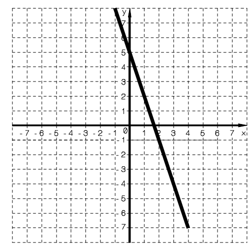
2.



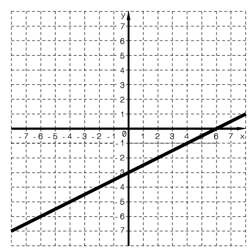
3.



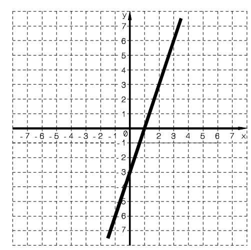
4.



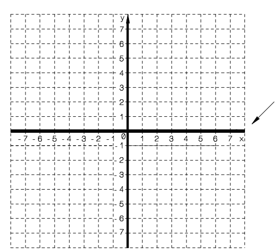
5.



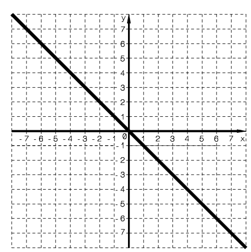
6.



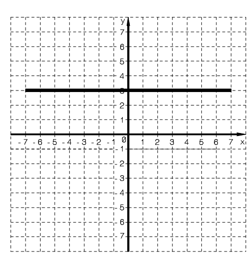
7.



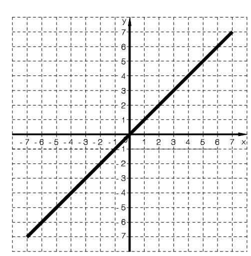
8.



9.



10.



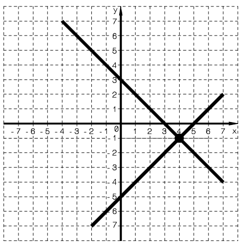
Chapter 13: Systems of Equations

Section 1: Equations and Solutions

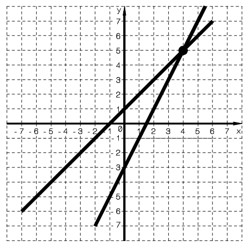
1. (4, 5), (1, 2), (3, 4), etc.
2. (0, −3), (2, 1), (5, 7), etc.

Section 2: Solving by Graphing

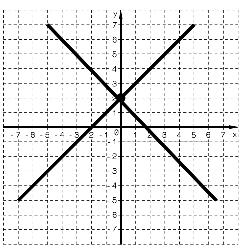
1. (4, −1)



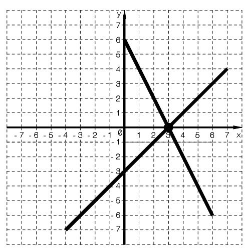
2. (4, 5)



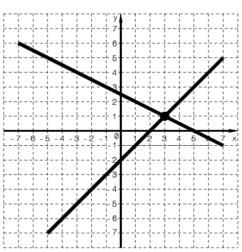
3. (0, 2)



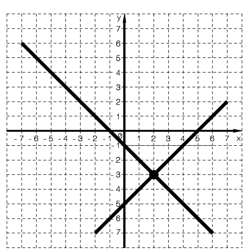
4. (3, 0)



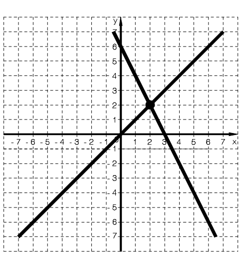
5. (3, 1)



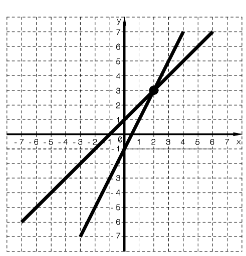
6. (2, −3)



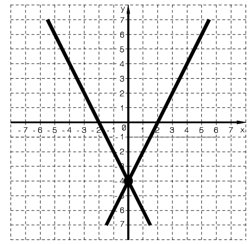
7. (2, 2)



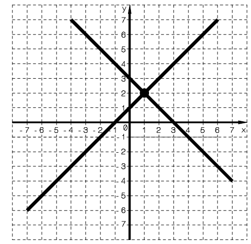
8. (2, 3)



9. (0, −4)



10. (1, 2)



Section 3: The Substitution Method

1. (7, 1)
2. (−1, 4)
3. (6, 1)
4. (0, 6)
5. (2, 7)
6. (−6, 1)
7. (1, 1)
8. (2, 4)
9. (3, 2)
10. (1, −3)
11. (−1, 1)
12. (−8, 23)
13. (4, 3)
14. $\left(\frac{1}{2}, \frac{1}{3}\right)$
15. (0, 0)

Section 4: The Addition Method

1. (4, 0)
2. (1, 2)
3. (3, 2)
4. (3, 0)
5. $\left(2\frac{2}{5}, 4\frac{1}{5}\right)$
6. (−1, 5)
7. (2, −3)
8. (2, 1)
9. (3, 4)
10. (5, −7)
11. (3, 4)
12. (2, −2)
13. (6, 2)
14. (9, 11)

15. (3, 1)
16. $(2, {}^-2)$
17. (5, 1)
18. (2, 2)

Section 5: Choosing a Method

1. $({}^-4, {}^-7)$
2. (2, 1)
3. (0, 1)
4. $({}^-2, 2)$
5. $({}^-4, 7)$
6. $({}^-3, {}^-1)$
7. (1, 2)
8. (3, 2)
9. $(5, {}^-7)$
10. $({}^-5, {}^-2)$
11. $\left(\frac{1}{2}, \frac{1}{3}\right)$
12. $({}^-1, {}^-2)$
13. $(1, {}^-2)$
14. (1, 3)
15. $({}^-1, {}^-2)$
16. (1, 2)

Section 6: Special Cases

1. (-10, 21)
2. $\left(2, -\frac{5}{2}\right)$
3. (8, 6)
4. $(4, {}^-3)$
5. $({}^-10, 3.1)$
6. $(4, {}^-3)$
7. (10, 12)
8. (25, 6)
9. Inconsistent (parallel)
10. $({}^-3, 2)$
11. Dependent (coinciding)
12. Inconsistent (parallel)
13. Dependent (coinciding)
14. (1, 2)

Chapter 14: Rational Expressions

Section 2: Simplifying Rational Expressions

1. 2
2. $\frac{1}{3}$
3. $\frac{3}{2}$
4. $\frac{x-4}{3}$
5. $\frac{1}{3}$
6. $\frac{2}{x}$
7. $1 + \frac{3}{x+2}$
8. $1 + \frac{-2}{x-1}$
9. $1 + \frac{-3}{x+5}$
10. $1 + \frac{8}{2x-5}$
11. $3 + \frac{4}{x-3}$
12. $2 + \frac{2}{x+2}$
13. $\frac{2}{x-2}$
14. $\frac{x+1}{x-5}$
15. $\frac{3(x-4)}{(x+2)(x+3)}$ (Does not reduce)
16. $\frac{x+5}{3}$
17. $\frac{2x+3}{x-5}$
18. $\frac{3}{x-7}$
19. $\frac{(x+4)(x-3)}{(3x-1)(x+2)}$ (Does not reduce)
20. $\frac{x}{x+5}$

Section 3: Division Using Chips

1. $x + 4 + \frac{-12}{x+2}$
2. $x - 6 + \frac{9}{x+1}$
3. $2x + 9 + \frac{21}{x-3}$
4. $3x - 3 + \frac{-2}{x+1}$
5. $x - 2 + \frac{-4}{x-2}$
6. $2x + 10 + \frac{15}{x-2}$
7. $x - 8 + \frac{25}{x+3}$
8. $2x - 11 + \frac{55}{x+5}$

