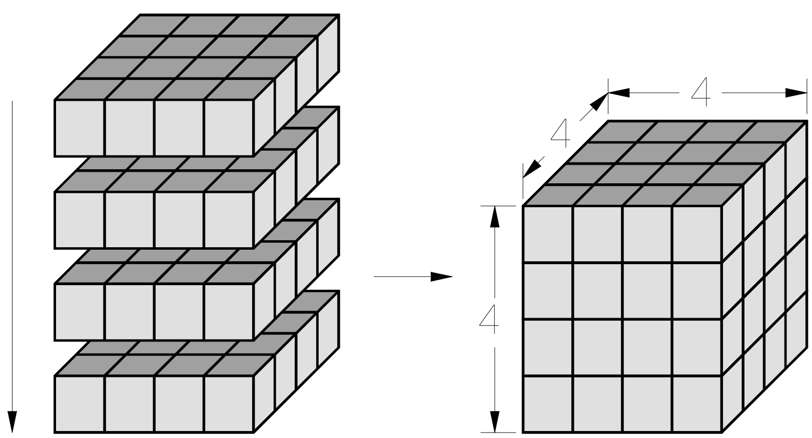
Chapter 8

Powers and Roots



Section 1
Introduction to Exponents

Exponents and Repeated Multiplication

Multiplication is often defined as repeated addition:

$$5 + 5 + 5 = 3 \cdot 5$$

$$8 + 8 + 8 + 8 + 8 + 8 + 8 = 7 \cdot 8$$

For repeating large numbers of additions, this notation becomes essential:

$$\mathbf{300 \cdot 5 =}$$

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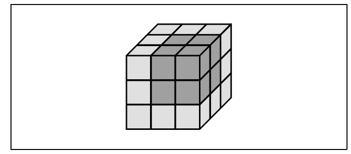
$$\mathbf{= 1500}$$

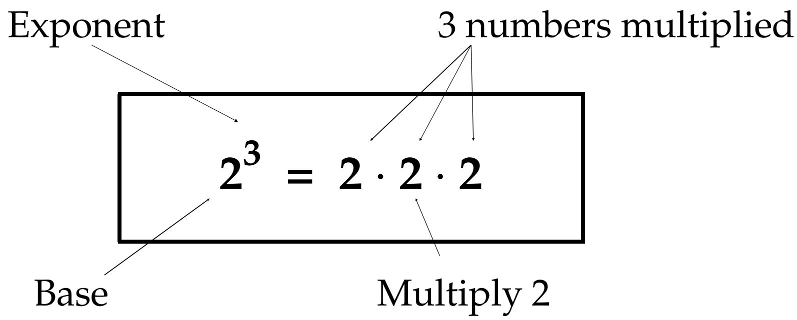
When we need to repeat *multiplications*, we invent a new notation to save time and space:

$$3 \cdot 3 \cdot 3 \cdot 3 = 3^4$$

$$2 \cdot 2 \cdot 2 = 2^3$$

The raised number is called a **power** or **exponent**—it stands for the number of quantities that we multiply. The number that is being multiplied is called the **base**. We read $\mathbf{2^3}$ as "Two to the third power." The operation of raising a number to a power is called **exponentiation**.





Notice that this is a new operation. 2^3 *is not the same as 2 times 3.*

Symbols and the Order of Operations

What is the meaning of:

$$2 \cdot 3^2 \ ?$$

Two operations are indicated—multiplication and exponentiation. Which comes first?

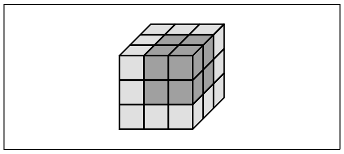
$$2 \cdot (3^2) \quad \text{or} \quad (2 \cdot 3)^2 \ ?$$

$$\begin{aligned} 2 \cdot (3^2) &= 2 \cdot (3 \cdot 3) \\ &= 2 \cdot 9 \\ &= 18 \qquad \textbf{(We agree that this is \textit{correct})} \end{aligned}$$

$$\begin{aligned} (2 \cdot 3)^2 &= 6^2 \\ &= 6 \cdot 6 \\ &= 36 \qquad \textbf{(We agree that this is \textit{incorrect})} \end{aligned}$$

The two alternatives have different answers! To avoid confusion and to save time, we agree that the first meaning is correct. *Exponentiation happens before multiplication and addition, unless parentheses indicate otherwise.*

$$2x^2 \quad \text{means} \quad 2 \cdot (x^2)$$



When there is no sign for an operation between two quantities, the meaning is the same as before—multiplication.

$$x^3 y^2 \quad \text{means} \quad x^3 \cdot y^2$$

$$(x-3)^2\,(y+6)^3 \quad \text{means} \quad (x-3)^2 \cdot (y+6)^3$$

Exponentiation happens *before* addition, subtraction, or negative signs:

$$\mathbf{-x^2} \quad \text{means} \quad \mathbf{-(x^2)}, \quad \textit{not} \quad (-x)^2$$

$$\mathbf{3+x^2} \quad \text{means} \quad \mathbf{3+(x^2)}, \quad \textit{not} \quad (3+x)^2$$

$$\mathbf{3-x^2} \quad \text{means} \quad \mathbf{3-(x^2)}, \quad \textit{not} \quad (3-x)^2$$

Exercises

Write each quantity as a multiplication problem, then calculate the answer:

1. 5^4
2. 4^5
3. 2^6
4. 10^3

Write each multiplication using powers:

5. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
6. $7 \cdot 7 \cdot 7$
7. $32 \cdot 32 \cdot 32 \cdot 32$
8. $0 \cdot 0$
9. $3 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 5$
10. $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$

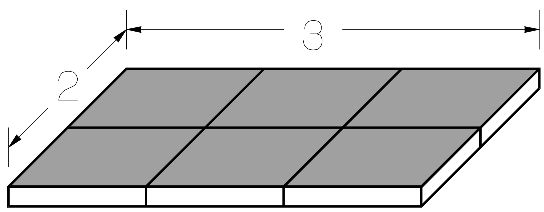
Calculate each answer:

11. $(-2)^3$
12. $(-2)^4$
13. $3 - 2^4$
14. $(3 - 2)^4$

Section 2
Squares

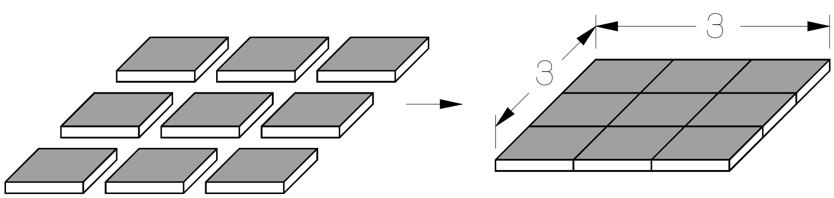
Squares and Second Powers

This section will present a visual explanation of raising numbers to a power of 2. In past chapters, we have considered multiplication of two numbers as the formation of rectangles:



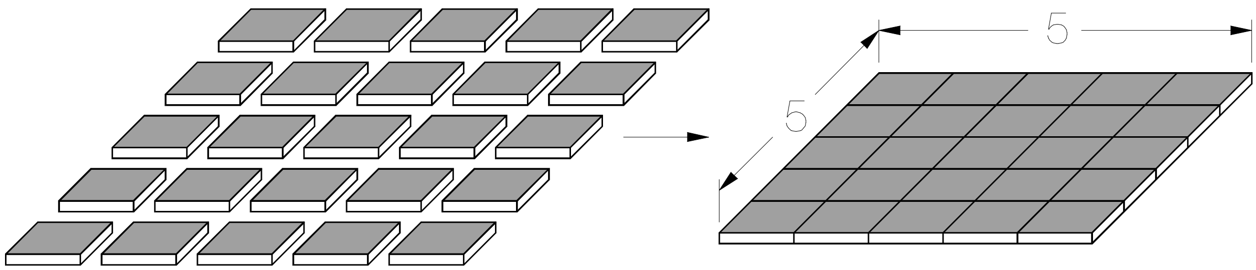
$2 \cdot 3 = 6$

Because $\mathbf{3^2}$ means $\mathbf{3 \cdot 3}$, raising numbers to the second power forms a square:

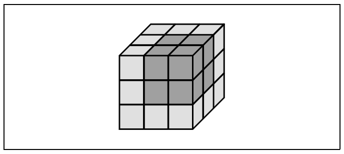


$3^2 = 3 \cdot 3 = 9$

When we raise 5 to the second power, we get $\mathbf{5^2}$ or $\mathbf{5 \cdot 5}$:



$5^2 = 5 \cdot 5 = 25$



This geometric property leads us to call 3^2 "three squared."

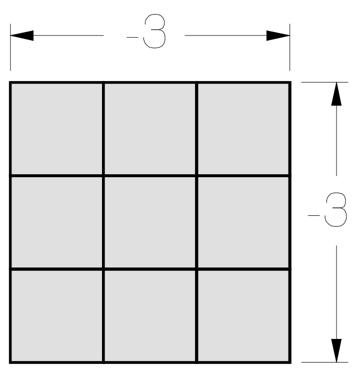
Raising a quantity to the second power

Make a square using the quantity for the length and width. The result is the area (number of unit chips) inside the square.

