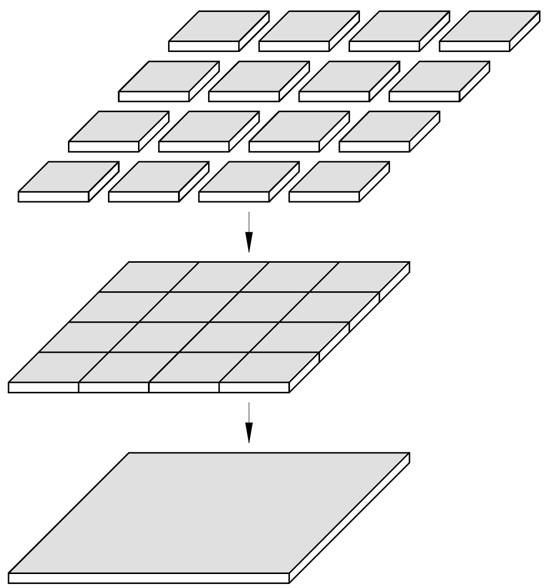
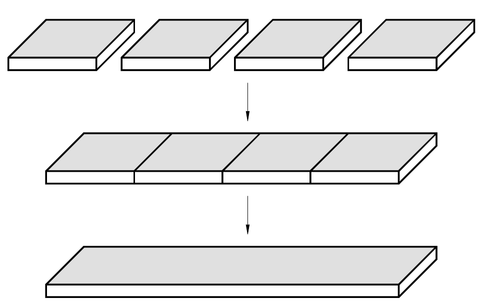
Chapter 9

Polynomials

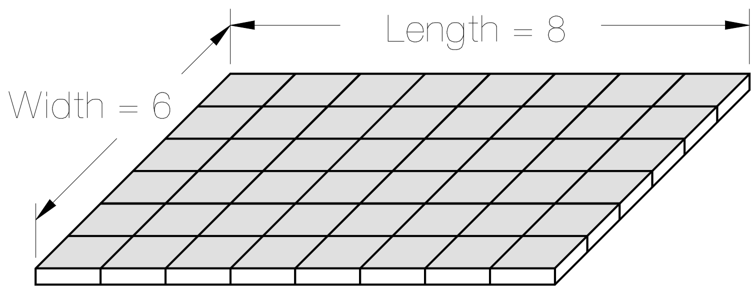




Section 1
Using Unknowns: 1, x, x^2

The Meaning of Multiplication

To do multiplication with unknowns we must remember how we do multiplication with positive and negative numbers. When we multiply numbers we are making rectangles, and the product (the answer to the multiplication problem) is the area of the rectangle:

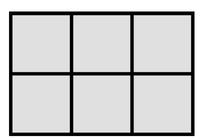


Length × Width = 8 · 6 = 48 units

To get the sign of the answer (product), we start with the colored side up, and then flip the chips once for each negative (-) sign in the problem.

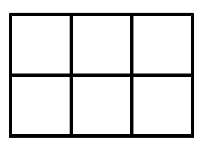
$(+3) \cdot (+2) = 6$

(No Flips)



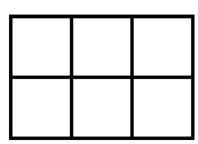
$(+3) \cdot ({}^{-}2) = {}^{-}6$

(One Flip)



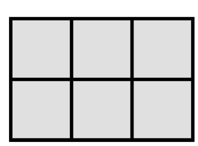
$({}^{-}3) \cdot (+2) = {}^{-}6$

(One Flip)

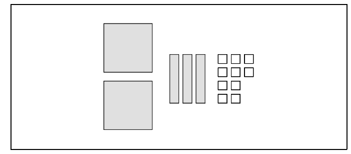


$({}^{-}3) \cdot ({}^{-}2) = +6$

(Two Flips)

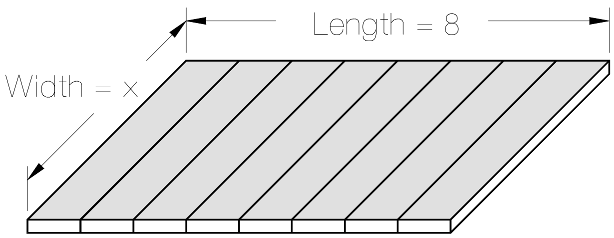


We should note here that the two numbers we are multiplying become the **dimensions** of the resulting rectangle. If just one of these dimensions is negative, then the rectangle ends up white side up (negative). If both dimensions are negative, then the product (the rectangle) will end up positive, with colored side up, just as it does when neither side is negative.



Multiplying With Unknowns

So far, we have been multiplying lengths and widths that are numbers. Can we make areas that have lengths or widths of x? Multiplication will still have the same meaning, but the sides may have dimensions involving x.



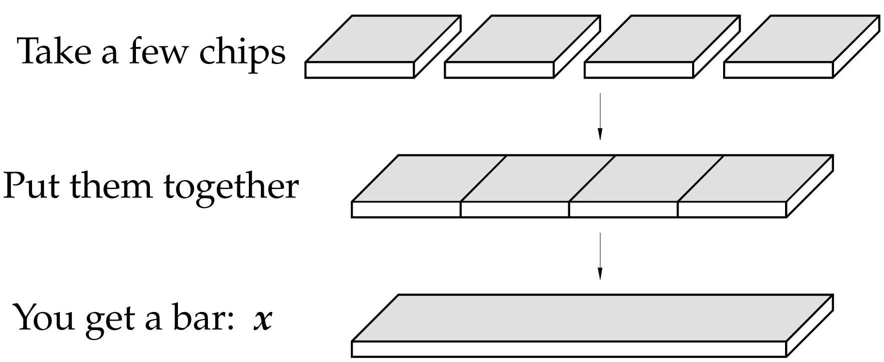
$$\text{Length times Width} = 8 \cdot x = 8x$$

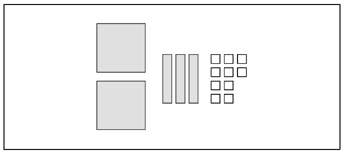
.
As we begin making rectangles using both numbers and unknowns, the process for determining the sign of the rectangle will remain the same. If just one dimension (side) of a rectangle is negative, the white side is up and the result is negative; if both or neither sides are negative, then the colored side is up and the answer is positive.

When we're using unknowns, we can still think of making rectangles, but now our rectangles will have bars as well as units.

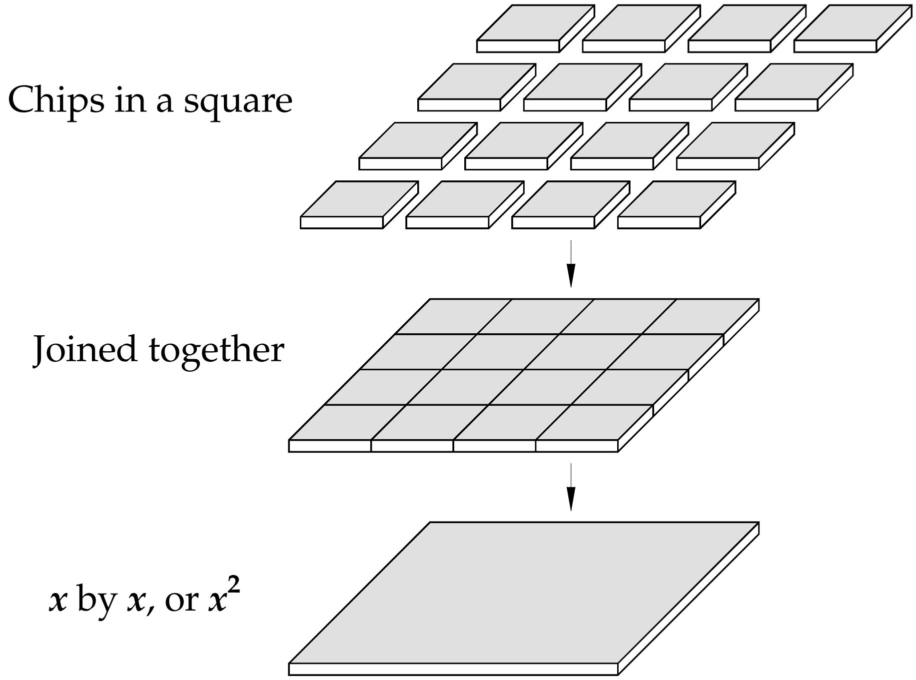
Definition of x and x^2

First, let's define x. Take a few chips and line them up in a row. Imagine that they are joined together in a bar, but then erase the boundaries so that all we see is a bar of unknown length. **This is x.**





Next, we will take a few chips and form a square. If we put the chips together and imagine that we cannot see exactly how many chips there are, we have built a square that is an unknown width and length. This is $x \cdot x$ or x^2:



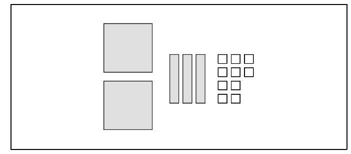
More on x and x^2

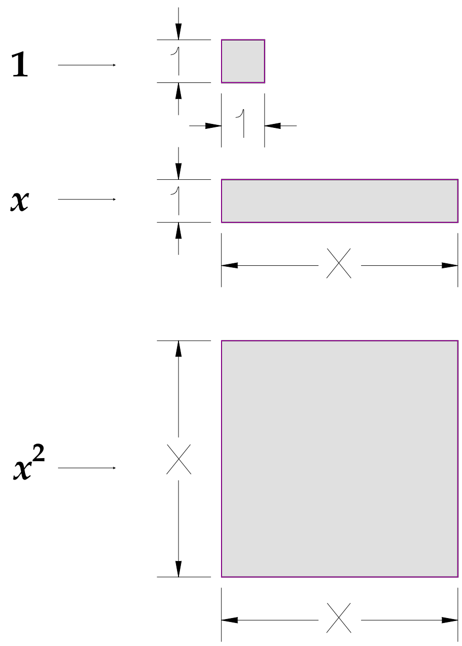
The following table shows the names and sizes of our new pieces. Note that the value of each piece is equal to its area.