Section 4: Long Division

1. $x + 5 + \frac{5}{x-2}$
2. $2x - 9 + \frac{26}{x+3}$
3. $x + 6 + \frac{-4}{x+1}$
4. $3x + 12 + \frac{53}{x-4}$
5. $x - 4 + \frac{8}{x+2}$
6. $5x + 32 + \frac{149}{x-5}$
7. $x + 1 + \frac{8}{x-3}$
8. $2x-- 13 + \frac{65}{x+5}$
9. $x - 4 + \frac{2}{x+1}$
10. $3x - 10 + \frac{41}{x+4}$

Section 5: Multiplication

1. $\frac{2(x+2)}{(x-2)(x+5)}$
2. $\frac{x+3}{2(2x+3)}$

3. $\frac{3(x+2)}{x-3}$

4. $\frac{5}{x+3}$

5. $\frac{x}{3(x-4)}$

6. $\frac{1}{(x-2)(x-2)}$

7. $\frac{x-1}{(x-2)(x-3)}$

8. $\frac{x-1}{x+6}$

9. $\frac{(x-4)^2}{3(x+4)}$

10. $\frac{(x+2)(x-1)}{(x-2)^2}$

Section 6: Division

1. $\frac{x}{2(x-3)}$

2. $\frac{x}{3}$

3. $\frac{(x+6)^2}{3(x+2)(x+1)}$

4. $\frac{(x+1)^3}{3(x+2)}$

5. $\frac{x-2}{3x(x+2)}$

6. $\frac{2(x-2)}{x(x+3)}$

7. $\frac{x+1}{(x+4)(x+2)}$

8. $\frac{2x+3}{x}$

9. 1

10. $\frac{(x+5)(x+3)}{3(x-4)}$

Section 7: Addition

1. $\frac{11}{12}$

2. $\frac{1}{10}$

3. $\frac{2}{21}$

4. $\frac{4}{3}$

5. $\frac{2x+9}{3x}$

6. $\frac{-1(x+3)}{x(x+1)}$

7. $\frac{3x^2+4x-2}{(x-2)(x+1)}$

8. $\frac{5x-24}{40}$

9. $\frac{13x-15}{6(x+3)}$

10. $\frac{x^2+8x-37}{(x-5)(x+2)}$

11. $\frac{-1(2x^2+3x+5)}{(2x+3)(x-1)}$

12. $\frac{5x-4}{x(x+6)}$

13. $\frac{x+2}{(x-2)^2}$

14. $\frac{x^2+3x+1}{(x-3)(x+2)}$

15. $\frac{7(x-1)}{(x+5)(x-2)}$

16. $^{-}1$

Section 8: Summary

1. $\frac{x^2+24}{8x}$

2. $\frac{2x+1}{3(x+2)}$

3. $\frac{x+3}{3}$

4. $\frac{-x+9}{x(x-4)}$

5. $\frac{x^2-x+1}{(x+1)(x-2)}$

6. $\frac{(x-2)^2}{(x+3)(x+1)}$

7. $\frac{1}{(x+1)(x+3)}$

8. $\frac{x^2-2x+4}{x(x-2)}$

9. $\frac{x+3}{x-2}$

10. $\frac{x^2-6x-7}{(x+3)(x-1)}$

11. $\frac{x(x-1)}{(x-2)(x+1)}$

12. $\frac{3x^2-2x-15}{x(x-5)}$

Appendices

Appendix 1: Division of Fractions, Part 2

1. $\frac{9}{8}$
2. $\frac{1}{8}$
3. $\frac{2}{12}$ *or* $\frac{1}{6}$
4. $\frac{3}{6}$ *or* $\frac{1}{2}$
5. 2

Appendix 2: Mixed Numbers

1. $\frac{15}{8}$ *or* $1\frac{7}{8}$
2. $\frac{28}{9}$ *or* $3\frac{1}{9}$
3. $\frac{49}{12}$ *or* $4\frac{1}{12}$
4. 6
5. $\frac{2}{12}$ *or* $\frac{1}{6}$
6. $\frac{2}{20}$ *or* $\frac{1}{10}$
7. $\frac{6}{20}$ *or* $\frac{3}{10}$
8. $\frac{10}{20}$ *or* $\frac{1}{2}$
9. $\frac{10}{12}$ *or* $\frac{5}{6}$
10. $\frac{20}{20}$ *or* 1

Appendix 3: The Function Game

1. Answers: 10, 17, $^-4$
2. Answers: 5, 11, $^-1$, 1
3. Answers: $^-10$, 0, $^-10$, $^-18$
4. Rule: $5x$
5. Rule: $x + 16$
6. Rule: $x^2 - 1$
7. x's: 17, $^-1$, 4
8. x's: 0, 4, $\frac{1}{3}$, $\frac{2}{3}$
9. x's: 1, $^-3$, 5, $^-7$
10. Rule: $14 - x$ x's: 11,2 answers: 14
11. Rule: $7x$ x's: 3 answers: 21, 14
12. Rule: $\frac{1}{2}x + 3$ x's: 1 answers: 4, $\frac{9}{2}$

Appendix 4: Functions and Maps

1. y's: 75, 75, 300, 27, 3, 3
2. y's: 4, 10, 7, $^-14$
3. y's: 0, 0, 0, 0, 0
7. Function
8. Not a Function
9. Not a Function
10. Function
11. Not a Function

Appendices

Appendix 1
Division of Fractions: Part 2

Dividing a Fraction by a Number

All of the examples in the FRACTIONS chapter covered situations of dividing an integer by a fraction. There are other cases such as:

$$\frac{1}{2} \div 6 \quad \text{or} \quad \frac{1}{3} \div \frac{2}{3}$$

The visual methods used before will also work for these cases, but the pictures will be a little more complex. Consider

$$\frac{1}{2} \div 6 = \frac{1}{2} \div \frac{6}{1}$$

Our rules tell us:

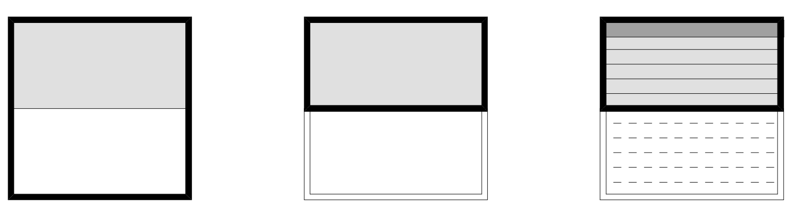
$$\frac{1}{2} \div \frac{6}{1} = \frac{1}{2} \cdot \frac{1}{6} = \frac{1 \cdot 1}{2 \cdot 6} = \frac{1}{12}$$

What does this problem look like? There are several ways to illustrate it:

- **Divide ½ into 6 equal parts. How large is each part?**
- **How many 6's can we make out of ½?**
- **Make a rectangle out of ½ unit. Arrange it with 6 in one direction. How long is the other direction?**

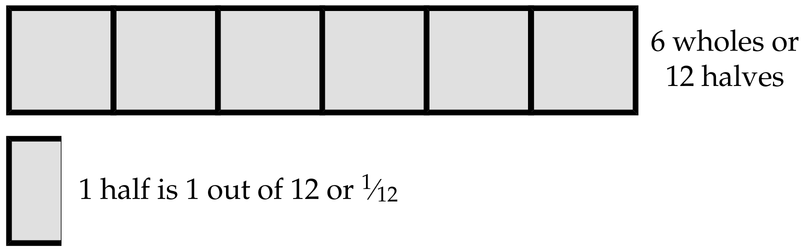
All of these are equivalent and of course give the same result. Because each method is useful in certain situations, you should work on understanding all of them.