Squares of Negative Numbers

We are already familiar with the meaning of multiplying two negative numbers. The square of a negative number is always positive; we have to "imagine" the two negatives in the original multiplication:

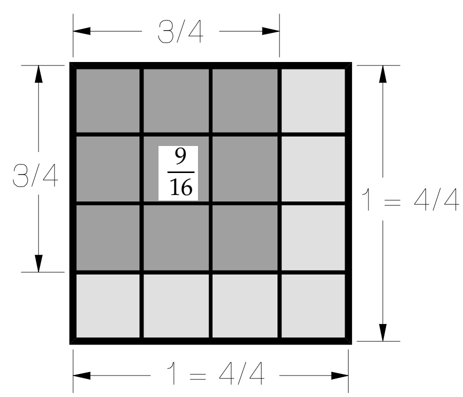
The result is $^{+}9$



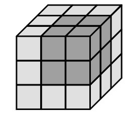
Squares Involving Fractions

Raising ¾ to the second power has the same meaning as raising a whole number to the second power—we build a square ¾ long by ¾ wide:

The result is $\frac{9}{16}$



Exercises



Use pictures or chips to illustrate and to answer these problems:

1. $(^-7)^2$
2. $(^-1)^2$
3. $\left(\frac{2}{3}\right)^2$
4. $\left(\frac{3}{5}\right)^2$
5. $\left(-\frac{4}{3}\right)^2$

Complete the operations of multiplication and exponentiation:

6. 15^2
7. $2^2 \cdot 7^2$
8. $(2 \cdot 7)^2$
9. $1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2$
10. $52 \cdot 1^{15}$

Write each number as the square of another number:

Example: 169
Solution: $169 = 13^2$

11. 121
12. 225
13. 10,000
14. 81
15. 144

Complete the multiplication and exponentiation:

16. $3 + (^-7)^2$
17. $3 - 7^2$
18. $3 \cdot 4^2$
19. $4 + 4^2 + 3^2$
20. $3^2\,(2^3)$

Section 3
Cubes

Cubes and Third Powers

If powers of 2 represent squares, what is the visual meaning of raising a number to a power of 3? Start by considering

$$4^3$$

This means the repeated multiplication of 4:

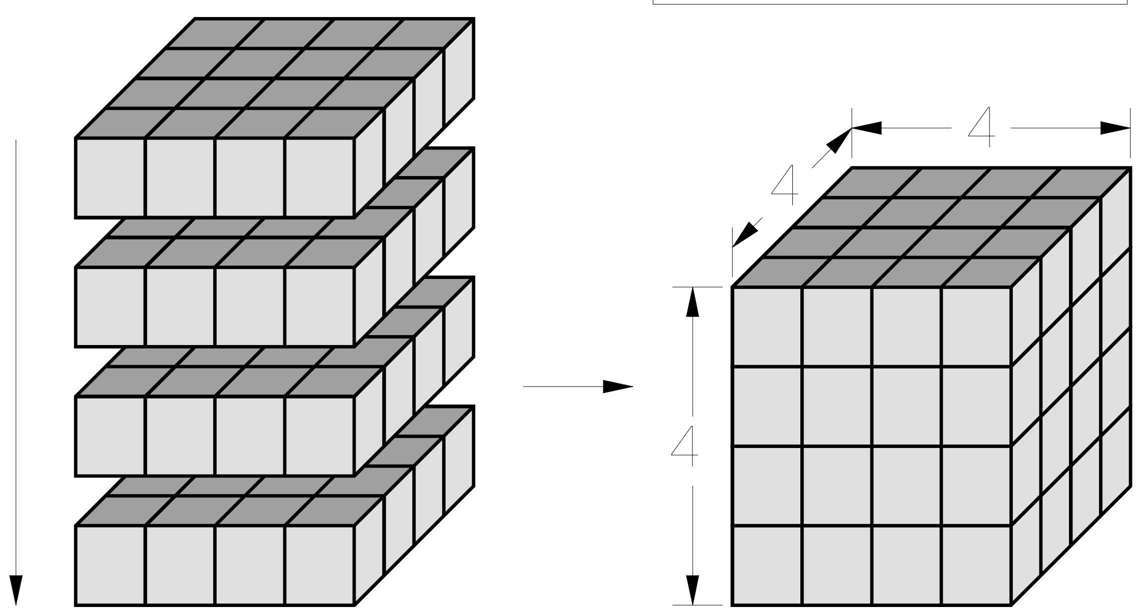
$$4 \cdot 4 \cdot 4$$

If we think of this as:

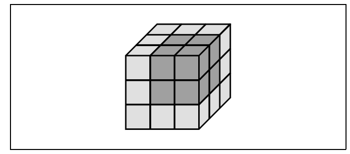
$$4 \cdot (4 \cdot 4)$$

then it is 4 squares, each 4 by 4. Using cubes, we can rearrange these to form a 4 by 4 by 4 cube:

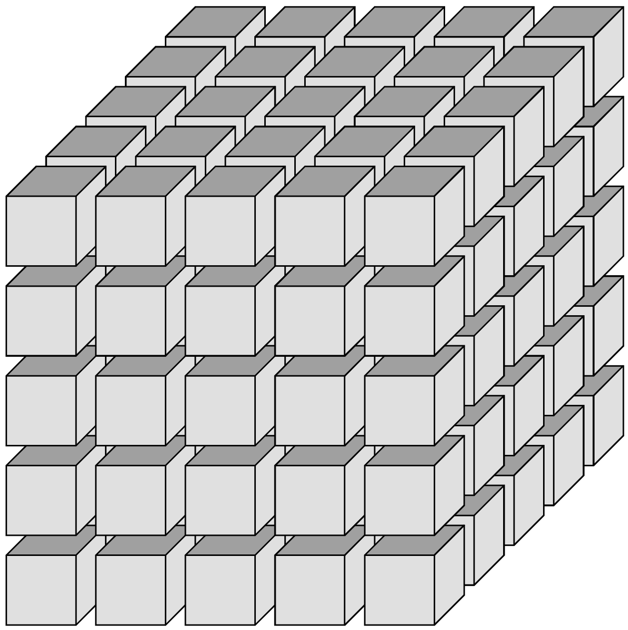
$$4^3 = 4 \cdot 4 \cdot 4 = 64$$



We read the symbol 4^3 as "four to the third power," "four to the power of three," or "four cubed." Visually, when we raise a number to the third power, we are building a larger cube composed of smaller unit cubes. The result of the multiplication is found by counting the number of unit cubes:



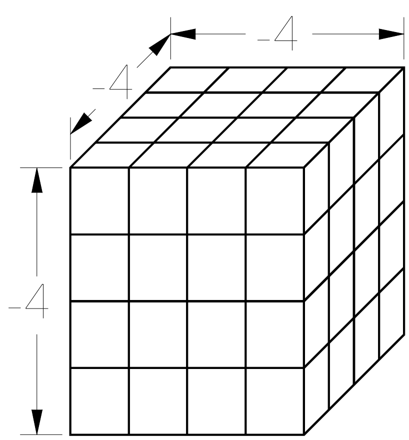
$$5^3 = 5 \cdot 5 \cdot 5 = 125$$

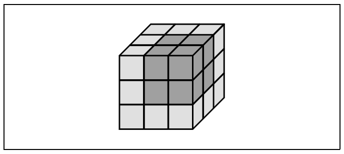


Cubes of Negative Numbers

We have already learned that the product of three negative numbers is negative. Therefore the cube of a negative number is also negative:

$$(^-4)^3 = (^-4) \cdot (^-4) \cdot (^-4) = {}^-64$$

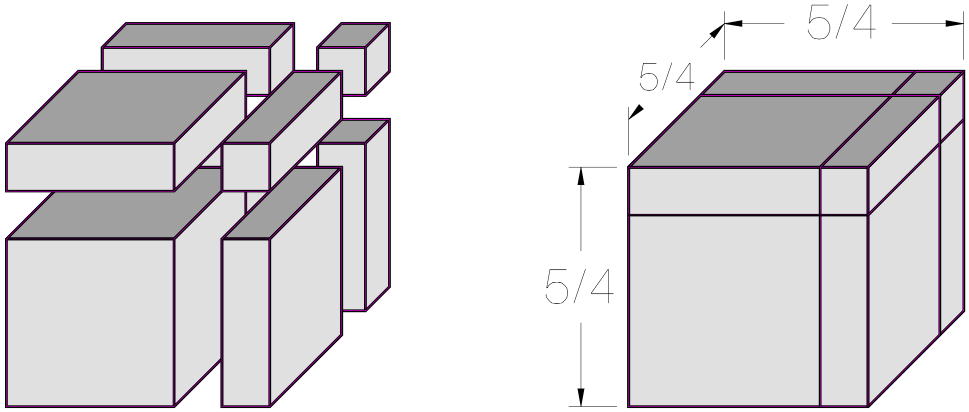




Cubes of Fractions and Mixed Numbers.

As with squares, there is no special difficulty with visualizing the cube of a fraction—for $(5/4)^3$, we build a cube that is $5/4$ on each side:

$$\left(\frac{5}{4}\right)^3 = \frac{5}{4} \cdot \frac{5}{4} \cdot \frac{5}{4} = \frac{125}{64}$$



Exercises

Draw a sketch and calculate:

1. $2^3 =$

2. $1^3 =$

Calculate the answer:

3. 7^3

4. 21^3

5. $\left(\frac{7}{3}\right)^3$

6. $2^2 \cdot 2^3$

7. 0^3

8. $291^3 \cdot 0^3$

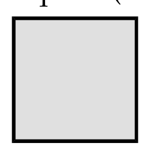
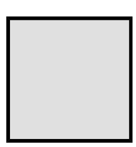
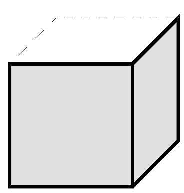
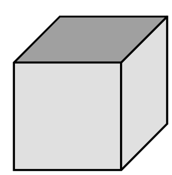
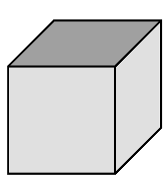
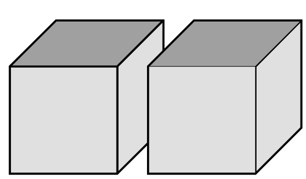
9. $\left(\frac{7}{3}\right)^3 \cdot 3^3$

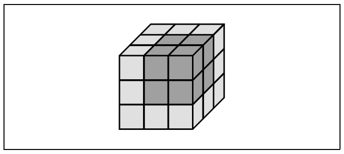
10. $3^2 \cdot 3^2$

Section 4
Higher Powers

Powers Greater Than Three

Our visual models become more difficult after the power of three. For each additional step from 1 to 2 to 3, we extended the model in another direction:

Start with:	Make 2	Connect to form:
Point (no dimensions)		Line (1 dimension)
Line (1 dimension)		Square (2 dimensions) 
Square (2 dimensions) 		Cube (3 dimensions) 
Cube (3 dimensions) 		?? (4 dimensions)