Piece	Value	Length ×	Width =	Area
Chip or Little Square	**1** (Unit)	1	1	1
Bar	x	x	1	x
Big Square	x^2	x	x	x^2

Some of the pieces have sides in common. The unit and the x both have a side of one. The x and the x^2 both have a side of x. The x does not represent a specific number of chips; it represents *any* unknown number of chips. If you try to match up unit chips along the long (x) side of the x bar, you will find that neither 5 nor 6 nor any number of chips fits exactly.

Our set of chips now looks like this:

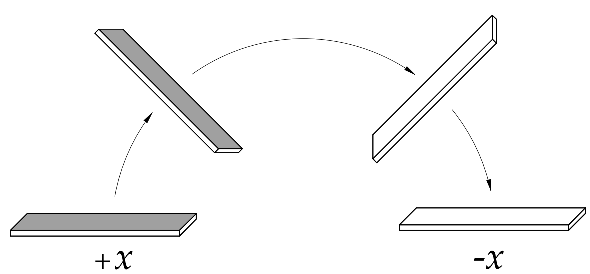




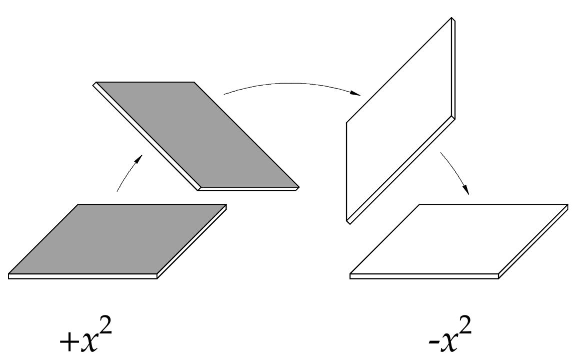
The Opposites of x and x^2

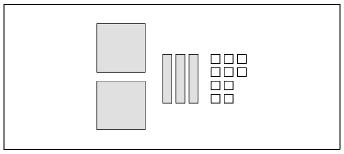
Unknowns can also have opposites. We have already been introduced to the idea that flipping a unit chip to the white side represents $^-1$; now we will put together the opposites of x and x^2.

First, let's review the idea of the opposite of the x bar. This will be written as **$-x$ a**nd will be called **negative x** or **the opposite of x**.



In the same way, we can construct **$-x^2$**:

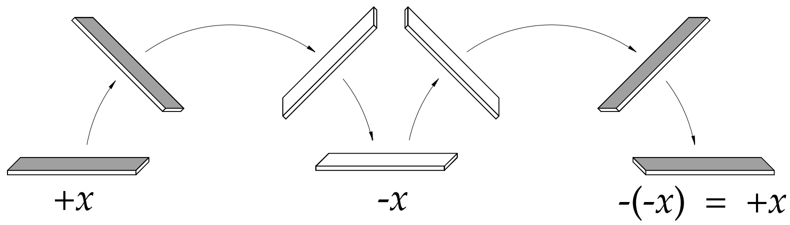




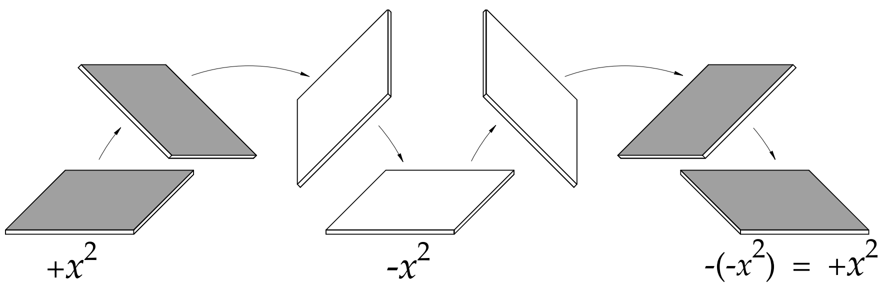
These new pieces behave in the same way as single chips—*Flipping an* x *or* x^2 *changes the sign.*

$$-(x) = -x$$
$$-(-x) = +x$$
$$-(x^2) = -x^2$$
$$-(-x^2) = +x^2$$

For x:



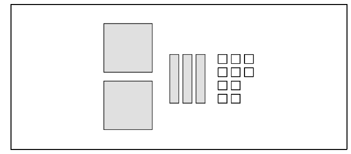
For x^2:

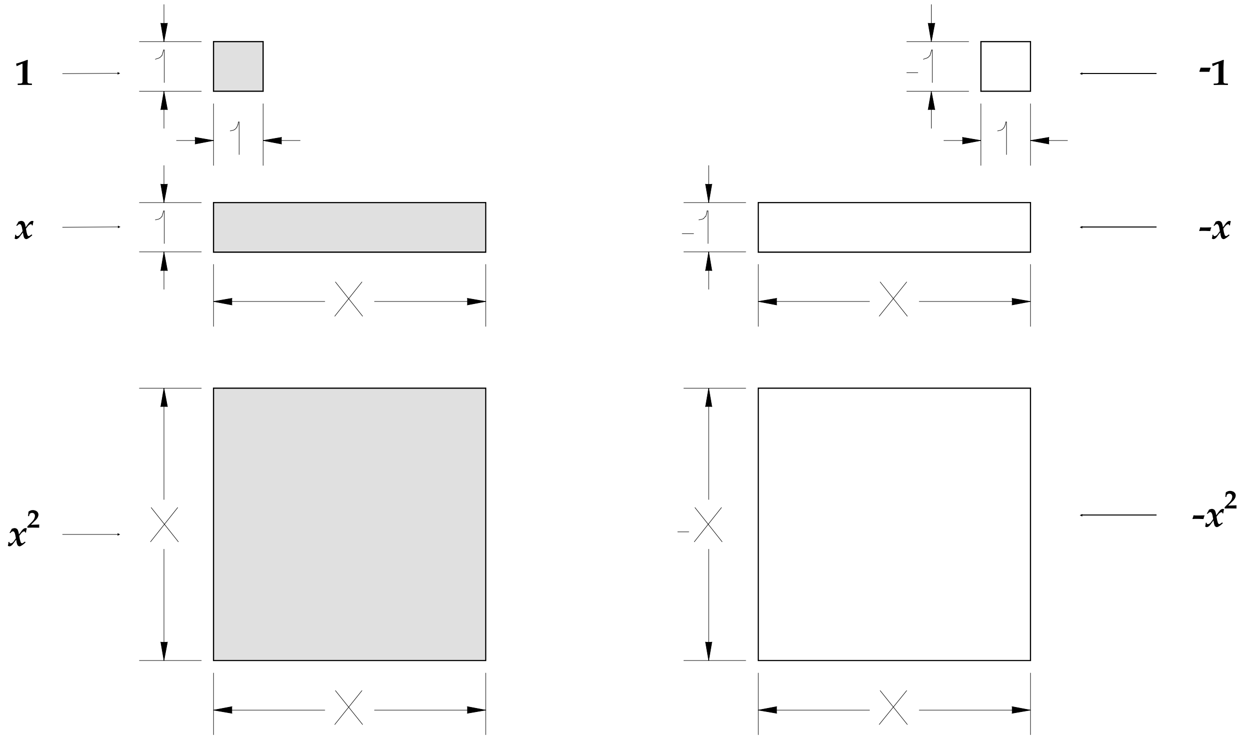


It is best to think of $-x$ and $-x^2$ as the opposites of x and x^2 ($-x$ *is not necessarily a negative number*!). Here is an expanded table of our pieces:

Piece	Value	Length ×	Width =	Area
Chip or Little Square	**1** (Unit)	1	1	1
Negative Chip	**-1** (Unit)	1	-1	-1
Bar	x	x	1	x
Opposite Bar	$-x$	x	-1	$-x$
Big Square	x^2	x	x	x^2
Opposite Big Square	$-x^2$	x	$-x$	$-x^2$

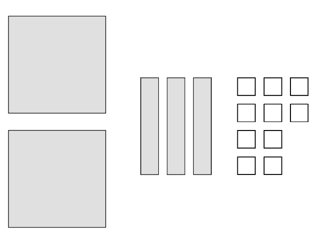
Finally, here are all of the new pieces:



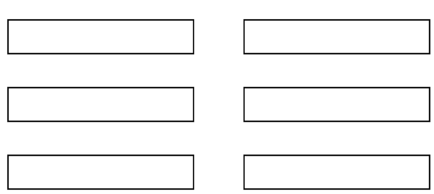


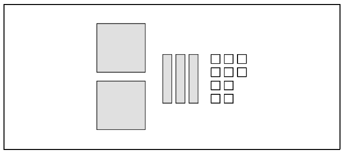
Polynomials

When we have an assortment of pieces such as units, x's, and x^2 chips, we call this a **polynomial**. A polynomial can have many types of pieces