First, let's divide ½ into 6 equal parts. The answer (each equal part) is $\frac{1}{12}$:

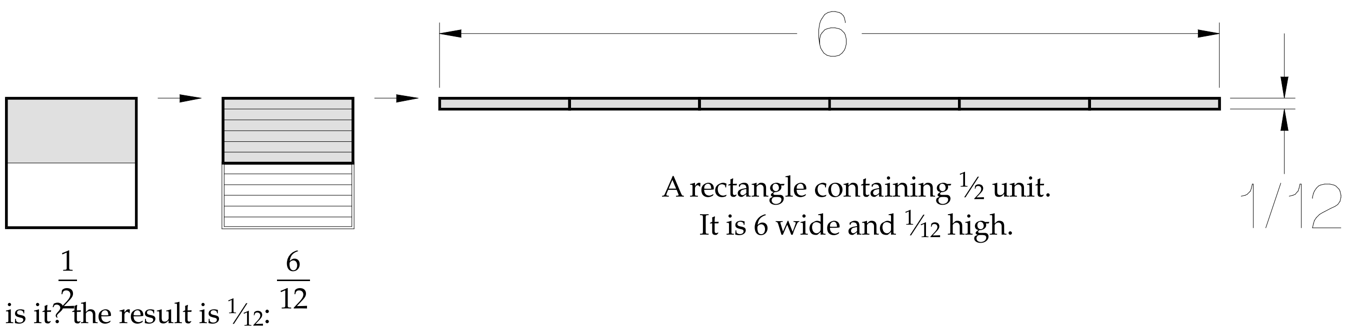


1 whole — $\frac{1}{2}$ — ½ divided into 6 groups of $\frac{1}{12}$.

Now, let's see how many 6's you can make out of ½. If you compare 6 and ½, it is clear that ½ is 1⁄12 of 6. You can make 1⁄12 of a 6 from ½:

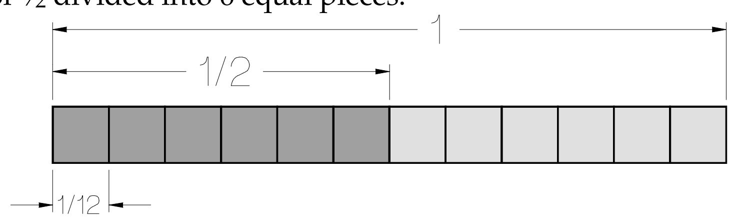


Finally, let's take ½ and arrange it into a rectangle that is 6 wide. How high is it? the result is 1⁄12:

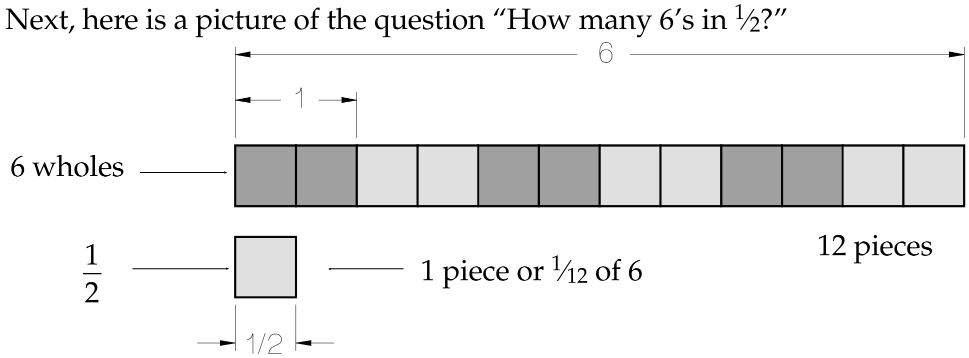


Using the Chips

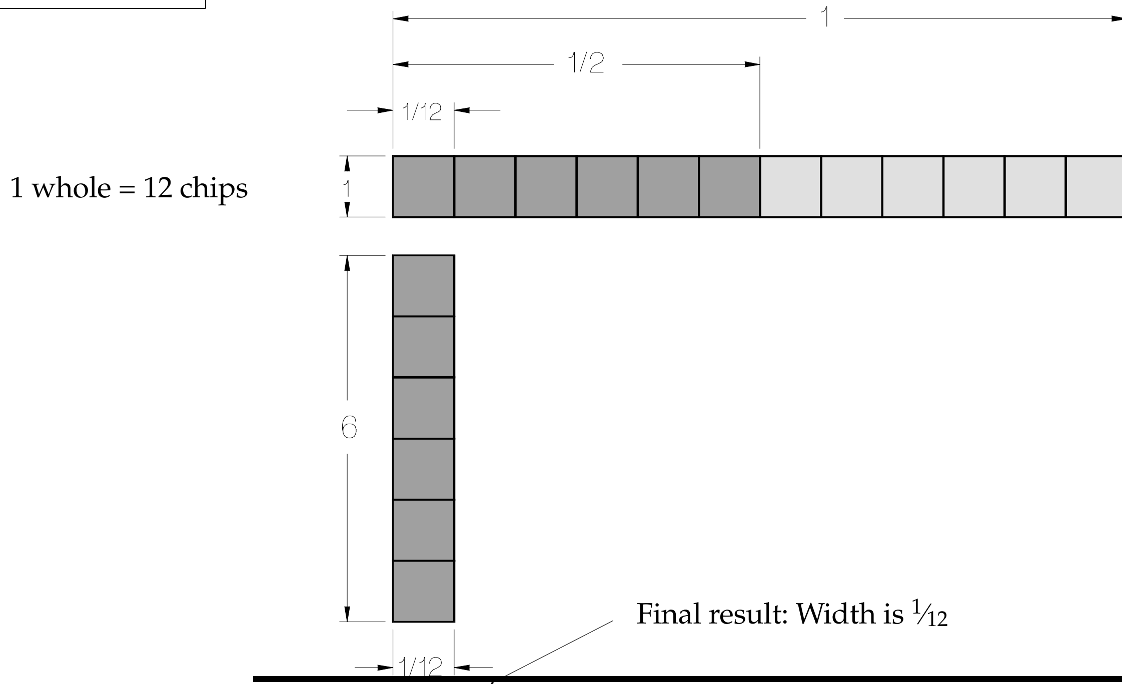
You can also use the chips to represent these division examples. *Remember that each chip takes on different dimensions in each problem*. First, here is a picture of ½ divided into 6 equal pieces:



Next, here is a picture of the question "How many 6's in ½?"