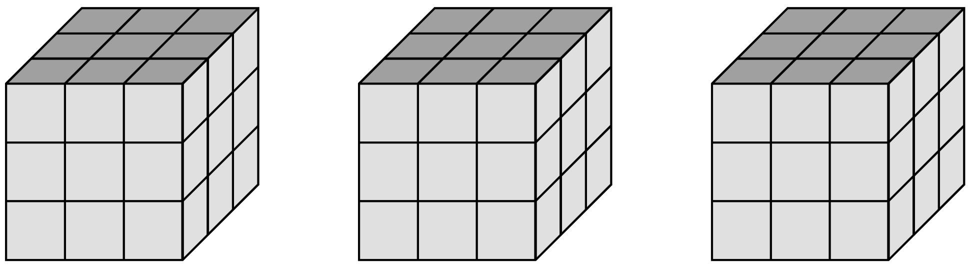


Because we cannot easily visualize a 4^{th} dimension, we will stop at this point. It may be useful with some topics, however, to consider a picture of the 4^{th} power of a number as a group of cubes:

$$\begin{aligned} 3^4 &= 3 \cdot 3 \cdot 3 \cdot 3 \\ &= 3 \cdot (3 \cdot 3 \cdot 3) \\ &= 3 \cdot (3^3) \end{aligned}$$

This would be 3 cubes:

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$



Levels of Exponents

We may need to understand the meaning of a more complex expression such as:

$$(2^3)^4$$

What does this mean? The outside exponent of 4 indicates that we are to multiply four of the quantity in parentheses:

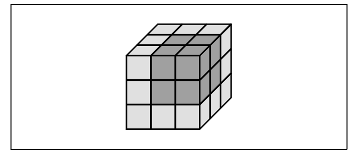
$$(\ )^4 = (\ ) \cdot (\ ) \cdot (\ ) \cdot (\ )$$

$$(2^3)^4 = (2^3) \cdot (2^3) \cdot (2^3) \cdot (2^3)$$

Even in complex expressions, the exponents have the same meaning—repeated multiplication. After multiplying the above expression out, we will have:

$$\begin{aligned} (2^3)^4 &= (2^3) \cdot (2^3) \cdot (2^3) \cdot (2^3) \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \\ &= 2^{12} \end{aligned}$$

Exercises



Complete the operations and simplify to one number:

1. $3^5 + 2^4$

2. $3^5 \cdot 3^4$

3. $(2^2)^2$

4. $(2 \cdot 3)^3$

5. $(2^3)^2 \cdot 5^2$

6. $(3^2)^2$

7. $(5^2)^2$

8. 7^4

9. $(^{-}1)^5$

10. $(^{-}1)^{36}$

11. $[(2^2)^2]^2$

12. $3^3 \cdot 2^2$

13. 2^{10}

14. 2^7

15. $(^{-}2)^{10}$

16. $^{-}(2)^{10}$

17. $5^3 + 6$

18. $1 + 5^3$

19. $1 - 5^3$

20. $(^{-}5)^3$

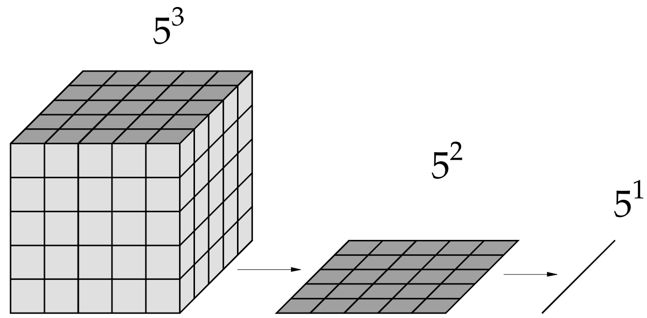
Section 5
Other Exponents: Negative Numbers, Zero, and One

The Power of One

What is the meaning of one as an exponent? When we raise a number to the power of one, we have the number only once. This means that *any number raised to the power of one is equal to itself*:

$$5^3 = (5) \cdot (5) \cdot (5)$$
$$5^2 = (5) \cdot (5)$$
$$5^1 = (5)$$

The progression of powers from 3 to 2 to 1 can be visualized in this manner: In order to extend this sequence, it will be helpful to think of every group of



multiplications as including a multiplication by the number 1; the 1 does not change the value.

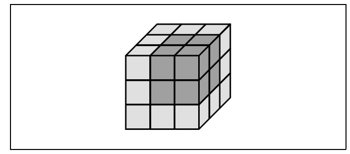
$$5^3 = (1) \cdot (5) \cdot (5) \cdot (5)$$
$$5^2 = (1) \cdot (5) \cdot (5)$$
$$5^1 = (1) \cdot (5)$$

The Power of Zero

What would be a sensible meaning for the power of zero?

$$5^0 = ? \qquad 3^0 = ?$$

From our discussion above, we can extend our idea to zero:



$$5^3 = (1) \cdot (5) \cdot (5) \cdot (5) \qquad \text{(3 fives)}$$
$$5^2 = (1) \cdot (5) \cdot (5) \qquad \text{(2 fives)}$$
$$5^1 = (1) \cdot (5) \qquad \text{(1 five)}$$
$$5^0 = (1) \qquad \text{(0 fives)}$$

Using the exponent to represent how many numbers to multiply, *the zero power must mean that we do not multiply any numbers at all.* For positive integers as bases, *any number raised to the power of zero is one.*

What would be the meaning of a negative exponent?

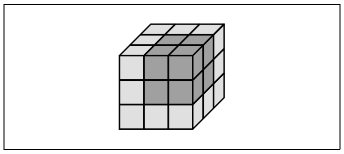
Negative Numbers as Exponents

Consider our familiar decimal system of place value. As we move to the left, each place or column is 10 times as large as the one before. As we move to the right, each column is $\frac{1}{10}$ as large; we divide the previous column by 10:

Place	**3**	**2**	**1**				
Value	1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
Exponent	10^3	10^2	10^1				

Because the place-values are multiples of 10, they can be represented by powers of 10 as shown above. If we add a column labeled “0” and use our new definition of $10^0 = 1$, the ones column will make sense.

Place	**3**	**2**	**1**	**0**			
Value	1000	100	10	**1**	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
Exponent	10^3	10^2	10^1	$\mathbf{10^0}$			



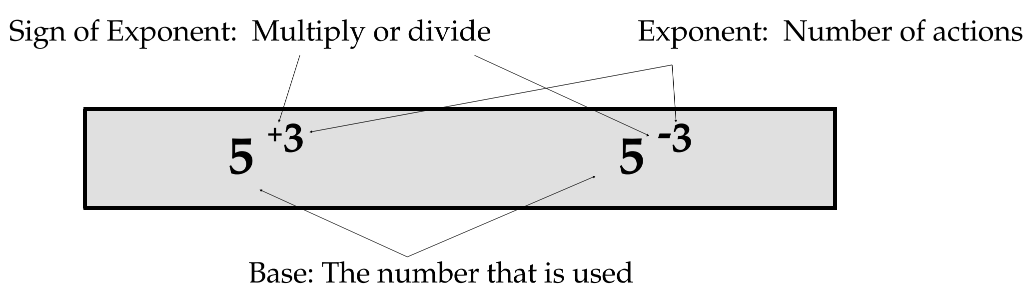
Finally, let's extend our system to the right and label columns as ⁻1 (10^{-1}), ⁻2 (10^{-2}), and so forth. This will preserve the pattern of multiplying by 10 when moving to the left and dividing by 10 when moving to the right:

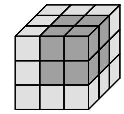
Place	**3**	**2**	**1**	**0**	**⁻1**	**⁻2**	**⁻3**
Value	1000	100	10	**1**	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
Exponent	10^3	10^2	10^1	$\mathbf{10^0}$	$\mathbf{10^{-1}}$	$\mathbf{10^{-2}}$	$\mathbf{10^{-3}}$

By this scheme, it seems sensible to define *negative* powers as *dividing* one by the base and *positive* powers as *multiplying* one by the base:

Exponent	Action (to 1)	Examples
Positive	Multiply	$10^3 = 1 \cdot 10 \cdot 10 \cdot 10 = 1000$
Zero	Nothing	$10^0 = 1 \cdot \_ = 1$
Negative	Divide	$10^{-2} = 1 \div 10 \div 10 = \frac{1}{100} = \frac{1}{10^2}$

It is important to notice that the *negative sign in the exponent* ***does not*** *mean that negative numbers are being multiplied or that the answer is negative*. Instead, it means that the base is on the *bottom* of the fraction.





Now we can consider the meaning of

$$0^0$$

Is there a reasonable meaning for this expression? If we use our idea of the progression of powers, we immediately run into a problem when we use zero as a base. As we move to the left, each number is zero times the one before; as we move to the right, we cannot divide by zero, so there is no clear answer. For this and many other reasons, *we leave 0^0 as* ***not defined.***

Place	**3**	**2**	**1**	**0**			
Value ?	0	0	0	**0?**	$\frac{1}{0}$	$\frac{1}{(0)(0)}$	$\frac{1}{(0)(0)(0)}$
Exponent	0^3	0^2	0^1	$\mathbf{0^0}$ **?**	**?**	**?**	**?**

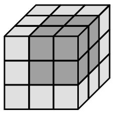
Summary

- **We can think of starting each exponential expression with the number one.**
- *Positive Exponents* **indicate that 1 is being multiplied by the base number several times. The exponent tells us how many times.**
- *Negative Exponents* **indicate that the starting number of 1 is being divided by the base number several times. The exponent tells how many times. Because a fraction indicates division, we often show these divisions as the denominator of a fraction.**
- *Zero Exponents* **indicate that we begin with 1 and then multiply by the base zero times (not at all). The result is 1.**

$$x^3 = 1 \cdot x \cdot x \cdot x$$

$$x^{-3} = \frac{1}{x \cdot x \cdot x} = \frac{1}{x^3}$$

$$x^0 = 1 \qquad \text{for } x \neq 0$$



- **Raising *zero to the zero power* is not defined.**

Zero to the Zero Power

0^0 is not defined

Exercises

Evaluate these expressions:

1. 999^1
2. 999^0
3. 6^{-3}
4. 1^{-5}
5. 5^{-1}
6. $(5^{-1}) \cdot 5^2$
7. $(10^{-2}) \cdot 10^2$
8. 4^{-3}
9. 5^{-4}
10. 1^{-1}
11. 5^0
12. $(^-3)^0$
13. 0^0
14. 273^1
15. 273.6^0
16. $(^-3)^{-4}$
17. $(^-3)^{-2}$
18. $2 + 2^{-1}$
19. $3 + 3^{-2}$
20. $16 + 4^{-2}$

Section 6
Properties of Powers

Introduction

In this section, we will examine some of the properties that allow us to restate exponential expressions. All of the properties have a clear basis; it is not necessary to memorize any of them. As you gain an understanding of these properties, you will find that you will remember them easily.