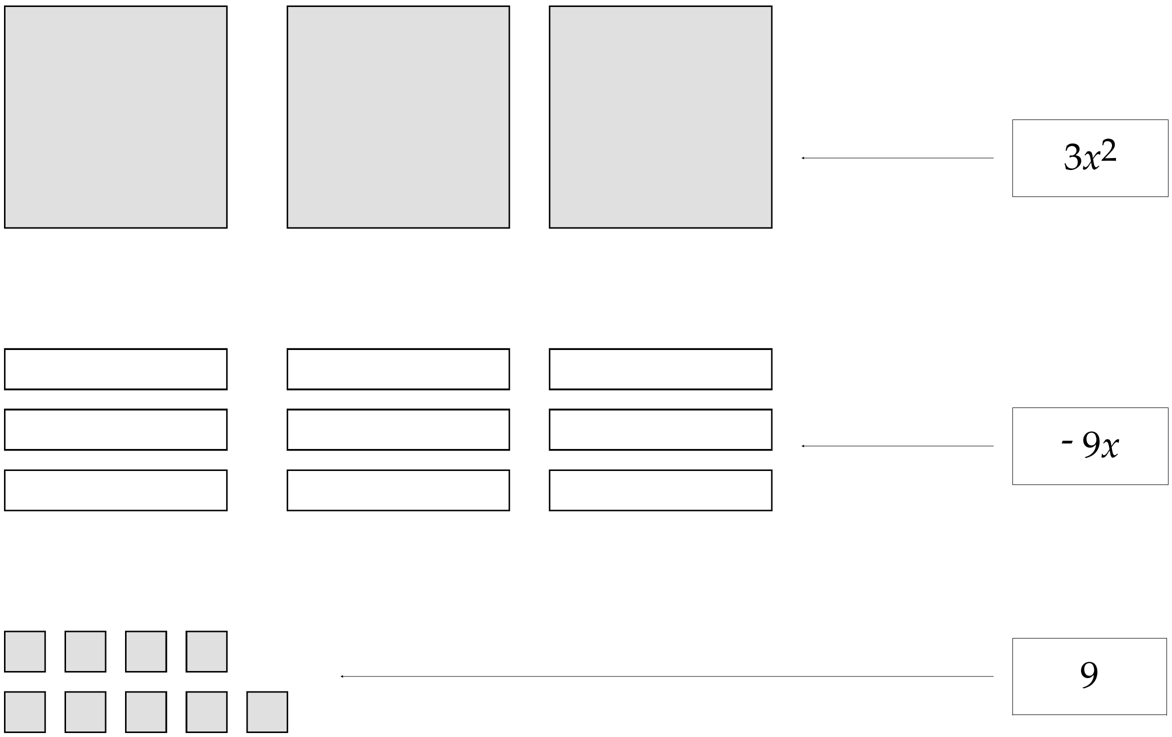


or just one kind of piece.





Each group of like shapes is called a **term**. When we have two bars we have two **x**'s, or **$2x$**. Three **x^2** pieces can be written as a **$3x^2$** term. Here are some examples of terms:



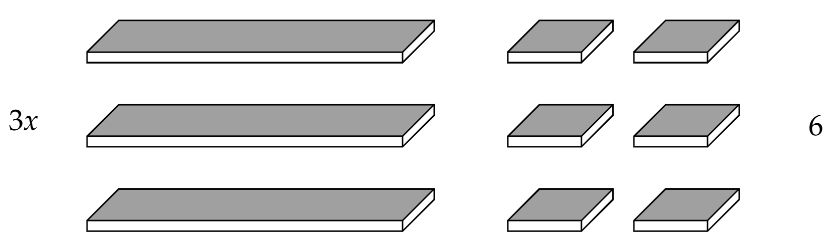
An expresion with *two* terms is called a **binomial**. An expression with *three* terms is called a **trinomial**.

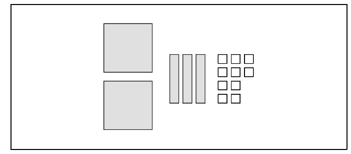
Exercises

Set up the following expressions with chips and identify individual terms:

Example: $3x + 6$

Solution: Terms are **$3x$** and **6.**





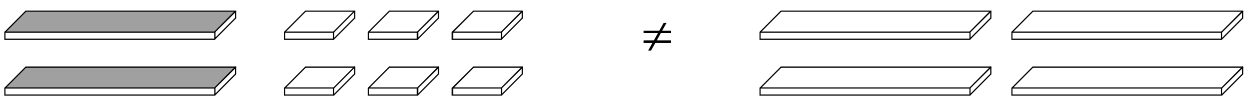
1. $7x$
2. $7x - 2$
3. $4x^2$
4. $3x^2 - 6$
5. $6 - 2x^2$
6. $2x^2 - 3x + 12$
7. $^{-}2x^2 - 5x - 1$
8. $^{-}0x^2$
9. $5 - 3x^2$
10. $2x + 3$
11. $x^2 - 5x + 6$
12. $2x - x^2 + 4$
13. $4x + 3x^2$
14. $2x^2 - 7$
15. $3x^2 - 5x + 2$

Section 2
Adding and Subtracting Polynomials

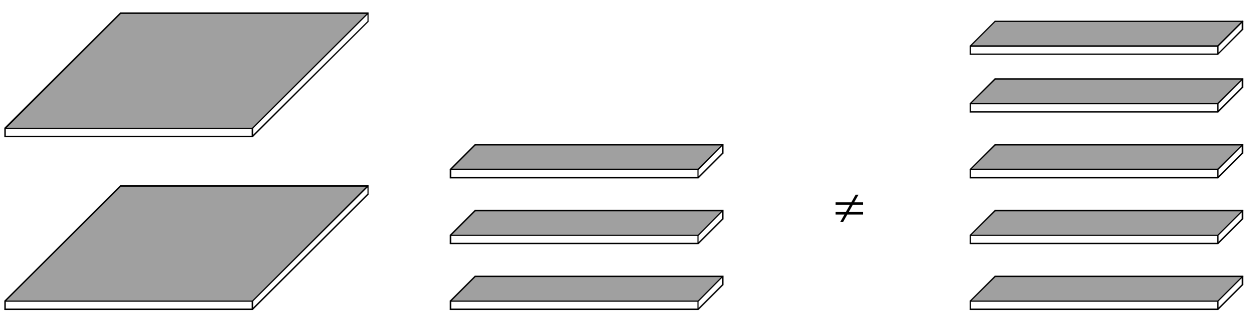
Combining Like Terms

If a polynomial has two separate kinds of pieces (bars and chips), they are not the same size and shape. This means that, in the symbolic language of algebra, we must also have ***two separate terms***; one with ***x***'s (bars) and the other with units (chips). These two terms are made up of ***different pieces*** and therefore they *cannot be combined*.

$2x - 6$ is not $^{-}4x$

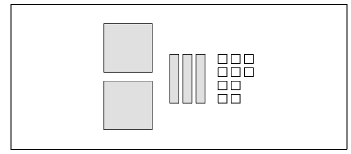


$2x^2 + 3x$ is not $5x$



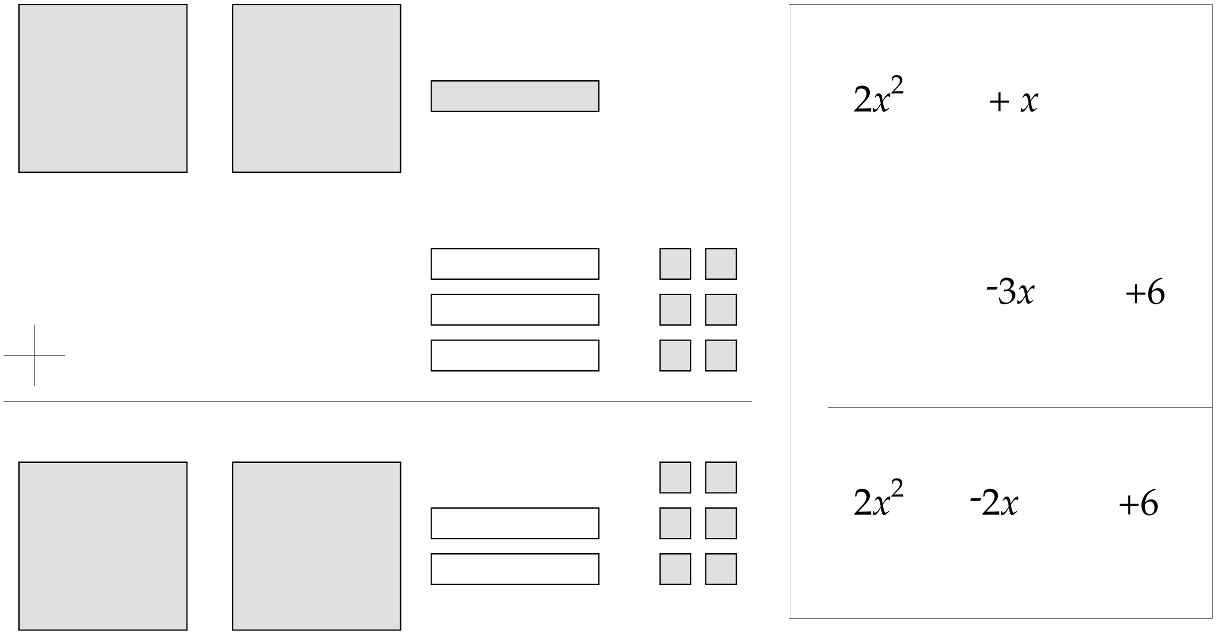
We cannot combine these terms because different shapes cannot be treated as if they are the same; they must be kept separate, ***x*'s** in one term and units in another.

If you use the chips and think of polynomials as groups of shapes, it will be easy to work with them without needing to memorize any rules. Just combine similar shapes.



Adding Polynomials

Adding two polynomials is done in the same way that we add units—we take similar pieces from each polynomial and combine like terms.