Finally, here is the rectangle approach where we arrange ½ into a rectangle that is 6 on one side and $\frac{1}{12}$ on the other:

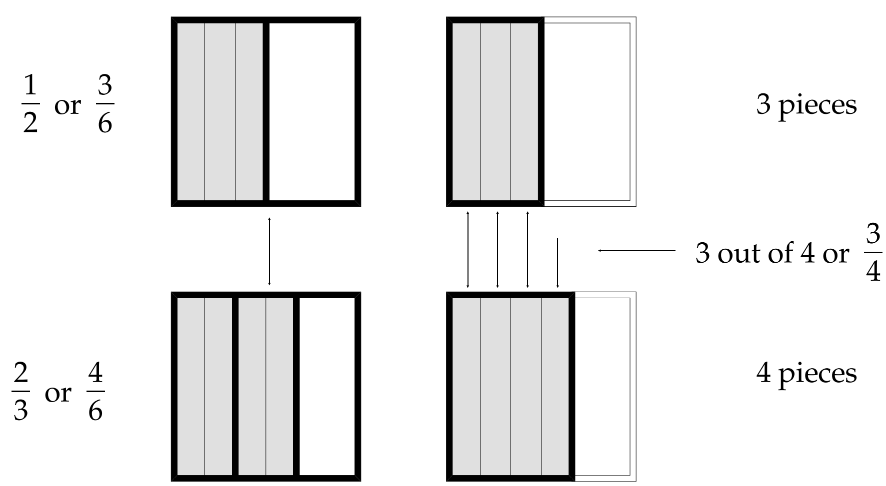


Dividing a Fraction by a Fraction

We will conclude with two pictures of

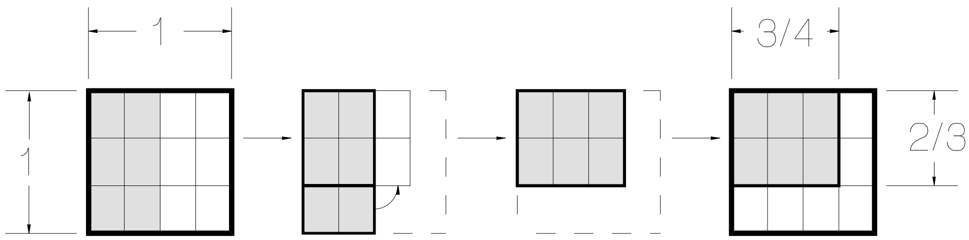
$$\frac{1}{2} \div \frac{2}{3}$$

If we think of the question as "How many ⅔'s can we make from ½," then we compare ½ to ⅔ and use one-sixth as a common unit.



We see 3 units (sixths) in ½ as opposed to 4 units in ⅔, so our result is that there are 3 out of 4 or ¾.

A second method is to build a rectangle from ½ that is ⅔ high. The width (¾) is our result:



Take $\frac{1}{2}$ Rearrange to be $\frac{2}{3}$ high Rectangle is $\frac{3}{4}$ wide

The rule of "invert and multiply" still works:

$$\frac{1}{2} \div \frac{2}{3} = \frac{1}{2} \cdot \frac{3}{2} = \frac{1 \cdot 3}{2 \cdot 2} = \frac{3}{4}$$

Now we see where the rule comes from. To summarize our method:

Dividing Two Fractions

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c} = \frac{ad}{bc}$$

Exercises

Use pictures and symbols to solve these problems. If chips are used, remember that each chip represents a different area in each problem.

1. $\frac{3}{4} \div \frac{2}{3}$

2. $\frac{1}{2} \div 4$

3. $\frac{2}{3} \div 4$

4. $\frac{1}{3} \div \frac{2}{3}$

5. $\frac{2}{3} \div \frac{1}{3}$

Appendix 2
Mixed Numbers

Multiplying Mixed Numbers

Fractions greater than one are sometimes called **mixed numbers** when written like this:

$$1\frac{1}{2}$$

and **improper fractions** when written like this:

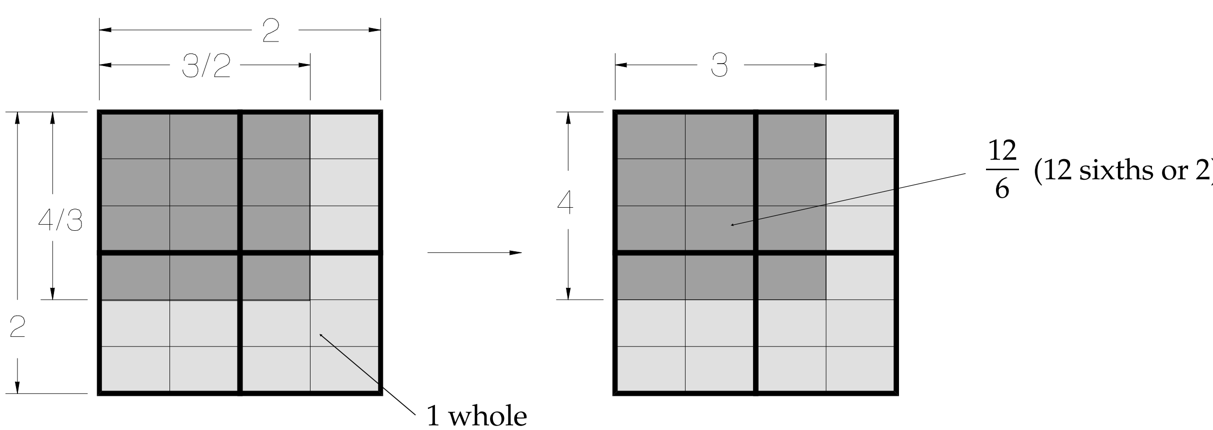
$$\frac{3}{2}$$

In most uses of algebra notation, we will be representing these numbers as one fraction rather than as a mixed number. *In these examples, always change mixed numbers into one fraction.*

Our examples so far have been limited to integers and fractions less than one. It is not difficult to extend our system to include more complicated fractions. Because of the number of chips involved, you may prefer to draw the pictures rather than use the chips. Consider the problem:

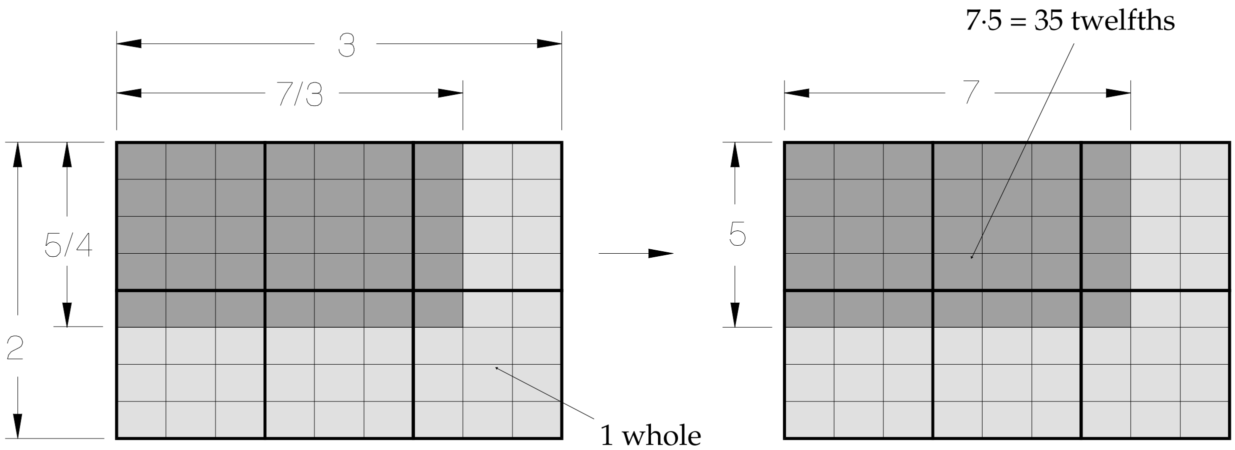
$$\frac{3}{2} \cdot \frac{4}{3}$$

This is a rectangle that is $^3\!/_2$ long and $^4\!/_3$ high. Because we have halves in one direction and thirds in another, the individual pieces are sixths (2·3). Because the resulting rectangle has four pieces in one direction and three in the other, we have 4·3 or 12 pieces. The result is 12 sixths ($^{12}\!/_6$ or 2):