Multiplying with the Same Base

Consider the expression

$$3^5 \cdot 3^4$$

From the symbols alone, it is difficult to tell if we can combine powers or rewrite terms. Is the answer

$$3^{20} \; ?$$

$$9^{20} \; ?$$

$$3^9 \; ?$$

$$6^9 \; ?$$

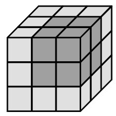
The best way to find out is to ask "What does it mean?"

$$\begin{aligned} 2^3 \cdot 2^4 &= (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) \\ &= (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \\ &= 2^7 = 128 \end{aligned}$$

or

$$2^{(3+4)}$$

Once we replaced the powers of 2 with their meaning in terms of multiplication, it was clear that *when multiplying two quantities with the same base raised to a power, the exponents add.* We add exponents because we are summing up the total number of factors.



To check, we calculate the value of $2^3 \cdot 2^4$ and compare this to the value of 2^7:

$$2^3 \cdot 2^4 = 8 \cdot 16 = 128$$

$$2^7 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = 128$$

Here are some other examples:

$$\begin{aligned} 2^3 \cdot 25 &= (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \\ &= (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \\ &= 2^{(3+5)} \\ &= 2^8 \end{aligned}$$

$$\begin{aligned} x^3 \cdot x^5 &= (x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x \cdot x) \\ &= (x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x) \\ &= x^{(3+5)} \\ &= x^8 \end{aligned}$$

Multiplying with the same base

$$x^a x^b = x^{a+b}$$

Dividing with the Same Base

A similar property exists when we divide two quantities where the same base is raised to a power. Consider:

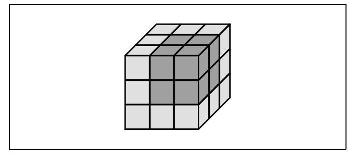
$$\frac{2^4}{2^3}$$

Again, if we think about the meaning of this expression, it will be easy to discover the property:

$$\begin{aligned} \frac{2^4}{2^3} &= \frac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} \\ &= \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} \cdot \frac{2}{1} \\ &= 1 \cdot 2 = 2 \end{aligned}$$

This suggests the idea that

$$\frac{2^4}{2^3} = 2^{(4-3)}$$
$$= 2^1$$
$$= 2$$



We subtract exponents because we are counting the number of factors that remain after cancelling to one. *When dividing two quantities where the same base is raised to a power, we subtract the bottom exponent from the top exponent.*

Here are some other examples:

$$\frac{3^4}{3^2} = \frac{3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3}$$
$$= \frac{3 \cdot 3}{3 \cdot 3} \cdot \frac{3 \cdot 3}{1}$$
$$= 3^{(4-2)}$$
$$= 3^2$$

$$\frac{x^4}{x^2} = \frac{x \cdot x \cdot x \cdot x}{x \cdot x}$$
$$= \frac{x \cdot x}{x \cdot x} \cdot \frac{x \cdot x}{1}$$
$$= x^{(4-2)}$$
$$= x^2$$

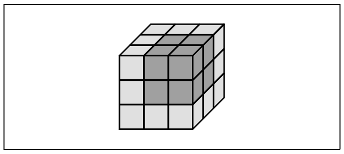
Dividing with the same base

$$\frac{x^a}{x^b} = x^{a-b}$$ **where x is not zero**

Zero and Negative exponents—Again

Let us return to our earlier definition that any non-zero number raised to the zero power is one. Using our latest property, look at

$$\frac{3^5}{3^5}$$



This is one, because any number divided by itself is one. By our property,

$$\begin{aligned}\frac{3^5}{3^5} &= \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} \\ &= 3^{(5-5)} \\ &= 3^0 = 1\end{aligned}$$

This is another reason that 30 must be equal to one. For 00, consider a similar example:

$$\begin{aligned}\frac{0^5}{0^5} &= 0^{(5-5)} \quad ? \\ &= 0^0 \qquad ?\end{aligned}$$

We cannot cancel out the quantities because we cannot divide by zero. This is another reason to decide that 0^0 is not defined.

We can think of a given quantity as if it were the result remaining from a fraction where the numerator and the denominator both had the same base. We see the result after common factors have cancelled.

- **If the exponent of the result is positive, there were more factors in the numerator.**
- **If the exponent of the result is negative, there were more factors in the denominator.**
- **If the exponent of the result is zero, there were equal numbers of factors that cancelled to 1.**

The property shows that our definition of negative exponents is sensible.

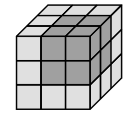
$$\begin{aligned}\frac{3^2}{3^5} &= \frac{3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} \\ &= \frac{3 \cdot 3}{3 \cdot 3} \cdot \frac{1}{3 \cdot 3 \cdot 3} \\ &= \frac{1}{3 \cdot 3 \cdot 3} = \frac{1}{3^3} \\ &= 3^{-3}\end{aligned}$$

By our property of subtracting exponents, this is

$$\frac{3^2}{3^5} = 3^{(2-5)} = 3^{-3}$$

Again, the property confirms our previous definition.

Two Levels of Exponents



When evaluating exponential expressions, we often encounter quantities like these:

$$(2^3)^4$$

$$(3^5)^2$$

$$(x^2)^3$$

By examining the meaning of these expressions, we can discover another useful property. First, as we discussed previously, raising a quantity to a power has the same meaning even if the quantity contains exponents:

$$(\quad)^4 = (\quad)\cdot(\quad)\cdot(\quad)\cdot(\quad)$$

$$(2^3)^4 = (2^3)\cdot(2^3)\cdot(2^3)\cdot(2^3)$$

If we expand this further, we see we have 4 groups, each containing 3 two's. The total number of two's is 3 · 4 or 12:

$$\begin{aligned}(2^3)^4 &= (2^3)\cdot(2^3)\cdot(2^3)\cdot(2^3)\\ &= (2\cdot 2\cdot 2)\cdot(2\cdot 2\cdot 2)\cdot(2\cdot 2\cdot 2)\cdot(2\cdot 2\cdot 2)\\ &= 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\\ &= 2^{12}\\ &= 2^{(3\cdot 4)}\end{aligned}$$

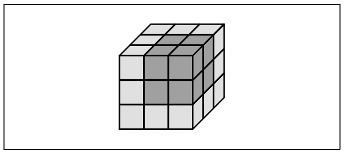
When a base is raised to a power, and the expression is again raised to a power, the result is the base raised to the product of the powers.

Two levels of exponents

$$(x^a)^b = x^{ab}$$

A Product Raised to a Power

If a product of two quantities is raised to a power, we can find another way to write the resulting expression:



$$\begin{aligned}(2 \cdot 3)^2 &= (2 \cdot 3) \cdot (2 \cdot 3)\\ &= 2 \cdot 3 \cdot 2 \cdot 3\\ &= (2 \cdot 2) \cdot (3 \cdot 3)\end{aligned}$$

Because the factors 2 and 3 both occur twice, the associative and commutative properties allow us to rearrange the numbers; the result is two of each.

This will clearly hold true for any quantities and any power:

$$\begin{aligned}(x \cdot y)^3 &= (x \cdot y) \cdot (x \cdot y) \cdot (x \cdot y)\\ &= (x \cdot x \cdot x) \cdot (y \cdot y \cdot y)\\ &= \quad x^3 \quad \cdot \quad y^3\end{aligned}$$

Any product raised to a power can be restated as the product of each factor raised to a power. Note that this pattern occurs because of the specific situation—there is no general "distributive" law that allows us to always take something on the outside and apply it to the inside.

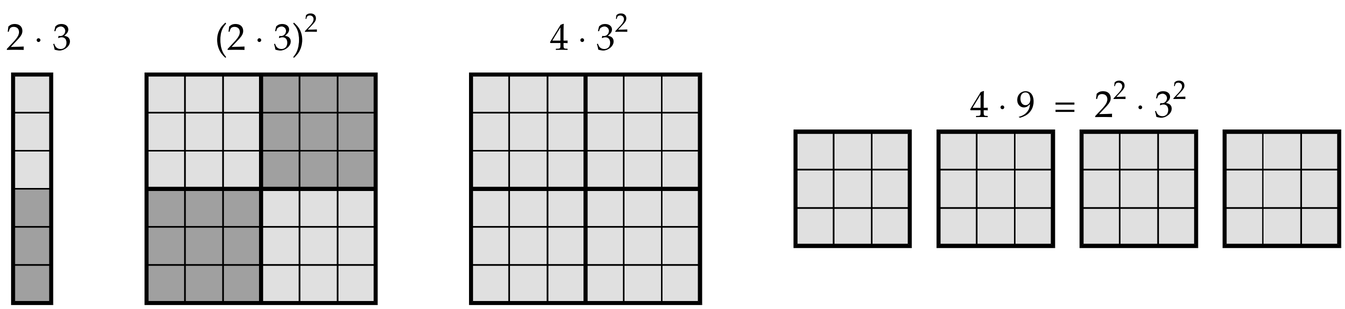
A product raised to a power

$$(xy)^a = x^a y^a$$

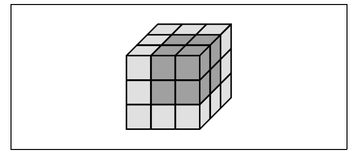
The picture of a simple example—

$$(2 \cdot 3)^2 = 2^2 \cdot 3^2$$

may help us to understand the meaning of this property. We start with the left side—a square that is 2·3 or 6 units on each side. We then show that this is the same as the right side—4 (or 2^2) groups of 9 (or 3^2):



Fractions and Exponents



There are two final properties involving fractions that we will find useful to discuss. Consider an expression where we are raising a fraction to a power:

$$\left(\frac{2}{3}\right)^2$$

We evaluate this by using the meaning of the exponent 2:

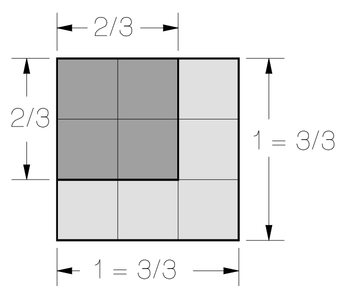
$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{2 \cdot 2}{3 \cdot 3} = \frac{4}{9}$$

When we square a fraction, we actually square both the top and bottom of the fraction. *When we raise a fraction to a power, we raise both the numerator (top) and the denominator (bottom) to the same power.*

A Fraction Raised to a Power

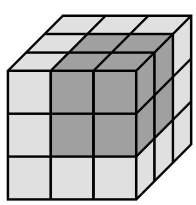
$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

One way of showing this visually is as follows:



$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$

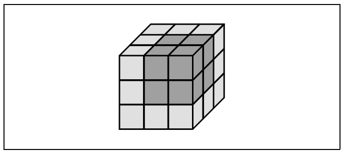
In three dimensions, here is $(2/3)^3$:



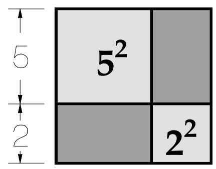
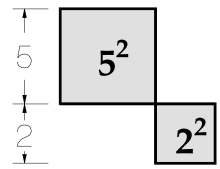
$$\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$$

Common Errors

The material in this section—properties of powers—is difficult for many students. Most errors result from attempting to memorize patterns of symbols instead of working to understand the concepts involved. *Properties can be learned as facts about real things rather than as meaningless patterns of symbols.*



Here are some of the common errors. Each is an attempt to apply a pattern of symbols to an inappropriate situation:

Error (False)	**Picture** (Why it's not true)	**Looks like:** (True)
$(5+2)^2$ *does not equal* $5^2 + 2^2$	$(5+2)^2$  $5^2 + 2^2$ 	$(5 \cdot 2)^2 = 5^2 \cdot 2^2$
$\frac{3^4}{4^3}$ *does not equal* $\left(\frac{3}{4}\right)^{4-3}$	**??** Top and bottom must have the same base to combine in this way.	$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$ $\frac{x^a}{x^b} = x^{a-b}$
$3^5 \cdot 2^6$ *does not equal* $(3 \cdot 2)^{5+6}$	**??** Both factors must have the same base to combine in this way.	$x^a y^a = (xy)^a$ $x^a x^b = x^{a+b}$

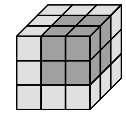
Summary

The properties we have learned, like any rules or shortcuts, are difficult to remember and use correctly unless you know where they come from. Starting with basic definitions, you can derive the properties for yourself any time that you need them.

Properties of Exponents

$x^1 = x$	$\frac{x^a}{x^b} = x^{a-b}$ **(x is not zero)**
$x^{-a} = \frac{1}{x^a}$	$(x^a)^b = x^{ab}$
$x^0 = 1$ **(x is not zero)**	$(xy)^a = x^a y^a$
$x^a x^b = x^{a+b}$	$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

Exercises



Use the properties in this section to simplify the expressions:

1. $a^3 \cdot a^7$

2. $\dfrac{x^{17}}{x^3}$

3. $(a^3)^6$

4. $(a^0)^{16}$

5. $2^3 \cdot 5^1$

6. $\dfrac{2^5 \cdot 5^{-1}}{2^4 \cdot 3^1}$

7. $\left(\dfrac{3}{5}\right)^3$

8. $\dfrac{x^3 y^{-5}}{x^2 y^2}$

9. $2^3 \cdot 3^2 \cdot 5^{-2}$

10. $\dfrac{x^3 x^1}{(x^3)^2}$

Decide whether each equation is true or false. If it is true, why?

11. $x^5 \cdot y^{-2} = (xy)^3$

12. $(a^3)^3 = a^9$

13. $3^5 \cdot 3^3 = 3^8$

14. $(3^5)^5 = 3^{25}$

15. $\dfrac{2^5}{2^{-2}} = 2^3$

16. $2^5 \cdot 2^{-2} = 2^3$

17. $(3^2 \cdot 5^3)^0 = 1125$

18. $\dfrac{4^0}{3^3} = \dfrac{1}{27}$

19. $\dfrac{a^2}{b^5} = \dfrac{1}{b^3}$

20. $(15)^4 = 3^4 \cdot 5^4$

Section 7
Simplifying Expressions

Using the Properties

Expressions often contain many levels of exponents and many different fractions, multiplications, etc. It is easy to combine and simplify expressions if we use the appropriate properties one at a time. For example:

$$\begin{aligned}(x^2y^3)^5 &= (x^2)^5 \cdot (y^3)^5 \\ &= x^{(2 \cdot 5)} \cdot y^{(3 \cdot 5)} \\ &= x^{10} \cdot y^{15} \\ &= x^{10} y^{15}\end{aligned}$$

It is sometimes helpful to temporarily ignore the quantity inside of a pair of parentheses if this makes the use of the properties more easily apparent:

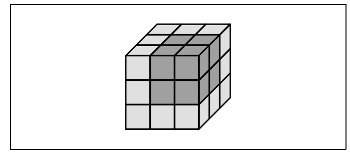
$$\begin{aligned}(x^2y^3)^5 &= (\quad)^5 \cdot (\quad)^5 \\ &= (x^2)^5 \cdot (y^3)^5 \\ &= x^{(2 \cdot 5)} \cdot y^{(3 \cdot 5)} \\ &= x^{10} \cdot y^{15} \\ &= x^{10} y^{15}\end{aligned}$$

When several different properties apply, it is often possible to simplify an expression in several ways; one way may be faster or easier, but it is not important which way we choose. For example:

$$\left(\frac{x^3}{x^2}\right)^5$$

If we begin by raising each part of the fraction to the 5^{th} power, it looks like this:

$$\begin{aligned}\left(\frac{x^3}{x^2}\right)^5 &= \frac{x^{3 \cdot 5}}{x^{2 \cdot 5}} \\ &= \frac{x^{15}}{x^{10}} \\ &= x^{(15-10)} \\ &= x^5\end{aligned}$$



If we simplify the fraction inside of the parentheses first, then the process is somewhat easier:

$$\left(\frac{x^3}{x^2}\right)^5 = (x^{(3-2)})^5$$
$$= (x^1)^5$$
$$= x^5$$

Here is the same problem done by cancelling common factors. This is a demonstration of why the rules work:

$$\left(\frac{x^3}{x^2}\right)^5 = \left(\frac{x \cdot x \cdot x}{1 \cdot x \cdot x}\right)^5$$
$$= \left(\frac{x}{1}\right)^5$$
$$= x^5$$

Whenever possible, simplify fractions and quantities in parentheses before raising quantities to a power.

Properties and Negative Exponents

We will now discover if the properties of the previous section will apply to quantities with negative exponents. For each property, we can evaluate the expression in two ways: first using the rule directly, and second, using the definition of negative exponents. For example:

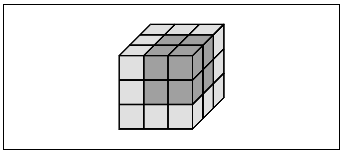
$$x^2 \cdot x^{-1}$$

By the property:

$$x^2 \cdot x^{-1} = x^{(2+-1)}$$
$$= x^{(2-1)}$$
$$= x^1$$

By the definition of negative exponents:

$$x^2 \cdot x^{-1} = x^2 \cdot \frac{1}{x^1}$$
$$= \frac{x^2}{x^1} = \frac{x \cdot x}{1 \cdot x}$$
$$= x^{(2-1)}$$
$$= x^1$$



We can see that both methods give the same answer; we can also see how adding a negative exponent gives the same result as subtracting a positive exponent. Here is a second example:

$$(x^2)^{-3}$$

By the property:

$$\begin{aligned}(x^2)^{-3} &= x^{(2 \cdot -3)} \\ &= x^{-6}\end{aligned}$$

By the definition of negative exponents:

$$\begin{aligned}(x^2)^{-3} &= \frac{1}{(x^2)^3} \\ &= \frac{1}{x^{(2 \cdot 3)}} \\ &= \frac{1}{x^6} \\ &= x^{-6}\end{aligned}$$

Again, the two methods give the same results.

Negative Exponents in the Denominator.

Consider the expression:

$$\frac{1}{x^{-2}}$$

Because a fraction can also represent a division problem, we can evaluate it like this:

$$\begin{aligned}\frac{1}{x^{-2}} &= 1 \div x^{-2} \\ &= 1 \div \frac{1}{x^2} \\ &= 1 \cdot \frac{x^2}{1} \\ &= x^2\end{aligned}$$

This will be true for all negative powers that are factors in the denominator.

Of course, we also know that a quantity in the numerator (top) of the fraction raised to a negative power can be rewritten as a positive exponent on the denominator (bottom) of the fraction:

$$\frac{z^{-3}}{1} = \frac{1}{z^{3}}$$

These two ideas can be used in the same fraction. If you wish, you can now rewrite all factors with negative exponents by inverting these factors and using all positive exponents. For example:

$$\frac{a^{-3}x^{-2}y^{5}}{z^{-3}b^{3}} = \frac{1}{a^{3}} \cdot \frac{1}{x^{2}} \cdot \frac{z^{3}}{1} \cdot \frac{y^{5}}{b^{3}}$$

$$= \frac{z^{3}y^{5}}{a^{3}x^{2}b^{3}}$$

Quantities having positive exponents do not change, but quantities with negative exponents are written in an inverted manner and the exponents become positive. Fractions having negative exponents (around the whole fraction) are inverted and the exponent becomes positive:

$$\left(\frac{a}{x}\right)^{-2} = \frac{a^{-2}}{x^{-2}} = \frac{x^{2}}{a^{2}} = \left(\frac{x}{2}\right)^{2}$$

- **A quantity with a negative exponent:**

  In the numerator—may be rewritten with a positive exponent in the denominator.

  In the denominator—may be rewritten with a positive exponent in the numerator.