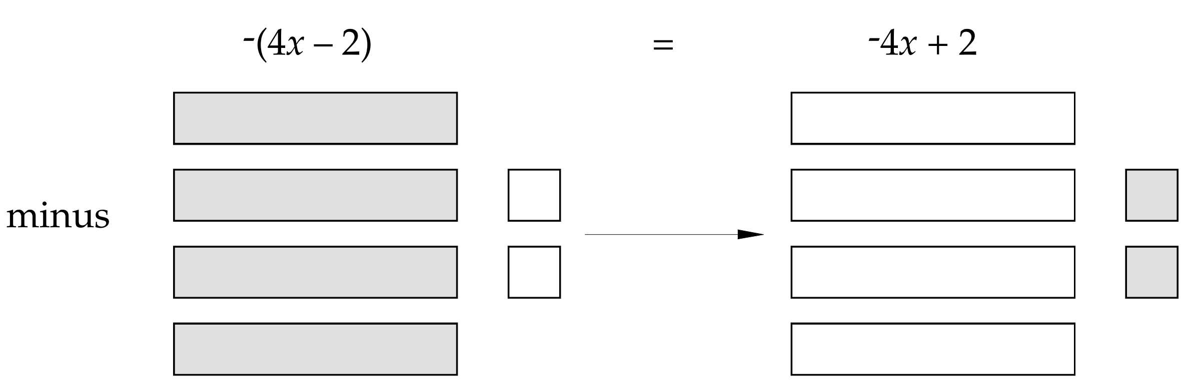
The algebra symbols show like *terms* being combined; the chips show like *pieces* being combined.

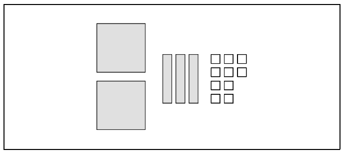
Subtracting Polynomials

To subtract a polynomial, we think of adding the opposite:

$$(3x - 5) - (4x - 2) = (3x - 5) + {}^{-}(4x - 2)$$

Just as with signed numbers, the negative sign means take the opposite, or *flip the chips*.



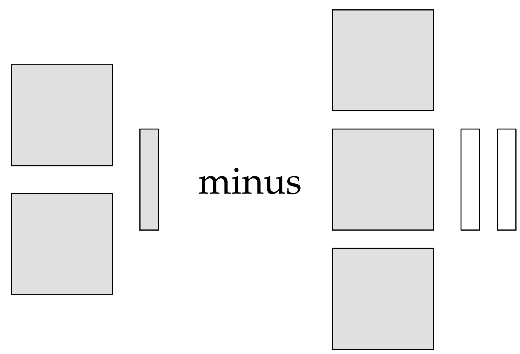


So for each subtraction, write the problem as an addition (flip the subtracted chips) and proceed as usual by combining like terms.

- **identify the two polynomials**
- **subtraction becomes adding the opposite**
- **find the opposite**
- **add**

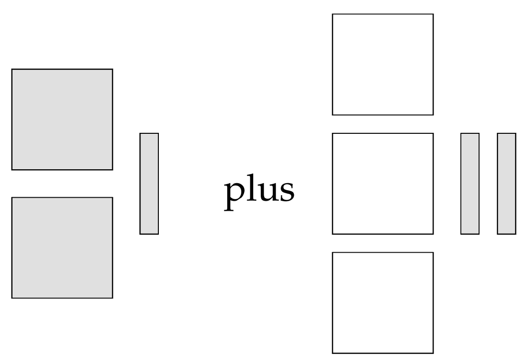
Here is an example of this process:

$(2x^2 + x) - (3x^2 - 2x)$

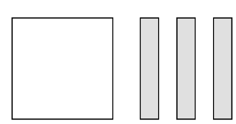


$(2x^2 + x) + {}^{-}(3x^2 - 2x)$

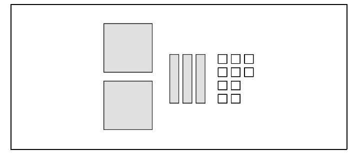
$(2x^2 + x) + {}^{-}3x^2 + 2x$



$-x^2 + 3x$



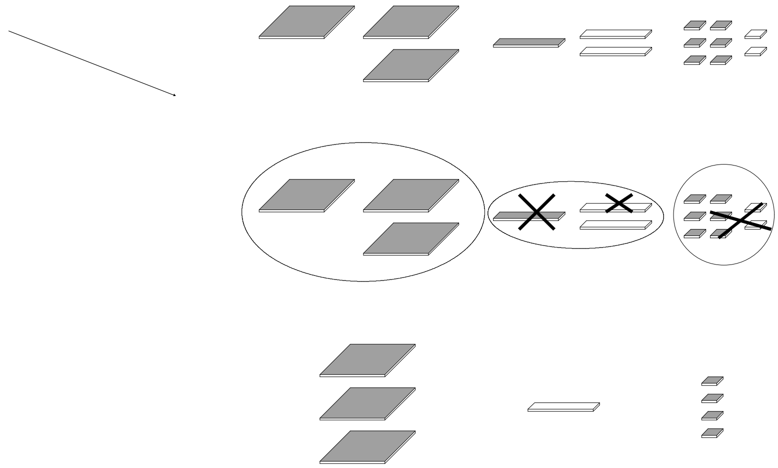
Exercises



Use your chips to set up the following problems. Combine similar shapes (terms).

Example: $x^2 + 2x^2 + x - 2x + 6 - 2$

Solution: $3x^2 - x + 4$



1. $3x - 2x$

2. $^{-}3x + 5 - 6x$

3. $5 - 4x^2 + x$

4. $2x^2 + x + x^2 + 3x$

5. $^{-}3x^2 - x^2$

6. $5x + 5 + (^{-}6x) + (^{-}3) + 2x^2$

7. $2x + x^2 - 5x - 3$

8. $^{-}3x^2 + 5 - x - 7$

9. $2x - 5 + x^2 + 3x$

10. $5 - (^{-}3x) + x + 7$

Use chips to complete these addition problems and write the algebra symbols as well:

11. $(3x - 2) + (5x - 6)$

12. $(x^2 + 3x + 3) + (2x^2 - x)$

13. $(^{-}2x^2 - x - 1) + (2x^2 + x - 1)$

14. $(2x - 5) + (x^2 + 3x + 2)$

15. $(x^2 - 3x + 1) + (x^2 - 7)$

16. $(^{-}5x + 3) + (2x^2 - 3x)$

17. $(x^2 + 3x - 2) + (3x^2 - x - 5)$

18. $(^{-}3x + 5) + (4x^2 - 5)$

Perform the following subtractions:

19. $(3x - 2) - (5x - 6)$

20. $(x^2 + 3x + 3) - (2x^2 - x)$

21. $(^{-}2x^2 - x - 1) - (2x^2 + x - 1)$

22. $(6x - 2) - (3 - 2x)$

23. $(x^2 + 3x - 1) - (x^2 - 2x + 5)$

24. $(2x + 3) - (x^2 - 5x)$

25. $(2x^2 - 5) - (x^2 + 5x - 6)$

26. $(3x^2 - 5x + 1) - (x^2 - 3x - 2)$

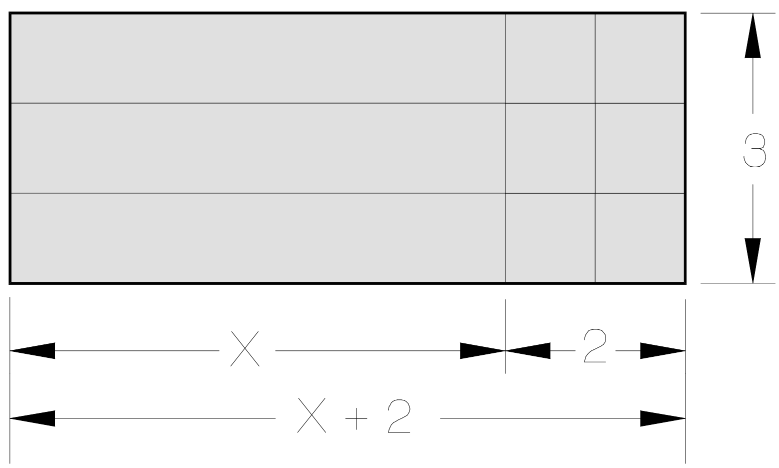
Section **3**
Multiplying Polynomials

Multiplying with One Unknown

If we have a product (multiplication) like

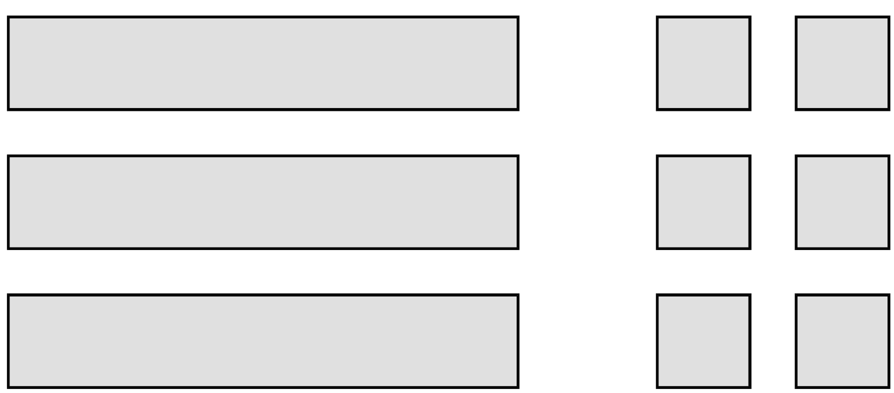
$$(3)\cdot(x + 2)$$

we make a rectangle with dimensions (sides) of **3** and ***x* + 2**, like this:

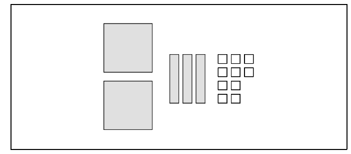


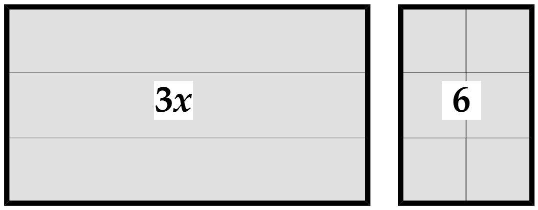
The product, or area, will just be the sum of the pieces we use, which is three bars (**3*x***) and six little squares (**6**):

$$3(x + 2) = 3x + 6$$

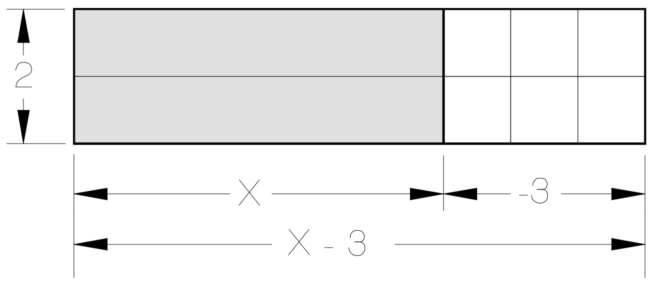


The product, or area, is **3*x* + 6.** We can think of this as being two smaller rectangles added together: one rectangle **3** by ***x***, and the other rectangle **3** by **2**. In this case both rectangles are positive.





If one piece of our product is negative, the product will look like this:



This time one of the smaller rectangles is positive ($2 \cdot x = 2x$) while the other smaller rectangle is negative ($2 \cdot {}^{-}3 = {}^{-}6$). Thus the product is.

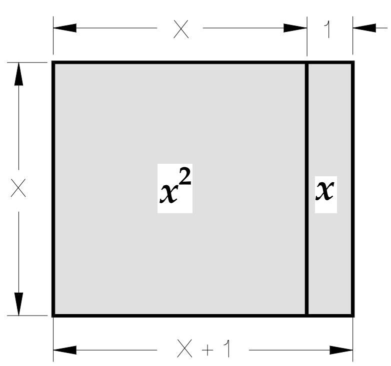
$$2(x - 3) = 2x - 6$$

Using Unknowns in Both Dimensions

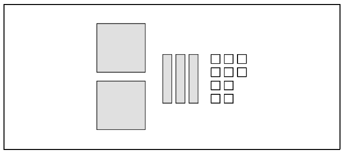
If we wish to find the product

$$x(x + 1)$$

we build a rectangle ***x*** wide and ***x* + 1** long. This is a rectangle made up of two smaller rectangles. One is ***x*** by ***x*** or x^2, the other is ***x*** by **1** or ***x*:**



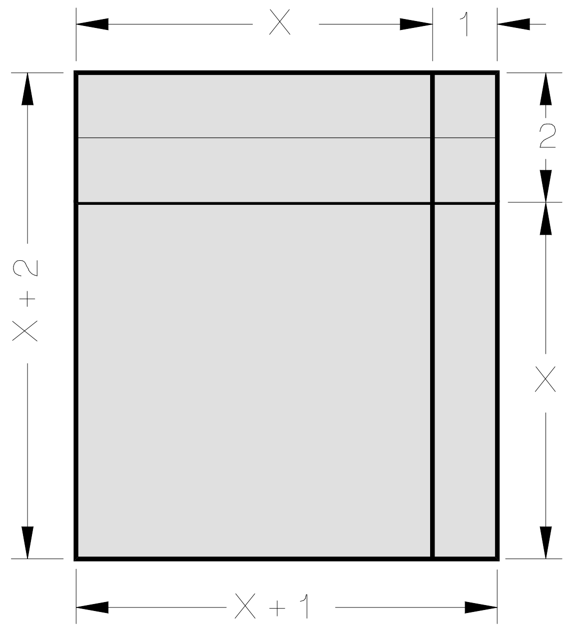
$$(x)(x + 1) = x^2 + x$$



If we wish to find the product

$$(x + 2)(x + 1)$$

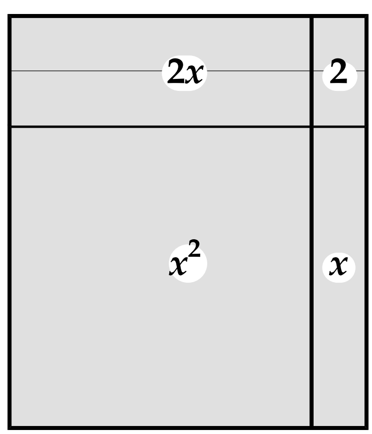
we must build a rectangle having each factor (**x + 2** and **x + 1**) as one dimension of length or width.



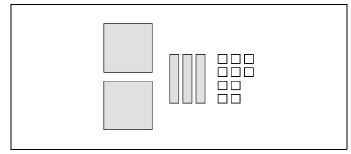
As can be seen in this illustration, the result is a large rectangle which can be subdivided into four smaller rectangular areas. In the upper right are two small chips, a rectangle 1 by 2 units. At the top left are two bars, defining a rectangle 2 by x. At the lower right is one bar, in a rectangle **1** by **x**. Finally, the large square forming the lower left corner is the rectangle with sides **x** by **x** and area **x^2**.

The area of the larger rectangle is the sum of these 4 areas:

$$(x + 2)(x + 1) = x^2 + x + 2x + 2$$



This time, two of the terms **are** made of the same size pieces; the x and the $2x$ are both made up of bars, so they can be combined giving



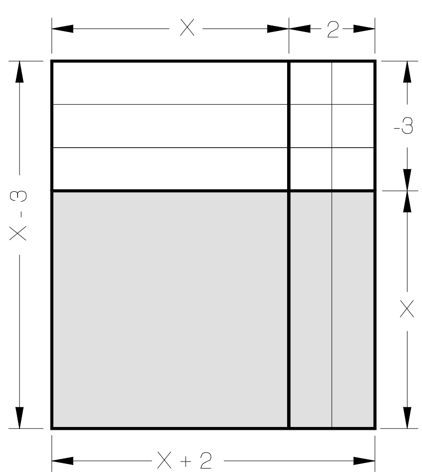
$$(x + 2)(x + 1) = x^2 + 3x + 2$$

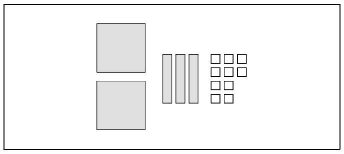
Each of the four smaller rectangles inside the large rectangle represents a piece of the product we are seeking. When using symbols, we can find these four smaller areas by using a technique called the **FOIL method**. This is defined as shown below:

F	**F**irst times **F**irst	$(x + 2)(x + 1)$	$x \cdot x = x^2$
O	**O**utside times **O**utside	$(x + 2)(x + 1)$	$x \cdot 1 = x$
I	**I**nside times **I**nside	$(x + 2)(x + 1)$	$2 \cdot x = 2x$
L	**L**ast times **L**ast	$(x + 2)(x + 1)$	$2 \cdot 1 = 2$

Each piece of the product, the x^2, the $1x$, the $2x$, and the 2, is one of the smaller rectangles in our figure.

If one or both sides of any of these smaller rectangles is negative, then we use our rules for signs to determine the sign of that particular rectangle. For example, let's illustrate the product of $(x - 3)(x + 2)$:



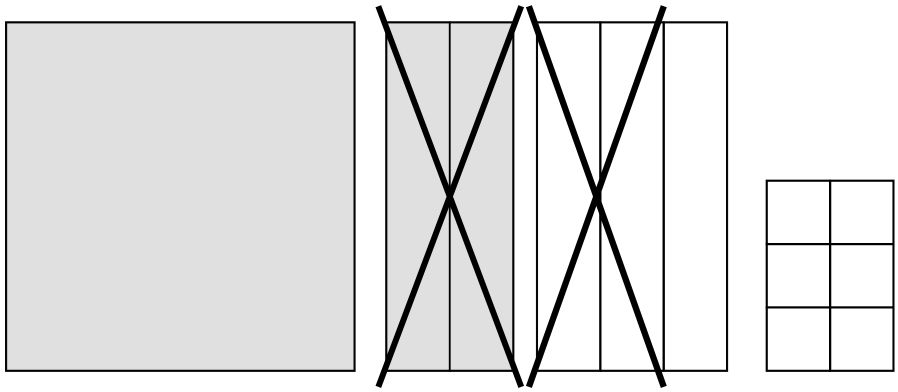


In this example, of the four rectangles within the figure, two are positive and two are negative (white side up).