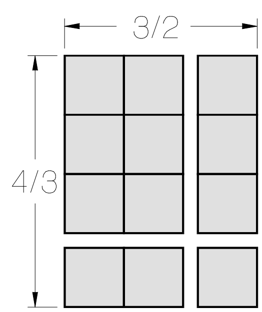
Here is another example:

$$\frac{7}{3} \cdot \frac{5}{4}$$



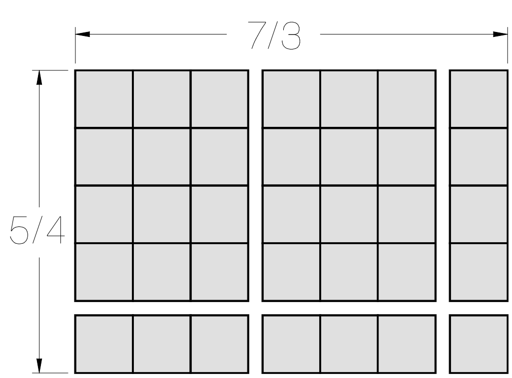
Using Chips for Multiplication

To use the chips, remember that each chip will represent a different area in each problem. Here is $\frac{3}{2} \cdot \frac{4}{3}$:



Chips are $\frac{1}{6}$ ($\frac{1}{2}$ by $\frac{1}{3}$).
We have 3·4 = 12 chips.
The result is $\frac{12}{6}$ or 2.

Here is $\frac{7}{3} \cdot \frac{5}{4}$:



Chips are $\frac{1}{12}$ ($\frac{1}{4}$ by $\frac{1}{3}$).
We have 7·5 = 35 chips.
The result is $\frac{35}{12}$.

Division of Mixed Numbers

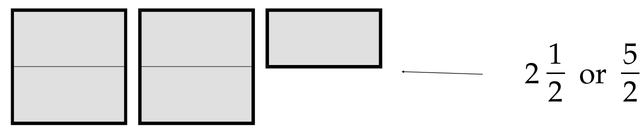
Like multiplication, division is not fundamentally different when done with fractions greater than 1. The meaning remains the same:

$\frac{5}{2} \div \frac{1}{2}$ means "How many one-halves in five-halves?"

or

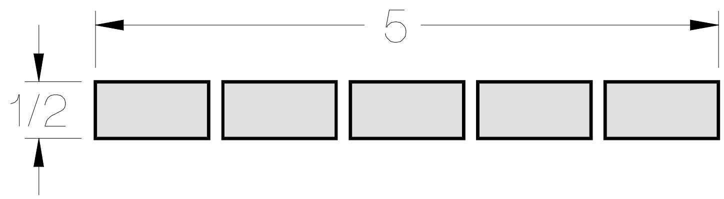
"Take $\frac{5}{2}$ and build a rectangle that is $\frac{1}{2}$ high. How wide is it?"

Here is an illustration:



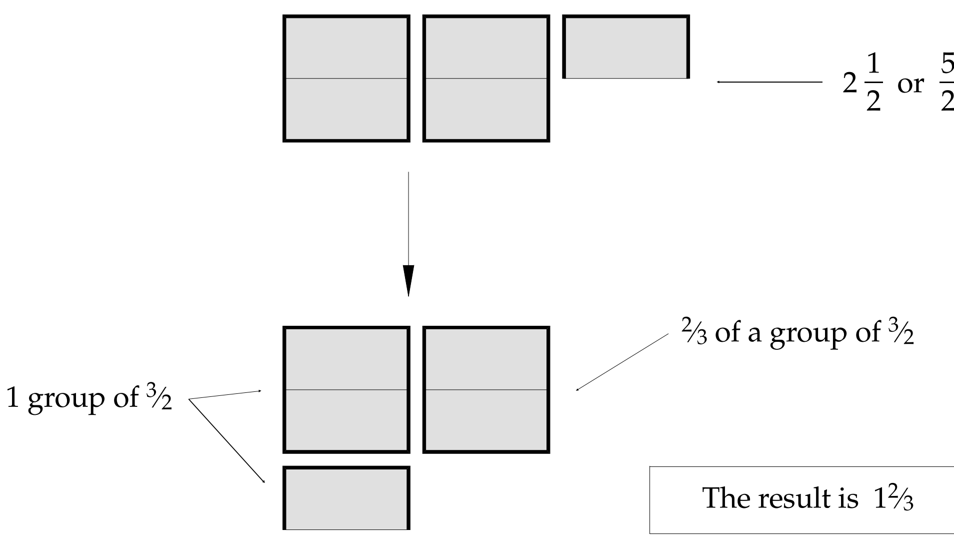
Build a rectangle $\frac{1}{2}$ high.

It is 5 wide. The result is 5.



Here is a second example:

$\frac{5}{2} \div \frac{3}{2}$ means "how many groups of 3/2 are in 5/2?"



The result is $1\frac{2}{3}$

Exercises

Draw pictures or use chips to solve the following problems. Write out your work in symbols as well.

1. $\frac{3}{2} \cdot \frac{5}{4}$

2. $\frac{7}{3} \cdot \frac{4}{3}$

3. $\frac{7}{4} \cdot \frac{7}{3}$

4. $\frac{3}{2} \div \frac{1}{4}$

5. $\frac{1}{4} \div \frac{3}{2}$

6. $\frac{1}{4} \div \frac{5}{2}$

7. $\frac{3}{4} \div \frac{5}{2}$

8. $1\frac{1}{4} \div \frac{5}{2}$

9. $1\frac{1}{4} \div 1\frac{1}{2}$

10. $1\frac{1}{4} \div 1\frac{1}{4}$

Appendix 3
The Function Game

Guess the Answer

In the game shown below, your job is to guess the answer to each number shown in the left side of the table. The original number is called x and the expression shown above is called the rule. *The rule tells you how to find the answer from x.*

Rule: $2x + 1$	
x	**Answer**
0	1
1	3
3	?
$^-1$	$^-1$
-2	?
3	?

Notice that the x's are not in order and that an x may be given more than once. To find the answer, you substitute the x in the rule expression and evaluate it as we did in the chapter on EXPRESSIONS:

For $x = 3$:

$$\begin{aligned} 2x + 1 &= 2(3) + 1 \\ &= 6 + 1 \\ &= 7 \end{aligned}$$

For $x = {}^-1$:

$$
\begin{aligned}
2x + 1 &= 2(^{-}1) + 1 \\
&= {}^{-}2 + 1 \\
&= {}^{-}1
\end{aligned}
$$

Here is another game. The rule is $10 - x$.
To find the missing answers, we evaluate the rule for each value of x:

Rule: 10 – *x*	
x	**Answer**
0	10
1	9
2	8
$^{-}1$	11
5	?
$^{-}2$	?

For $x = 5$:

$$
\begin{aligned}
10 - x &= 10 - (5) \\
&= 5
\end{aligned}
$$

For $x = {}^{-}2$:

$$
\begin{aligned}
10 - x &= 10 - (^{-}2) \\
&= 10 + 2 \\
&= 12
\end{aligned}
$$

Be careful to substitute the exact value for x (positive *or* negative) wherever x occurs in the rule.
Here is one more game with a more complicated rule. Can you calculate the missing answers for $^{-}3$, 1, 0, and 4?
The answers are 11, 7, 2, and 46.

The tables shown above illustrate the idea of a **function.** A function is a group of number pairs where each pair contains x and a matching answer. These functions have a rule which tells us how to calculate the answer. The rule is based only on x; the order that x's are given has no effect on the answer for any specific x. The answer for a specific x is always the same.

Rule: $2x^2 + 3x + 2$

x	**Answer**
0	2
2	16
-3	?
1	?
0	?
4	?