Format and Symbols

Many students are not sure whether it is necessary to change all factors with negative exponents into factors with positive exponents. For the purposes of this book, there are two acceptable methods; it is not important which way you do it, as long as you are consistent:

$$x^{-6} \quad \text{or} \quad \frac{1}{x^{6}}$$

$$x^{2}y^{-6} \quad \text{or} \quad \frac{x^{2}}{y^{6}}$$

Being consistent means that you show the result in one of two ways:

- **Method 1:** ***With fractions, using only positive exponents*. Each expression with x or y is shown on the top or the bottom of the fraction, whichever will result in a positive exponent.**

  Examples: $\dfrac{x^3}{y^2}$, $\dfrac{8}{x^3 y^2}$

- **Method 2:** ***Without fractions, using both positive and negative exponents.***

  Examples: $x^3 y^{-2}$, $8x^{-3} y^{-2}$

Preferred		**Not Preferred**
Method 1	**Method 2**	
$\dfrac{x^3}{y^2}$	$x^3 y^{-2}$	$\dfrac{y^{-2}}{x^{-3}}$
$\dfrac{128}{x^2}$	$128x^{-2}$	$\dfrac{x^{-2}}{2^{-7}}$
$\dfrac{1}{8x^3}$	$\dfrac{1}{8} x^{-3}$	$2^{-3} x^{-3}$
$\dfrac{a^3 b^2 c^5}{x^2 y^3 z^4}$	$a^3 b^2 c^5 x^{-2} y^{-3} z^{-4}$	$\dfrac{x^{-2} y^{-3} z^{-4}}{a^{-3} b^{-2} c^{-5}}$

Other forms may be called "not preferred," but they are not wrong. It is convenient to agree upon standard forms so that we will be able to compare our results with others. Even if you prefer one form over the other, it is still useful to practice both formats. Our agreement on the final form is given in the following summary:

Summary

To simplify complicated expressions:

- **Apply each property separately.**
- **If the properties seem confusing, return to the most basic definitions and work through every step.**
- **Simplify fractions and expressions inside of parentheses first, before raising the expressions to a power.**
- **Use the properties with negative as well as positive exponents.**
- **If desired, factors with negative exponents can be rewritten with positive exponents by using the ideas that:**

$$x^{-a} = \frac{1}{x^{a}}$$

$$\frac{1}{x^{-a}} = x^{a}$$

- **Where negative exponents or fractions occur, write the result consistently with one of these methods: no fractions *or* no negative exponents. Do not mix fractions and negative exponents.**
- **Write small common numbers (8, 16, 25) as integers, not exponential expressions (2^3, 2^4, 5^2).**

Exercises

Evaluate these expressions by combining and simplifying. Write your answers in one of the two forms described above:

1. $x^{-2}y^{-5}x^{3}$

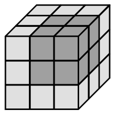
2. $\dfrac{(3xy)^2}{x^{-2}y^{3}}$

3. $(a^{2})^3(a^{0})^2(a^{-5})$

4. $\dfrac{(2m)^3}{(mn^{5})^2}$

5. $\dfrac{6^2 \cdot 5 \cdot 3^{-3}}{2^4}$

6. $\dfrac{x^{2}y^{-3}z^{5}}{x^{4}y^{-5}z^{3}}$



7. $(2 \cdot 5)^4$

8. $(3^2)^{-3}$

9. $(-2x^{-3})^3$

10. $\dfrac{1}{2^2 \cdot 3^{-2}}$

11. $(x^2)^5$

12. $(x^{-2})^5$

13. $\dfrac{y^2}{(y^3)^4}$

14. $\dfrac{(x^{-9})^0 (x^4)^{-3}}{(x^{-1})^5 (x^5)^{-2}}$

15. $\dfrac{15x^3 y^2}{20x^2 y}$

16. $\dfrac{12x^3 y^6}{12x^3 y^7}$

17. $\dfrac{a^3 b^5 c^7}{a^{-3} b^{-5} c^{-7}}$

18. $\dfrac{1}{a^{-3} b^{-5} c^{-7}}$

19. $\left(\dfrac{(a^3 b^4)^{-2}}{a^3 c^5}\right)^{-1}$

20. $\left(\dfrac{a^3 b^{-2}}{(c^5 a^2)^{-1}}\right)^2$

21. $\left(\dfrac{(x^2 y^{-2})^3}{(x^3 y^2)^{-2}}\right)^2$

22. $\left(\dfrac{a^{-1} x^{-1}}{a^{-1} x^{-1}}\right)^{10}$

Section 8
Roots and Radicals

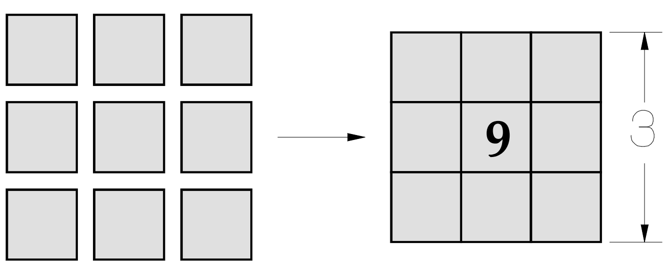
Square Roots

We have defined raising 3 to the second power as follows:

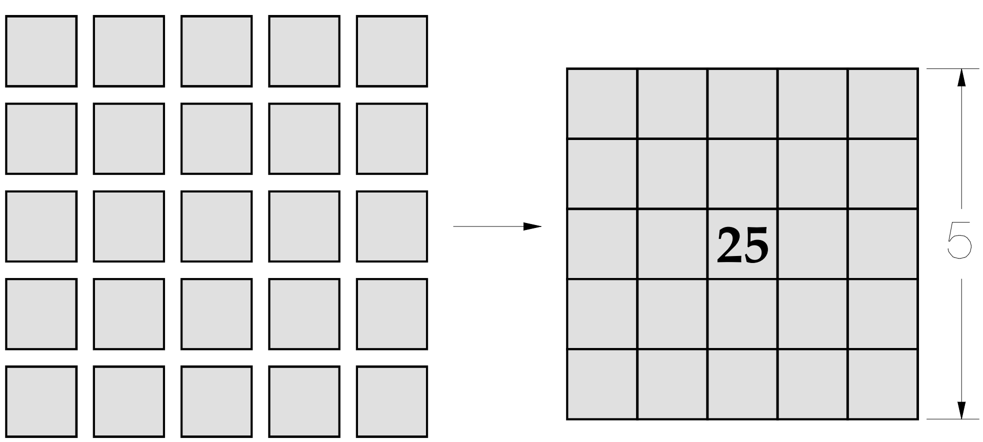
Make a square that is 3 long by 3 wide.
Count the number of squares inside.
This is the result: 9.

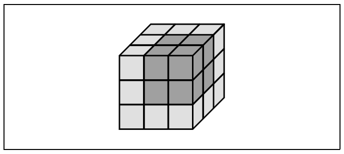
We could also consider the opposite problem:

Count out 9 unit chips.
Form the chips into a square.
Measure the length of the side.
This is the result: 3.



Try it now with 25 chips:





The result is 5. We call this process **taking the square root**. It is indicated as:

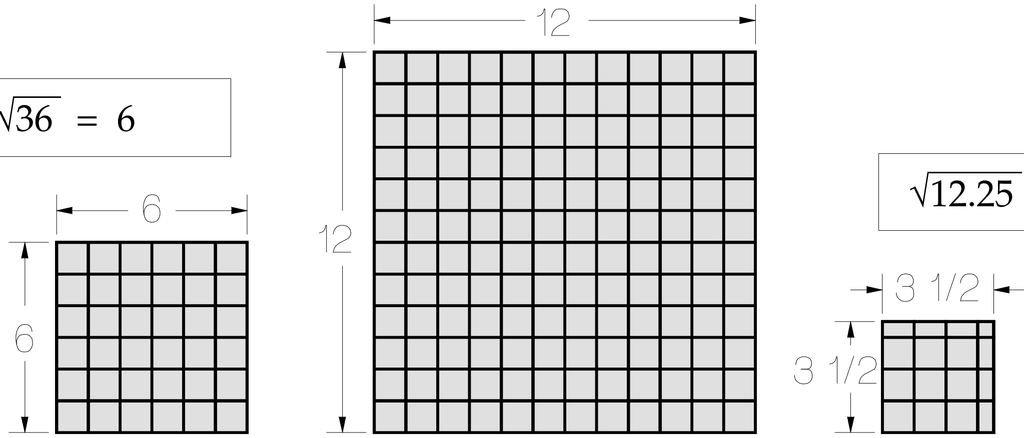
$$\sqrt{25} = 5$$

The new symbol $\sqrt{}$ is called a **radical sign**. Here are some more examples:

$$\sqrt{144} = 12$$

$$\sqrt{36} = 6$$

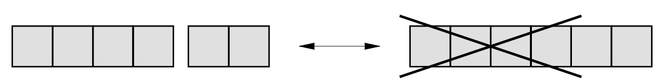
$$\sqrt{12.25} = 3.5$$



Taking the square root of 25 can also be stated as the question "What positive number can be multiplied times itself to get 25?" Notice that to avoid confusion, we have ignored the possibility of choosing $^-5$ as the answer, even though $(^-5)^2$ is 25. *The square root is always a positive number.*

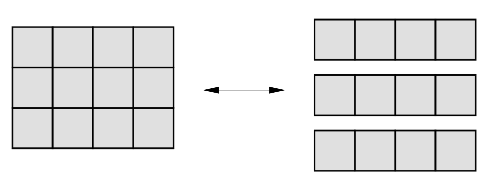
If the difference between the square and the square root seems hard to grasp, you have probably noticed that the two operations are very similar. In fact, they are opposite or inverse operations in the same way that we discussed addition/subtraction and multiplication/division as opposites:

$4 + 2 = 6$



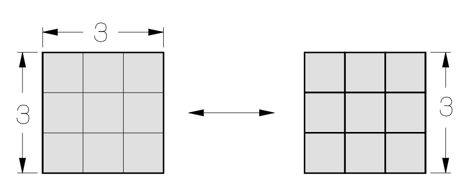
$6 - 4 = 2$

$3 \cdot 4 = 12$



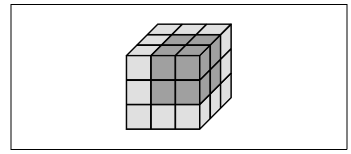
$12 \div 4 = 3$
or
$12 \div 3 = 4$

$3^2 = 9$



$\sqrt{9} = 3$

Cube Roots



Again, we can look at raising 2 to the third power as:

Build a cube 2 long by 2 wide by 2 high.