$$(x-3)(x+2) = x^2 + 2x - 3x - 6$$

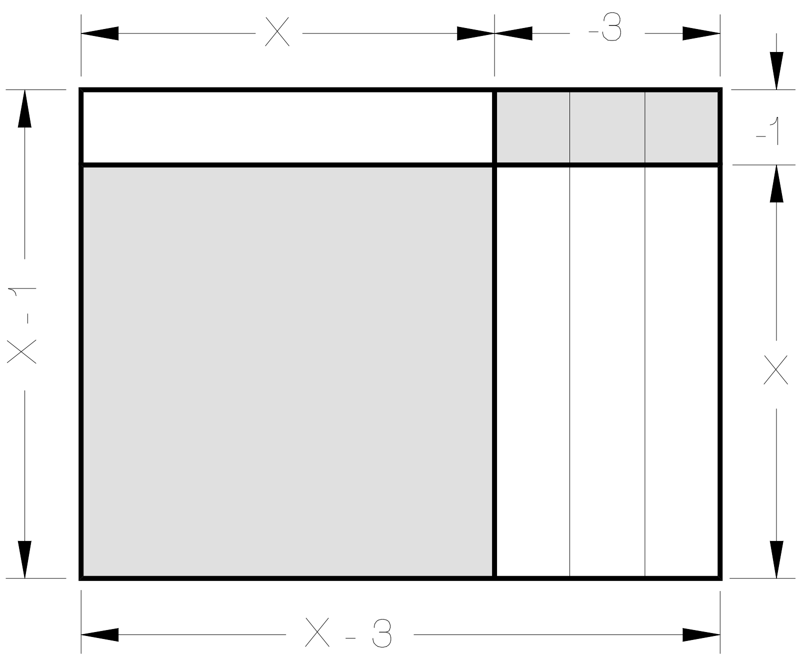
If we combine like terms, the *x*'s (positive bars) will be cancelled out by the negative bars leaving:

$$(x-3)(x+2) = x^2 - x - 6$$

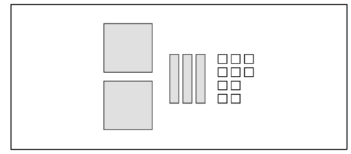


Again, each piece of the rectangle comes from one piece of the product when using the FOIL method. When some parts of our area are positive and other parts are negative, we can think of the product, or area of the figure, as being the difference of the areas, or the area left over when the white is taken away from the colored area.

Here's an example having two negative terms:

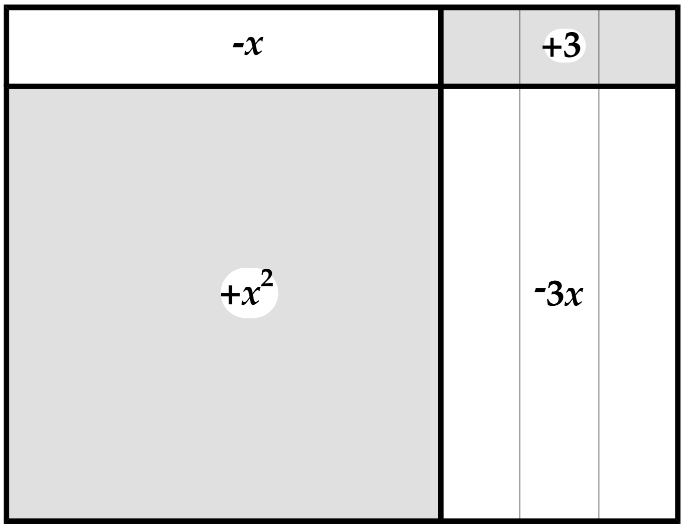


This is



$$(x - 1)(x - 3)$$

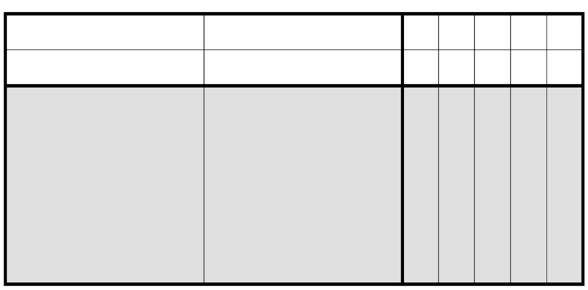
Can you explain why the ***x*-bars** are turned white side up, and the three chips are turned colored side up?



Here is one final example. Find the product

$$(x - 2)(2x + 5)$$

This requires forming a rectangle of dimensions **(*x* – 2)** by **(2*x* + 5):**



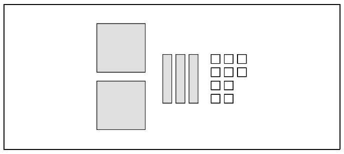
From this we see

$$(x - 2)(2x + 5) \; = \; 2x^2 + 5x - 4x - 10$$

Combining like terms gives

$$(x - 2)(2x + 5) \; = \; 2x^2 + x - 10$$

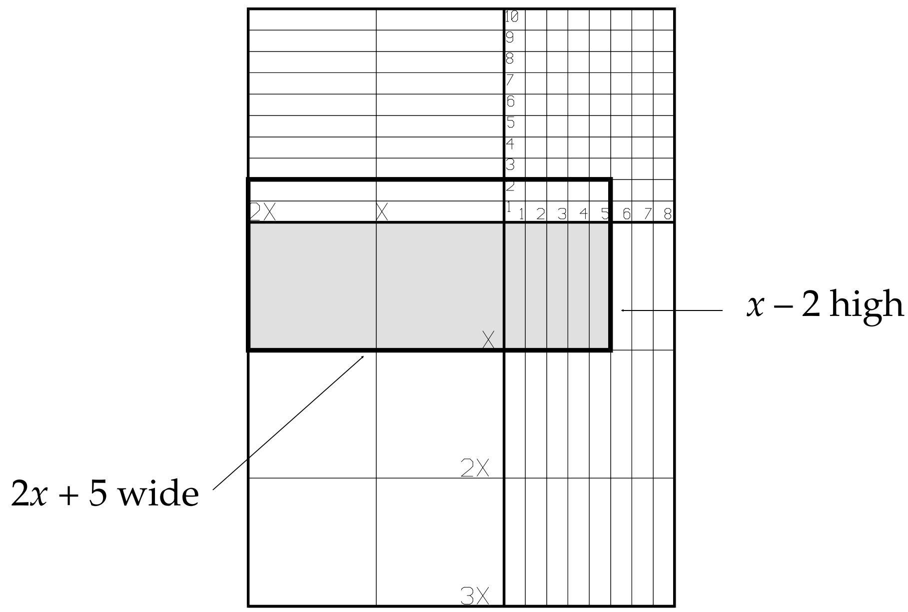
Do you understand where the signs on each term came from?



A plastic grid is included with this book. You can use the grid instead of the chips to plot multiplication of polynomials. Use a water-based marker to outline the rectangles or chips you want to use. You can also mark areas as positive or negative.

The grid is ruled in units of x's and **ones.** The darker lines across the grid (one horizontal and one vertical) are the lines which separate the four smaller rectangles within the larger figure. Remember that each of these smaller rectangles has its own sign and represents one term of the product.

Here is the example above, using the grid:

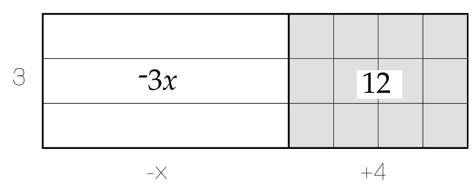


Exercises

Look at the example products and then use your chips to do the following multiplications.

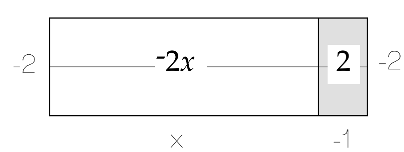
Example: $3(-x + 4)$

Solution: $^-3x + 12$



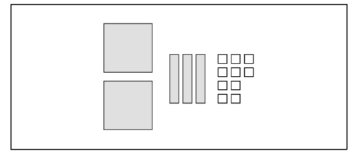
Example: $^-2(x - 1)$

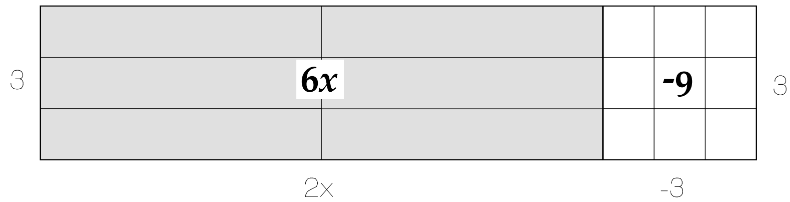
Solution: $^-2x + 2$



Example: $3(2x - 3)$

Solution: $6x - 9$





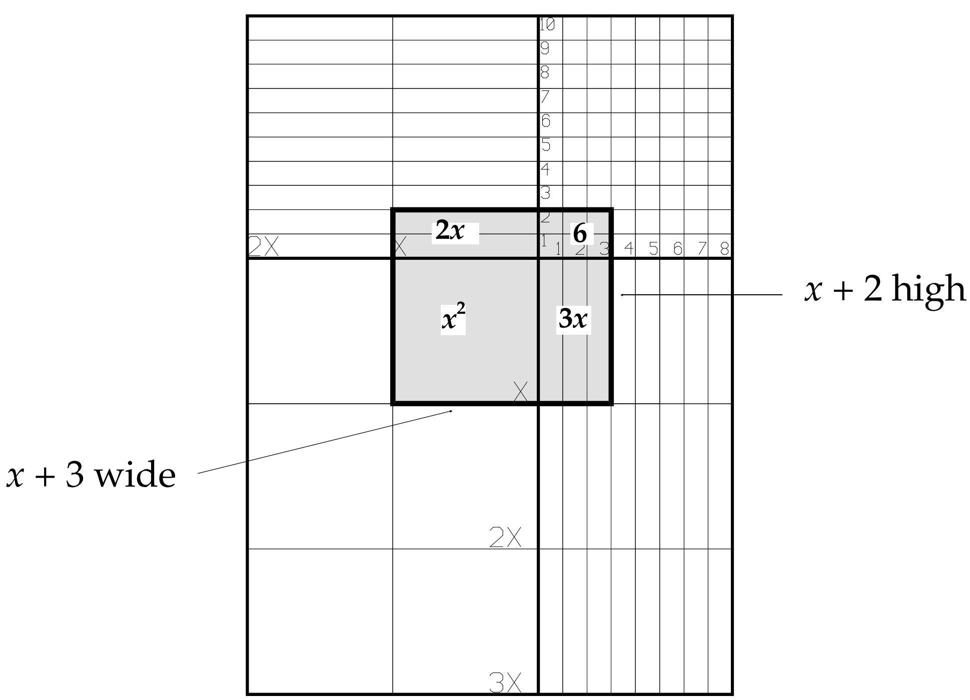
Multiply:

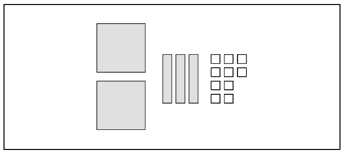
1. $2(x - 4)$
2. $3(2x + 1)$
3. $3(-x + 1)$
4. $^{-}2(x - 3)$
5. $^{-}2(-x - 1)$
6. $2(3x - 1)$
7. $^{-}3(-x + 3)$
8. $^{-}2(2x - 5)$

Try these problems using chips or the grid:

Example: $(x + 3)(x + 2)$

Solution: $x^2 + 3x + 2x + 6 = x^2 + 5x + 6$





9. $5(2x - 3)$

10. $-3(x - 5)$

11. $2(-2x + 1)$

12. $-5(-2x - 3)$

13. $-5(3x - 2)$

14. $2(5 - 3x)$

15. $-4(3 - x)$

16. $(x + 4)(x + 1)$

17. $(x - 3)(x + 4)$

18. $(x - 1)(x - 5)$

19. $(x + 5)(x - 3)$

20. $x(x - 6)$

21. $(2x + 1)(x - 4)$

22. $-x(3x - 2)$

23. $(2x - 3)(x - 2)$

24. $(x + 3)(x - 5)$

25. $(x - 2)(x - 6)$

26. $(x + 3)(2x - 1)$

27. $(2x - 3)(x + 2)$

28. $-x(3 - 2x)$

29. $(x - 2)(2x + 1)$

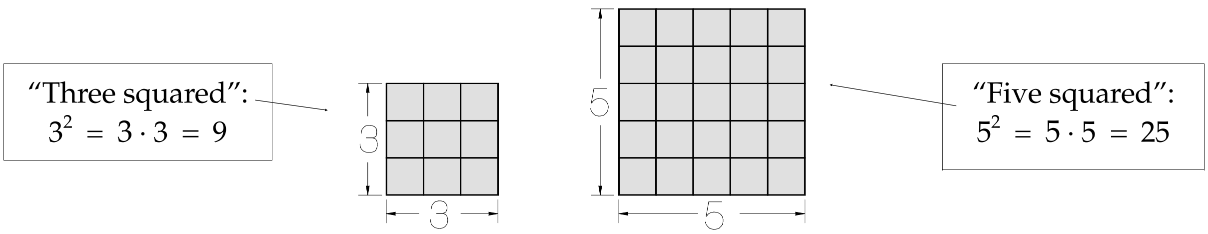
30. $(2x - 1)(2x + 3)$

Section 4
Special Products

Perfect squares

Two types of polynomials are considered special. These special polynomials are called **perfect squares** and **the difference of two perfect squares**.

Any time we make a rectangle where the length and width are the same, we get a square. This is obviously true if the sides of the square are just numbers.



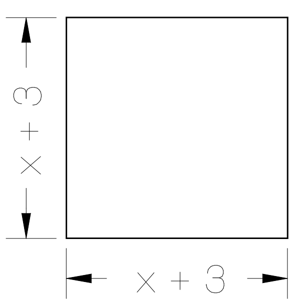
In fact, numbers which can be made into a square in this way are called **perfect square** numbers. The first six perfect square numbers are

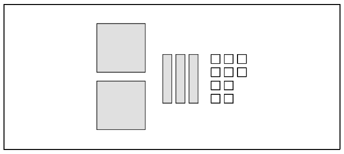
$$1,\ 4,\ 9,\ 16,\ 25,\ 36$$

Can you name the next six perfect square numbers in the series? If you take a number of chips from this list you will be able to arrange them into a perfect square, just as the name suggests.

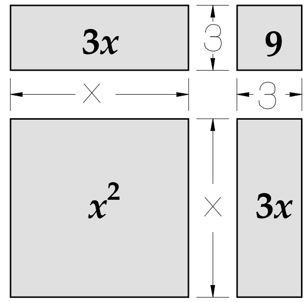
In the same way, if we multiply a polynomial having two terms (a **binomial**) times itself, we get a rectangle which has the same length and width: a *perfect square*.

Using chips, if we multiply the quantity $(x + 3)$ times itself, giving $(x + 3)^2$ or x plus three, quantity squared, we will be making a rectangle having the same length and width: a square.





Breaking this square into its four smaller areas we find that two of them, the units and the x^2's, are smaller squares.



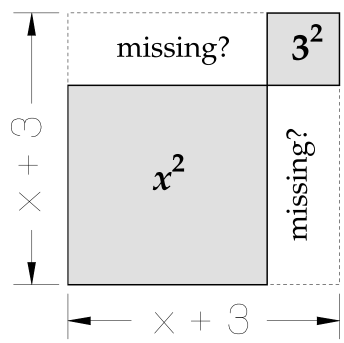
The remaining rectangles, the x's, are both the product of one side of each of these smaller squares ($x \cdot 3$).

Although this example seems obvious when working with chips, it is important to remember when using the symbolic language of algebra, that

$(x + 3)^2$ ***is not*** $x^2 + 3^2$

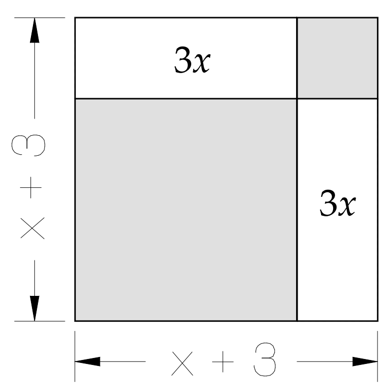
With the chips, we can see that these two expressions cannot be equal:

$(x + 3)^2$ is not just x^2 plus 3^2



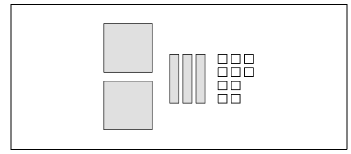
We must include the two rectangles which each have area $3x$. Remember the FOIL method:

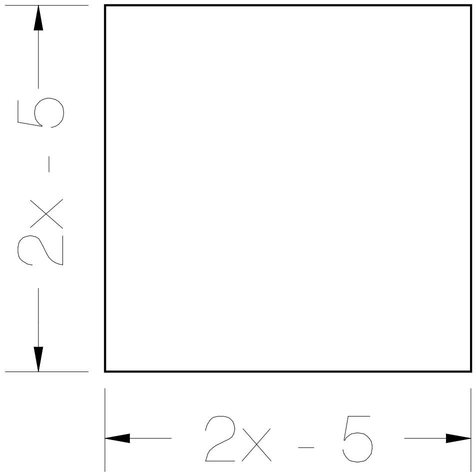
$$
\begin{aligned}
(x+3)^2 &= (x+3)(x+3) \\
&= x^2 + \mathbf{3x} + \mathbf{3x} + 9 \\
&= \mathbf{x^2 + 6x + 9}
\end{aligned}
$$



When we include all four of the areas shown we get the correct result.

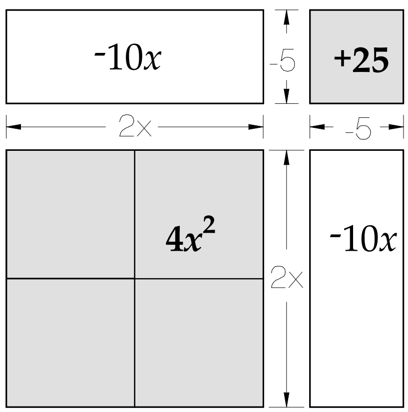
Now consider a second example of multiplying a binomial by itself to form a perfect square: $(2x - 5)^2$, or two x minus 5, quantity squared:





$$(2x - 5)^2 = (2x - 5)(2x - 5)$$

Filling in the four smaller rectangles within this diagram we again find two squares and two rectangles.