Guess the Rule

In the previous examples, we were given the rule and were asked to guess the answer. Now we will attempt to guess the rule when only x's and answers are given:

How did you guess the rule? The best method is to make a guess and try it for each x. Is the rule to add to x? Subtract? Multiply? Add and multiply? In the case above, the rule was to add 3 to x. We write this as $\mathbf{x + 3}$.

Rule: ?

x	**Answer**
0	3
1	4
-3	0
-1	2
2	5

Here is another example. Guess the rule:
In this game, the rule involved two operations—adding and multiplying. Each x was doubled and then 3 was added. This is written as $\mathbf{2x + 3}$.

Try one more game and guess the rule:

Rule: ?

x	**Answer**
0	3
1	5
-3	-3
-1	1
2	7

As you probably discovered, this rule used squaring and adding. Each x was raised to the second power and then 1 was added: $\mathbf{x^2 + 1}$.

Rule: ?

x	**Answer**
0	1
1	2
-3	10
-1	2
2	5

Guess the *x*

Our third and final game requires us to guess the x when we are given the rule and the answer. To solve the problem, we must look at the answer in each pair and guess which number would have given us that answer. Look at the table on the next page:

How can you discover the missing x's? One way is to think of the answer as the result of tripling an unknown number and adding one. To work backwards, we subtract one and divide by three:

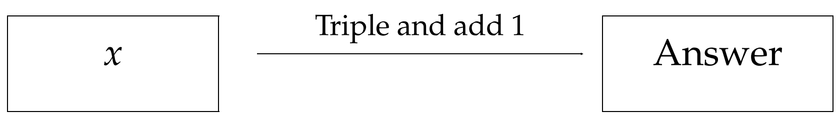
We can also find the x more formally by setting the rule expression equal to

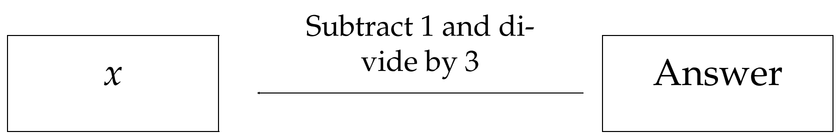
Rule: $3x + 1$

x	**Answer**
?	16
?	$^{-}2$
?	10
?	7

the answer:

For 16:





$$
\begin{aligned}
3x + 1 &= 16 \\
3x &= 15 \\
x &= 5
\end{aligned}
$$

For $^{-}2$:

$$
\begin{aligned}
3x + 1 &= {}^{-}2 \\
3x &= {}^{-}3 \\
x &= {}^{-}1
\end{aligned}
$$

Can you solve for x when the answer is 10?
Here is another example. Determine the missing x's by working backwards or by solving the appropriate equation:
Using the equation method:

For $x = 5$:

$$2x - 1 = 5$$
$$2x = 6$$

Rule: $2x - 1$

x	**Answer**
?	5
?	$^{-}1$
?	11
?	7

$$x = 3$$

For $x = {}^{-}1$:

$$2x - 1 = {}^{-}1$$
$$2x = 0$$
$$x = 0$$

The other missing answers are shown below in the completed table:
Finally, the next page shows a game with a more complicated rule. If you work backwards or solve the equation, how will you "undo" all of these operations?

Here is how to solve for x when the answer is 13:

Rule: $2x - 1$

x	**Answer**
3	5
0	$^{-}1$
6	11
4	7

$$\textbf{Rule: } \frac{7x+5}{2}$$

x	**Answer**
?	13
?	$^{-}1$
?	20
?	6

$$\frac{7x+5}{2} = 13$$

$$2\left(\frac{7x+5}{2}\right) = 2(13)$$

$$7x+5 = 26$$

$$7x = 21$$

$$\frac{7x}{7} = \frac{21}{7}$$

$$x = 3$$

The completed table is shown below:

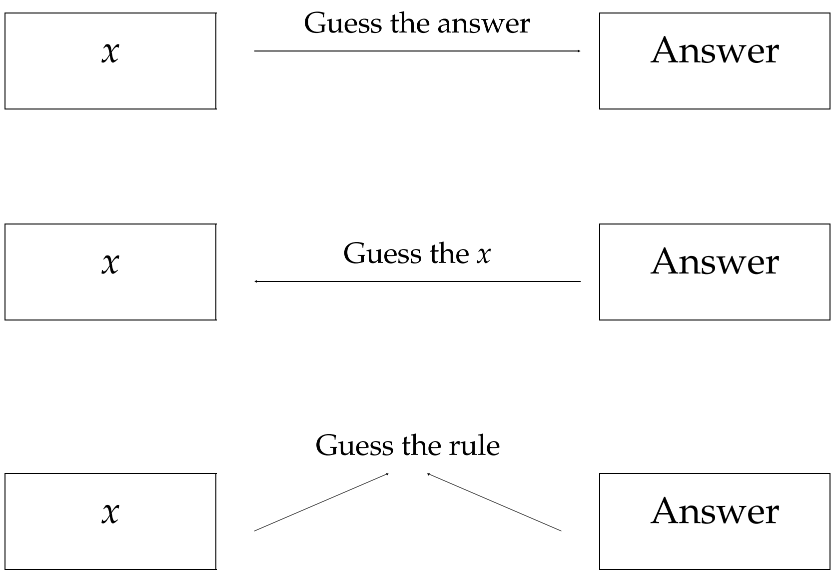
This version of the Function Game is really just the Equation Game in another form.

$$\textbf{Rule: } \frac{7x+5}{2}$$

x	**Answer**
3	13
$^{-}1$	$^{-}1$
5	20
1	6

The three games we have played are useful to help you understand the idea of a function:

Exercises



Guess the answer:

1. Rule: $7x - 4$

x	**Answer**
1	3
2	?
3	?
0	?

2. Rule: $x^2 + 3x + 1$

x	**Answer**
1	5
2	?
$^-1$	?
$^-3$	?

3. Rule: $^-5 - x$

x	**Answer**
5	?
$^-5$	?
5	?
13	?

Guess the rule:

4. Rule: ?

x	**Answer**
1	5
2	10
3	15
0	0

5. Rule: ?

x	**Answer**
1	17
2	18
$^-1$	15
$^-3$	13

6. Rule: ?

x	**Answer**
5	24
$^-5$	24
3	8
13	168

Guess the missing x's:

+

7. Rule: $x - 4$

x	Answer
-5	-9
?	13
?	-5
?	0

8. Rule: $3x - 5$

x	Answer
?	-5
?	7
?	-4
?	-3

9. Rule: $\frac{x+3}{4}$

x	Answer
?	1
?	0
?	2
?	-1

Discover the rule and fill in the missing x's and answers:

10. Rule: ?

x	Answer
-5	19
1	13
19	-5
3	?
12	?
?	0

11. Rule: ?

x	Answer
-1	-7
0	0
1	7
3	?
2	?
?	21

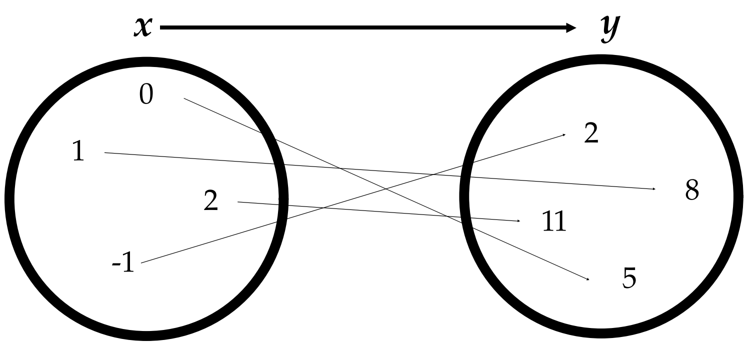
12. Rule: ?