Count up the number of unit cubes inside.

This is the result: 8.

To reverse the process:

Start with 8 unit cubes.

Arrange them into a larger cube.

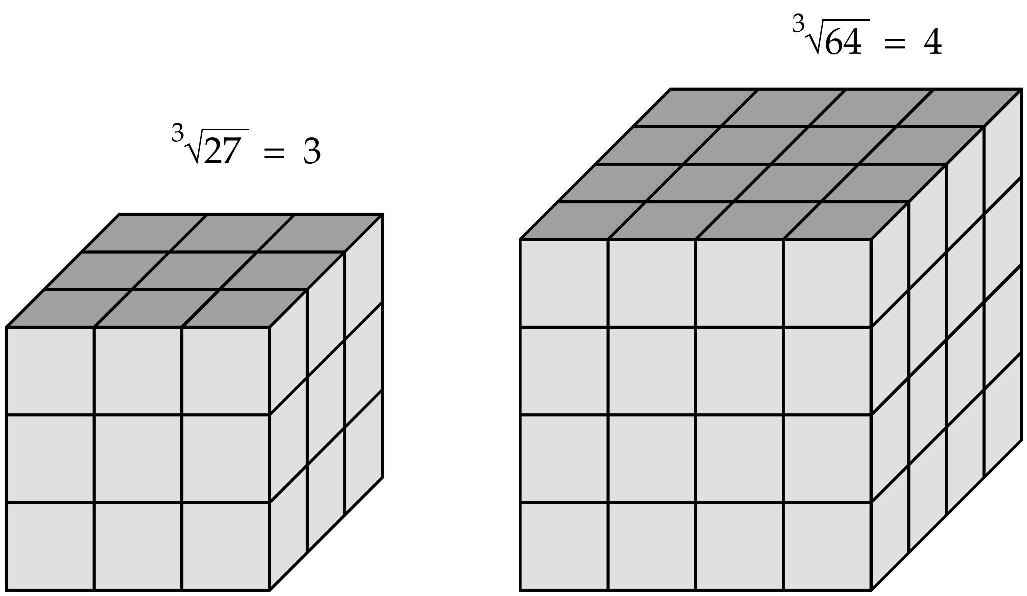
How large is the side of the cube?

The result is 2.

This process is called **taking the third root or cube root.** Taking the cube root of 8 can also be stated as “What number to the 3^{rd} power gives 8?” The cube root of 8 is indicated by:

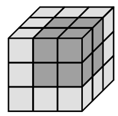
$$\sqrt[3]{8} = 2$$

The small 3 indicates the type of root. Here are some other examples:



Other Roots

Just as we agreed to define exponents of 4, 5, or any positive integer, we can define other kinds of roots. For example, the 5^{th} root of 32 is the number that that is raised to the 5^{th} power to give 32. The correct choice is 2.



The following chart shows some other examples:

Symbol	Meaning	Result	Check
$\sqrt[5]{32}$	What number to the 5th power gives 32?	2	$2^5 = 32$
$\sqrt[4]{16}$	What number to the 4th power gives 16?	2	$2^4 = 16$
$\sqrt[4]{81}$	What number to the 4th power gives 81?	3	$3^4 = 81$

Roots with Negative Bases

Is it sensible to define the square or cube root of a negative number?

$$\sqrt{^-25} = ?$$

$$\sqrt{^-16} = ?$$

$$\sqrt[3]{^-8} = ?$$

First, consider the square roots of negative numbers. If we ask the question "What number times itself equals $^-25$," we know from our study of integers that neither negative nor positive numbers will fit; any number times itself results in a positive number or zero. We conclude that *the square root of a negative number is not defined.*

Next, consider cube roots of negative numbers. Our question is "What number to the 3^{rd} power gives a result of $^-8$?" Because three negative numbers multiplied together will result in a negative number, there is a possible answer: $^-2$. Unlike the difficulty with square roots, there is no problem in deciding that *the cube root of a negative number must be negative.* Here are some more examples:

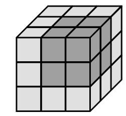
$$\sqrt{^-73} = \text{not defined}$$

$$\sqrt[3]{^-64} = {}^-4$$

$$\sqrt[5]{^-32} = {}^-2$$

Because odd numbers of negatives multiply to give negative results, you can see that *odd-numbered roots of negative numbers have a solution, but even-numbered roots do not.*

Summary



- **Taking roots is the inverse operation of exponentiation.**

- **The radical sign indicates the operation of taking the root. A small raised number indicates the type of root.** ***If there is no number, we agree that the quantity will be a square root.***

- **The square root of a given number is interpreted as taking that number of unit chips, building a larger square, and measuring the length of the side. It is also interpreted as the answer to the question "What number can be multiplied by itself to result in the given number?"**

- **The cube root of a given number is interpreted as taking that number of unit cubes, building a larger cube, and measuring the length of the side. It is also interpreted as the answer to the question "What number raised to the 3^{rd} power will result in the given number?"**

- ***The square root (or any even root) of a negative number is not defined.*** **The cube root (or any odd root) of a negative number will be negative. Any root of a positive number will be positive.**

Exercises

Evaluate these roots. If necessary, simplify the radicals and complete the multiplication or addition.

1. $\sqrt{64}$

2. $\sqrt[3]{125}$

3. $\sqrt[3]{1000}$

4. $\sqrt[3]{1}$

5. $\sqrt{10{,}000}$

6. $\sqrt{1}$

7. $\sqrt{0}$

8. $\sqrt[3]{-1000}$

9. $\sqrt[3]{-125}$

10. $\sqrt{-25}$

11. $\sqrt{25} + \sqrt{36}$

12. $\sqrt{25} \cdot \sqrt{36}$

13. $\sqrt[3]{8} \cdot \sqrt[3]{64}$

14. $\sqrt[3]{8 \cdot 64}$

Section 9
Irrational Numbers

The Square Root of 10

Most of the examples of square roots we have been considering have answers that are positive integers. A positive integer with an integer square root is called a **perfect square.**

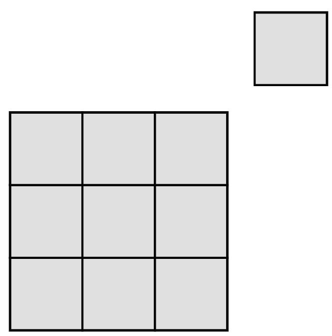
16, 25, 144, and 100 are perfect squares because

$$16 = 4^2,\ 25 = 5^2,\ 144 = 12^2,\ \text{and } 100 = 10^2.$$

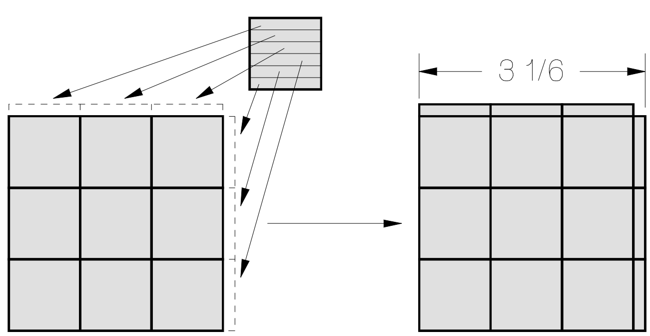
Can we extend the idea of a square root to numbers that are not perfect squares? Let us consider this expression:

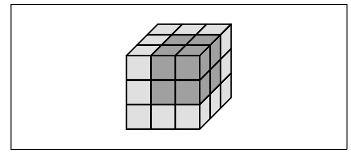
$$\sqrt{10}$$

By our previous definition, we should take 10 unit chips and rearrange them to form a larger square. After we use up 9 chips, we have 1 left over:



If we cut up this chip (you might want to use a paper chip), we can rearrange the pieces to get closer to a square; since we need to add equally to the height and width, 3 + 3 or 6 pieces will work best:

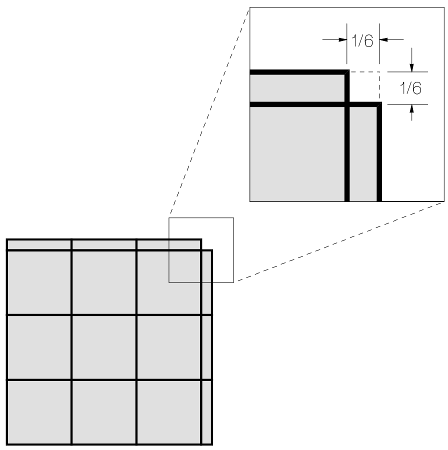




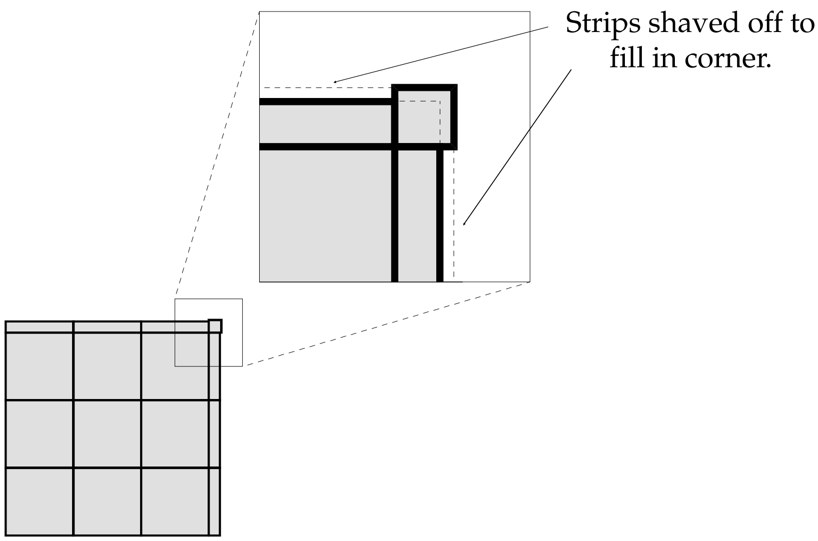
Each piece we added was ⅙ thick, so the figure is now 3⅙ wide and 3⅙ high. To check our work, we convert 3⅙ to a decimal and then square it:

$$\left(3\frac{1}{6}\right)^2 = (3.1667)^2 = 10.03$$

This is a good estimate, but it is not exact. Why not?—because there is a small area in the top right corner that is not filled in:



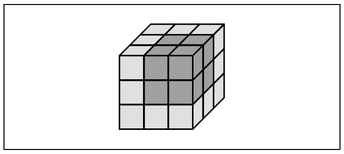
If we wish, we can continue to try to get an exact fit by shaving off a little from the top and right sides; we then use this to fill in the missing corner:



$$\text{Missing area} = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$$

$$\text{Shaved from top and right (6 pieces)} = \frac{1}{36} \div 6 = \frac{1}{216}$$

$$\text{New length of side} = 3 + \frac{1}{6} - \frac{1}{216} = 3.1620$$



To check our work:

$$(3.1620)^2 = 9.998$$

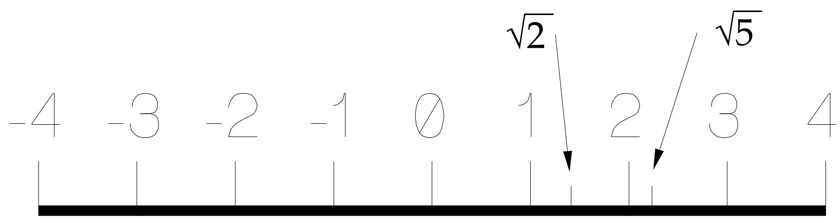
This is even closer, but the answer is now a little too small because we have cut off too much; the corner square now sticks out a little past the sides.

We can see that this process will get us increasingly more accurate answers in terms of sums and differences of fractions, but that we will never get the exact value. *The square root of 10 is not a fraction.*

Fractions and integers are called **rational numbers** because they can be expressed as ratios of integers. If an integer is not a perfect square, we can see that its square root is not a fraction. The square roots of 2, 3, 5, 6, 7, 8, and 10 are not fractions; we call numbers **irrational** if they cannot be represented as fractions.

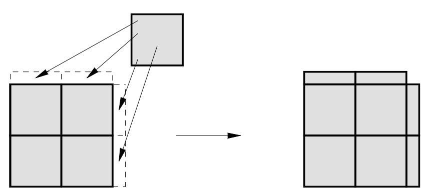
Square roots of positive integers are either integers or irrational numbers.

On a number line, irrational numbers are exact lengths just as integers are exact lengths. We can draw a line that is $\sqrt{2}$ units long just as accurately as we can draw a line that is 2 units long, but we can't write the value of $\sqrt{2}$ as an exact fraction or decimal.

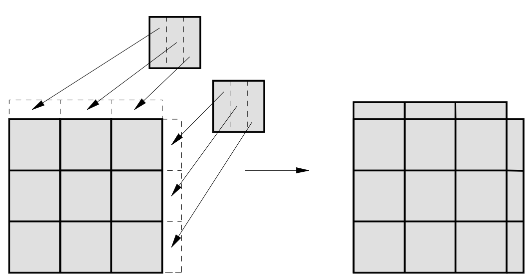


Here are simple estimates of other square roots:

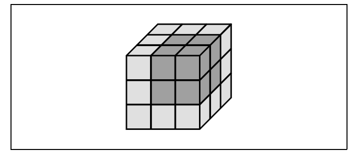
$\sqrt{5}$ is about $2\frac{1}{4}$



$\sqrt{11}$ is about $3\frac{1}{3}$



Summary



- **Perfect squares are integers that have exact integer square roots.**
- **All positive numbers have square roots.**
- **If an integer is not a perfect square, its square root is not a fraction—it is irrational.**

Exercises

Decide whether these square roots are integers or irrational numbers:

1. $\sqrt{17}$

2. $\sqrt{121}$

3. $\sqrt{82}$

4. $\sqrt{1000}$

5. $\sqrt{144}$

Using the method of this section, make a first estimate of these square roots. Square your answer to determine its accuracy:

6. $\sqrt{17}$

7. $\sqrt{8}$

8. $\sqrt{26}$

9. $\sqrt{38}$

10. $\sqrt{6}$

11. $\sqrt{12}$

Section 10
Properties of Roots

The Root of a Product or Fraction

When we examine the square root of (4 · 9), we can discover a useful property by getting the result in two different ways:

$$\begin{aligned} \sqrt{4 \cdot 9} &= \sqrt{(2 \cdot 2) \cdot (3 \cdot 3)} \\ &= \sqrt{(2 \cdot 3) \cdot (2 \cdot 3)} \\ &= \sqrt{(2 \cdot 3)^2} \\ &= (2 \cdot 3) = 6 \\ &= \sqrt{4} \cdot \sqrt{9} \end{aligned} \qquad \textit{or} \qquad \sqrt{4 \cdot 9} = \sqrt{36} = 6$$

To find the square root of a product, we find the square roots of both factors and then multiply them to get the result. *The square root of the product is the product of the square roots.* Both methods give the same result. We check the property by finding the square root of the product. Here is another example:

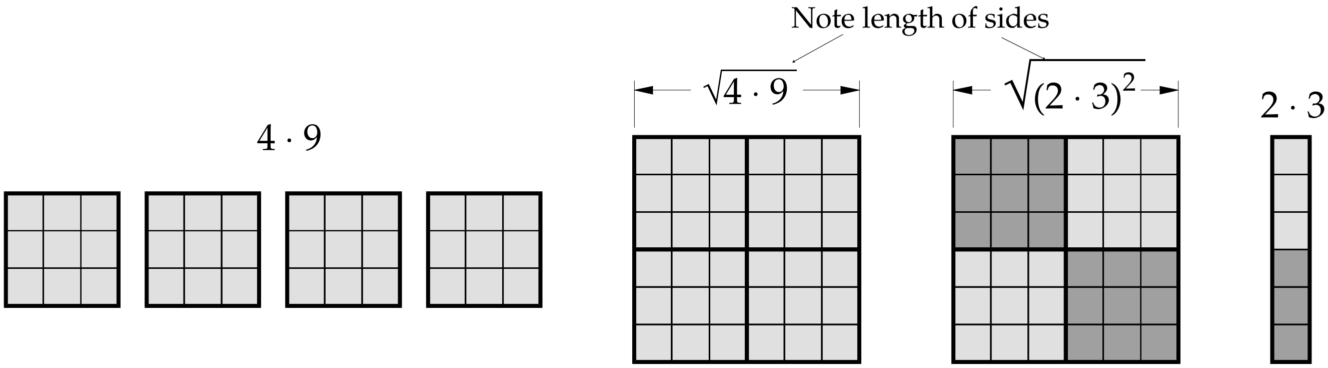
$$\begin{aligned} \sqrt{16 \cdot 25} &= \sqrt{16} \cdot \sqrt{25} \\ &= 4 \cdot 5 \\ &= 20 \end{aligned} \qquad \textit{or} \qquad \begin{aligned} \sqrt{16 \cdot 25} &= \sqrt{400} \\ &= \sqrt{20 \cdot 20} \\ &= 20 \end{aligned}$$

We can summarize this property as follows:

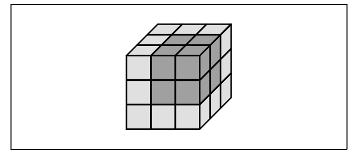
The square root of a product

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

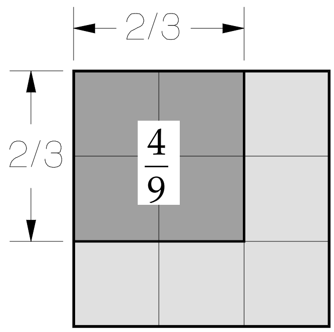
To demonstrate the first example, we begin with 4 groups of 9 (4 · 9) and then take the square root by arranging the groups of 9 in a 2 by 2 square. Each 9 is 3 by 3, so the resulting side (square root) is 2 groups of 3 or 2 · 3:



When we take the square *root* of a fraction, the same property applies:



$$\sqrt{\frac{4}{9}} = \sqrt{\frac{2}{3} \cdot \frac{2}{3}}$$
$$= \frac{2}{3}$$
$$= \frac{\sqrt{4}}{\sqrt{9}}$$



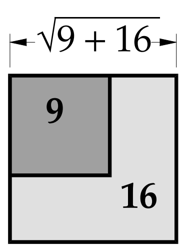
In summary:

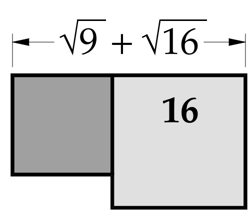
The square root of a fraction

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

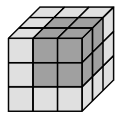
Common Errors

Be careful. Taking the square root is a factoring process. The square roots of products and quotients can easily be factored and simplified. The same is *not* true for the square roots of sums.

<table>
<tr><th>Error (False)</th><th colspan="2">Picture (Why it's not true)</th><th>Looks like: (True)</th></tr>
<tr><td>$\sqrt{9+16}$
does not equal
$\sqrt{9}+\sqrt{16}$</td><td>

= 5</td><td>

= 7</td><td>$\sqrt{9 \cdot 16} = \sqrt{9}\sqrt{16}$</td></tr>
</table>



Exercises

Use the properties from this section to simplify these expressions:

1. $\sqrt{100 \cdot 16}$

2. $\sqrt{4 \cdot 36}$

3. $\sqrt{9 \cdot 25}$

4. $\sqrt{16 \cdot 49}$

5. $\sqrt{\frac{36}{25}}$

6. $\sqrt{\frac{100}{9}}$

7. $\sqrt{\frac{49}{81}}$

8. $\sqrt{\frac{64}{25}}$

Factor the larger numbers given below into perfect square factors and then simplify:

9. $\sqrt{400}$

10. $\sqrt{2500}$

11. $\sqrt{4900}$

12. $\sqrt{8100}$

Show that these exercises have the same result if you simplify first and then take the square root or if you take the square root first and then simplify:

13. $\sqrt{\frac{25}{100}}$

14. $\sqrt{\frac{36}{4}}$

15. $\sqrt{\frac{144}{81}}$

16. $\sqrt{\frac{100}{16}}$