x	Answer
6	6
4	5
0	3
2	?
3	?
?	$7/2$

Appendix 4
Functions and Maps

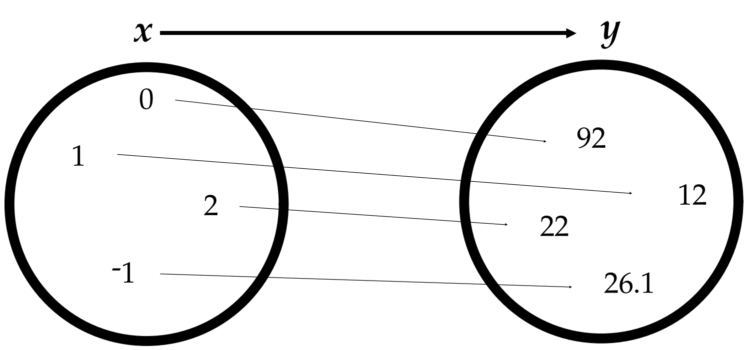
Maps

We can illustrate a function with a **map.** A map is a diagram of the x's and y's with each x connected to its correct y. The connection is shown by an arrow to remind you that x is first and y is the answer. The rule determines which x goes to which y.

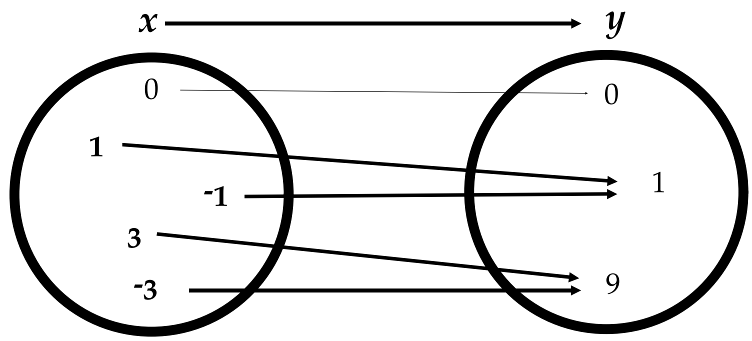


We can draw a map from a list of pairs, even if we do not know the rule:

Pairs are: (0, 92), (1, 12), (2, 22), and (-1, 26.1)



Here is a map of the function x^2:



With some functions, different x's are mapped to the same y.

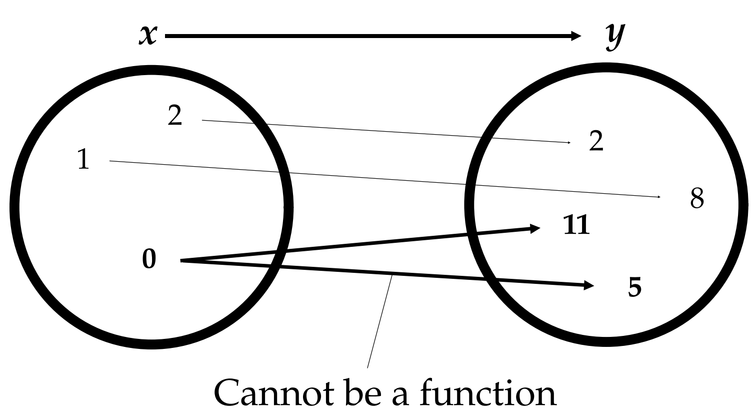
Maps that are Not Functions

Not all maps or lists of pairs are functions. By our previous definition, *the rule had to give the same answer for x each time.* In the game, this means that if 1 is given as x once or several times, the answer will always be the same. *Each x can have only one y as an answer.* We must understand, however, that the chart may have two x's which share the same y.

The following chart does *not* represent a function because 0 has two *different* answers:

x	y
0	**5**
1	8
0	**11**
2	2

The corresponding map looks like this:



Summary

+

Here is an illustration of what is *not a function*:

- **A *table*, where an x has more than one y:**

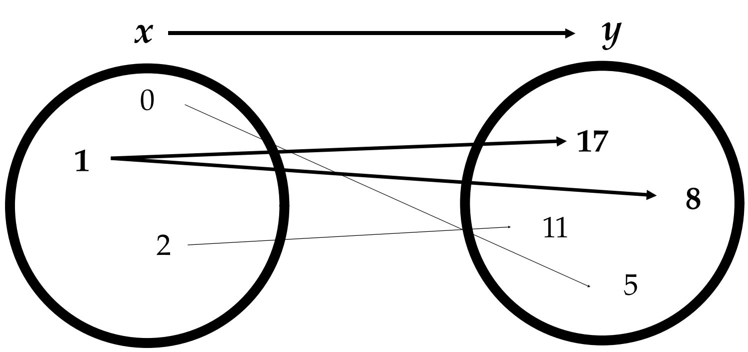
x	y
0	5
1	**8**
2	11
1	**17**

These are *not* functions.

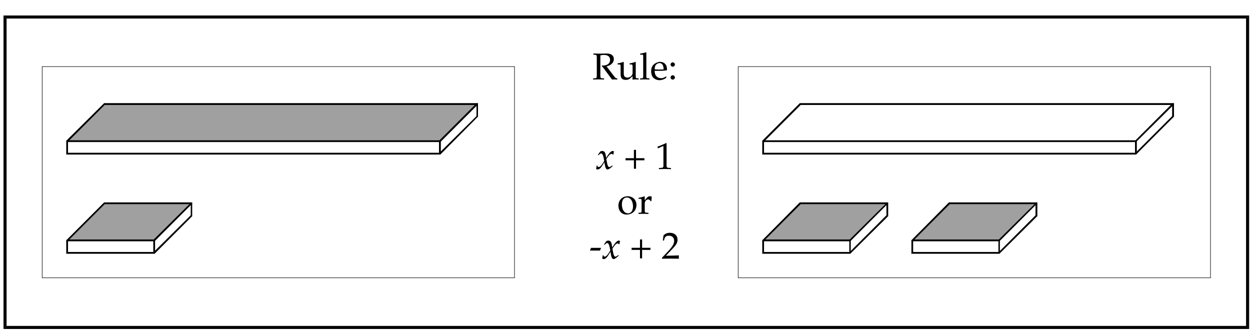
- **A list of *ordered pairs*, where an x has more than one y:**

(0, 5), **(1, 8)**, (2, 11), **(1, 17)**

- **A *map*, where an x is paired with more than one y:**



- **A machine, where the rule is not the same every time:**



Exercises

Finish the tables:

1. $y = 3x^2$

x	y
$^{-}5$	75
5	?
10	?
3	?
1	?
$^{-}1$	?

2. $y = x + 2x + 1$

x	y
$^{-}1$	$^{-}2$
0	1
1	?
3	?
2	?
$^{-}5$	?

3. $y = -x + x$

x	y
6	0
$^{-}4$	?
0	?
2	?
$^{-}3$	?
17	?

Draw a map for each list of ordered pirs:

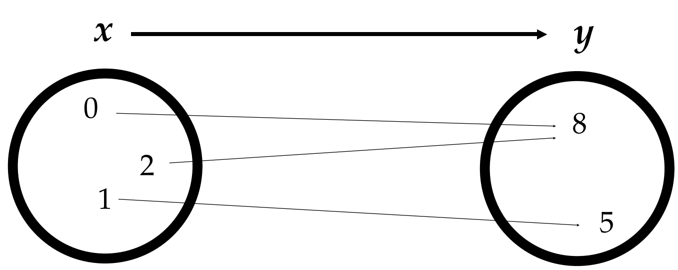
4. (0, 5), (1, 6), (2, 7), (3, 8)

5. (0, 5), (3, 5), ($^{-}3$, 5), (16.3, 5)

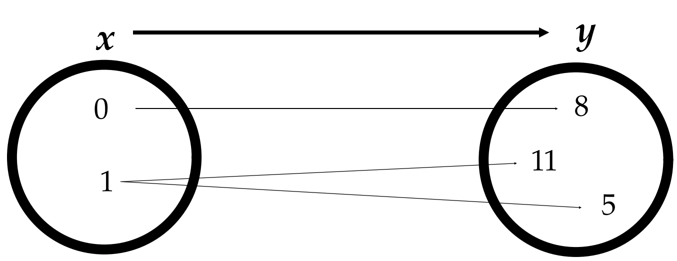
6. ($^{-}3$, 11), (6, $\sqrt{17}$), $\left(\frac{3}{2}, 5\right)$

Which of the following maps or charts are functions?

7.



8.



9. $x = 3y^2$

x	y
75	-5
75	5
300	10
12	2
0	0
$^-1$	?

10.

x	y
1	17.5
0	17.5
1	17.5
3	17.5
2	17.5
5	17.5

11.

x	y
6	-6
6	4
6	3/2
6	6
1	5
$^-1$	5

+

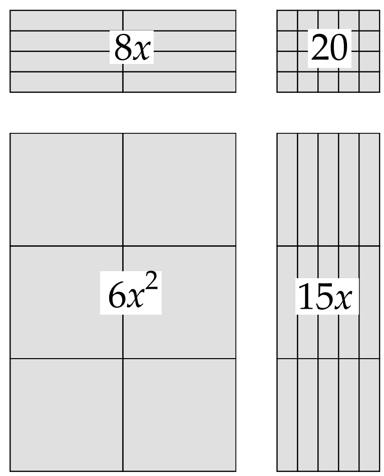
Appendix 5
Factoring By Grouping

The Shortcut Method

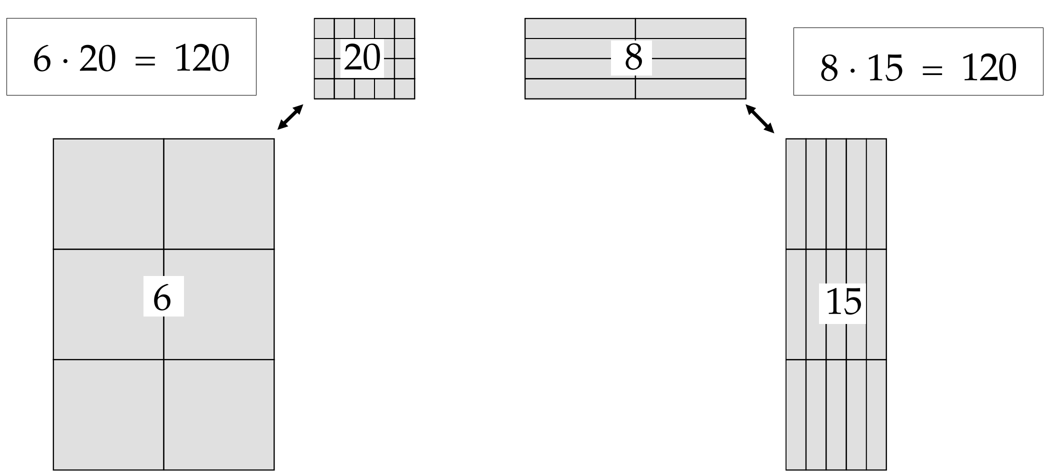
In FACTORING POLYNOMIALS, Section 6, we introduced a shortcut method of factoring. This section is a more detailed explanation of why this method works.

Below is a picture of the rectangle formed when we multiply

$$(3x + 4)(2x + 5) = 6x^2 + 15x + 8x + 20$$

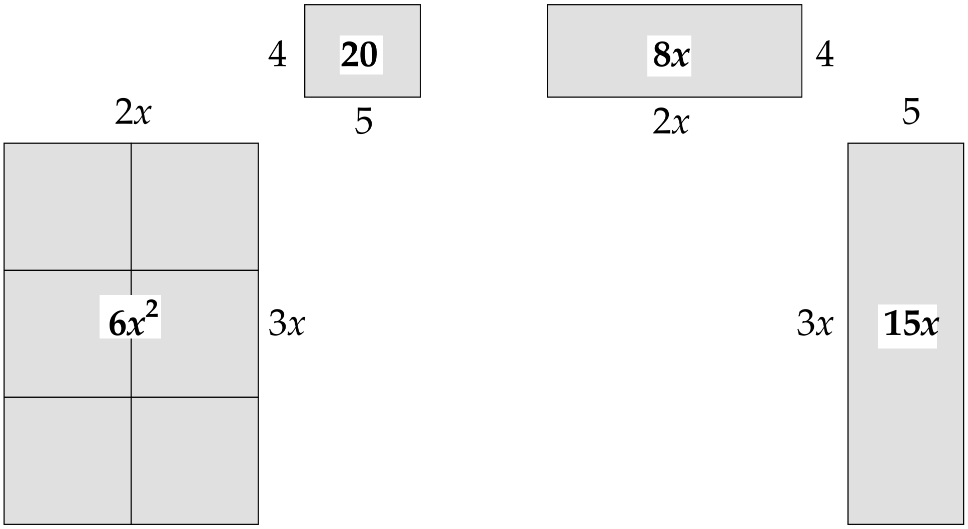


You should notice that the number of big squares times the number of small squares (6·20) equals the number of top x-bars times the number of side x-bars (8·15).



If we look at the edges of each rectangle we will understand why these products will always be equal.

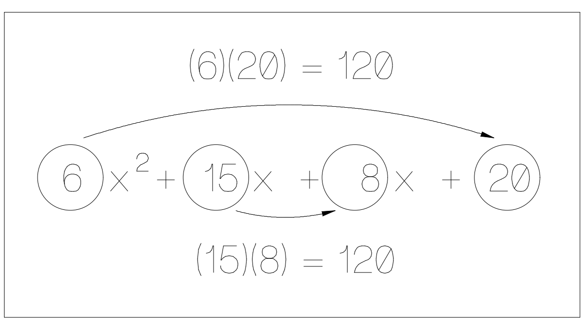
+



$6 \cdot 20 = (2 \cdot 3) \cdot (4 \cdot 5)$ $\qquad$ $8 \cdot 15 = (2 \cdot 4) \cdot (5 \cdot 3)$

Both products are the same because they both have all the same factors, just arranged in a different order.

Looking at the four terms of our product, this means that the number of x-squared pieces (first term) times the number of units (last term) will always equal the product of the numbers of x-bars (middle terms).



Now when the two middle terms are combined, giving

$$6x^2 + 23x + 20$$

we can see that the numbers which added to give us 23 must also multiply to give 120.

Starting with the combined form and working backwards to factor, we can use the method described in the chapter text to break the middle term into its two parts.

Then we can take the common factor from the first two terms and from the last two terms; this results in an amount in parentheses which is the same in both cases. This common factor is one of the factors of the original expression; the other factor is the sum of the pieces multiplying this common factor.

$$6x^2 + 23x + 20$$

$$6x^2 + (15x + 8x) + 20$$

$$(6x^2 + 15x) + (8x + 20)$$

$$3x(2x + 5) + 4(2x + 5)$$

$$(3x + 4)(2x + 5)$$

(6)(20) = 120	
120	1
60	2
40	3
30	4
24	5
20	6
15	**8**
12	10

This method is called **factoring by grouping**; it works for factoring expressions having any number of x-squared pieces.

Polynomial Plotter