The two squares are both positive (colored side up),

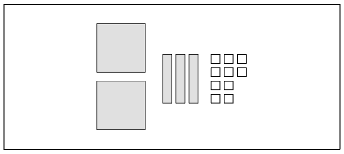
$$(2x)(2x) = 4x^2$$

$$(^-5)(^-5) = {}^+25$$

but this time the x-bars are negative (white side up) since

$$(^-5)(2x) = {}^-10x$$

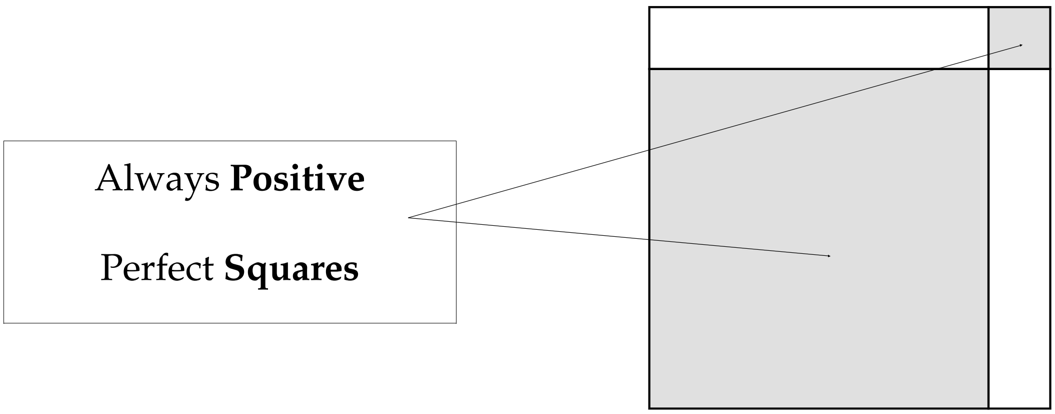
$$(2x)(^-5) = {}^-10x$$



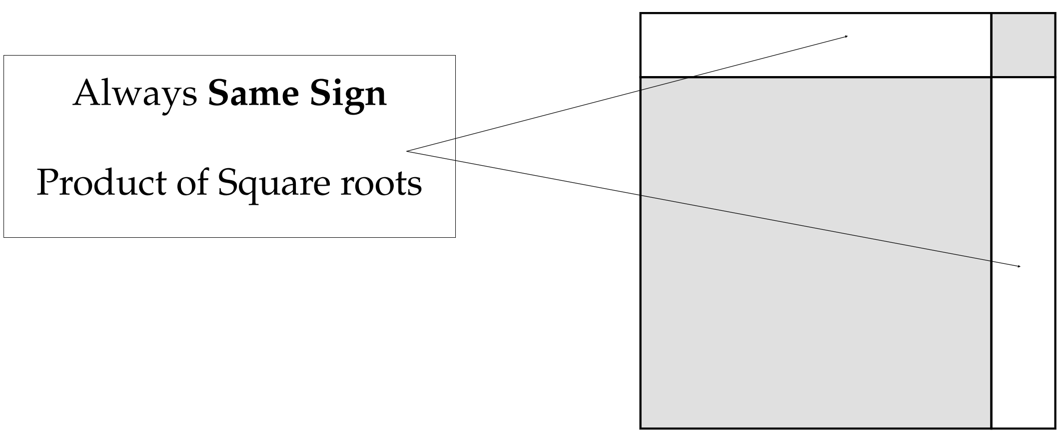
Again, using the FOIL method of symbol multiplication we get all four of the included areas and the correct result:

$$\begin{aligned}(2x-5)^2 &= (2x-5)(2x-5)\\ &= (2x)(2x)+(2x)(-5)+(-5)(2x)+(-5)(-5)\\ &= 4x^2-10x-10x+25\\ &= \mathbf{4x^2-20x+25}\end{aligned}$$

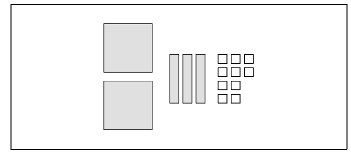
The smaller squares (x^2 pieces and units) within a perfect square are always positive in value (colored side up). This is because we get both of them by multiplying a number times itself, which always gives a positive result.



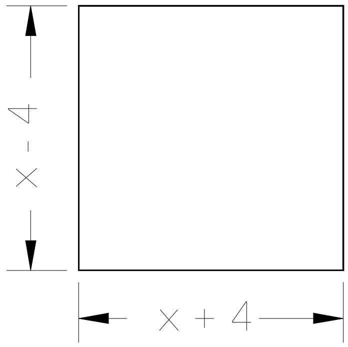
As the two examples demonstrate, the x-bars in a perfect square trinomial can sometimes be positive and sometimes be negative,. But in any one perfect square, all of the x-bars must be the *same sign*, either all plus or all minus. The number of x-bars will always equal the product of the square roots of the units square and the x^2 square, times two (because there are two groups of x-bars).



The Difference of Two Perfect Squares

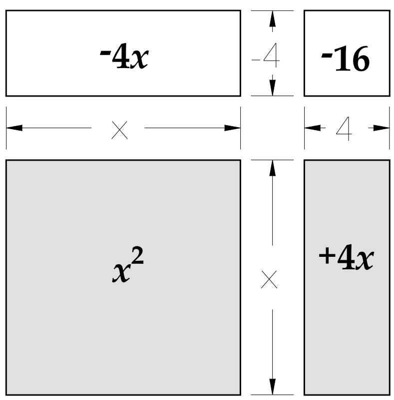


The two binomials $(x + 4)$ and $(x - 4)$ look very similar to each other; their only difference is the sign on the second term. If we multiply these two binomials together we get an interesting result.

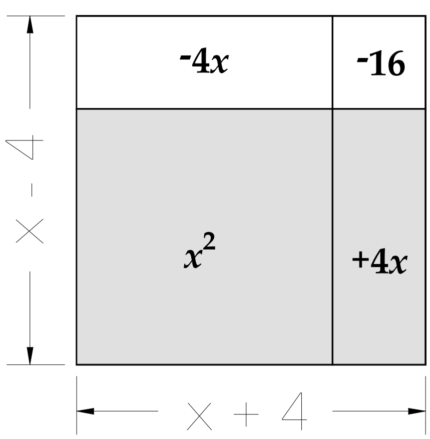


$$(x + 4)(x - 4)$$

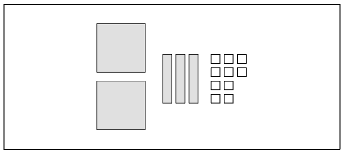
We again have a figure which appears square, but this time the two sides will have some pieces of different colors.



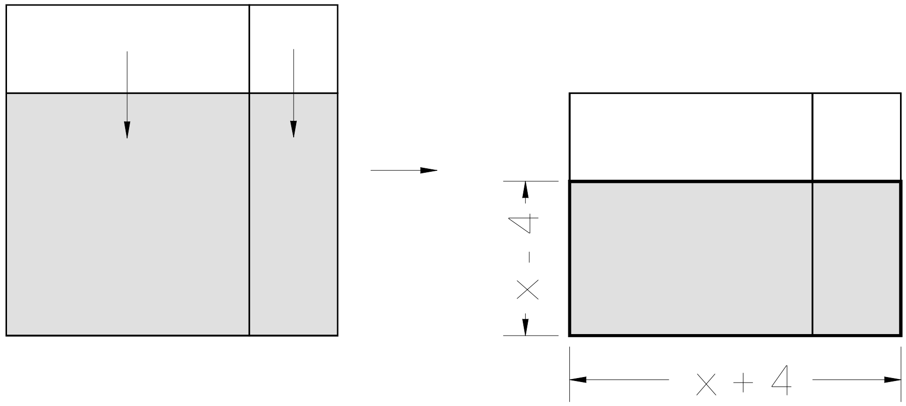
Shown as one rectangle, our example now looks like:



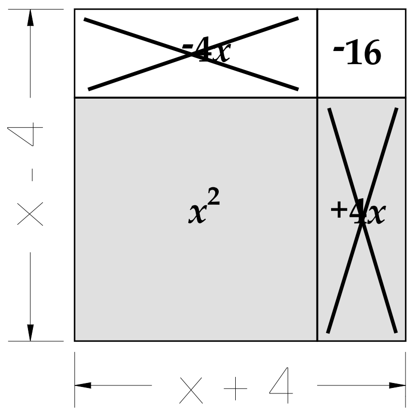
$$(x + 4)(x + 4) = x^2 - 4x + 4x - 16$$



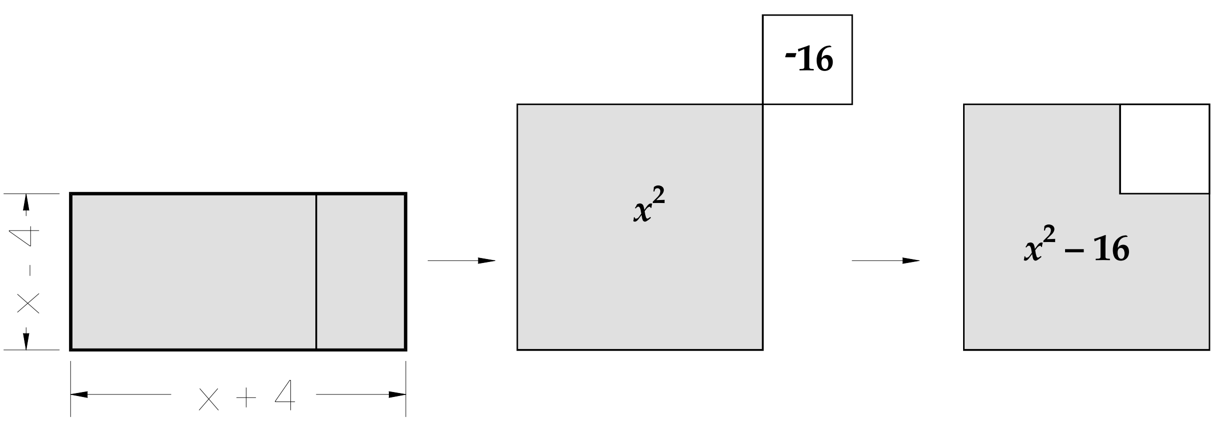
If we overlay the pieces and subtract the negative areas from the positive areas, we see that the resulting area is not really a square, but a rectangle having dimensions $(x + 4)$ and $(x - 4)$.

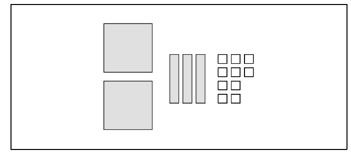


If we let the positive and negative pieces cancel in a different way we get an equivalent and surprising result.



The ^+4x and the ^-4x cancel each other out, leaving only x^2 and $^-16$. So we see





Now we can see why such products, products of binomials which differ only in the sign on the second term, are called *the difference of two perfect squares*; they are one square subtracted from another. We can use the FOIL method of multiplying symbols to verify this result.

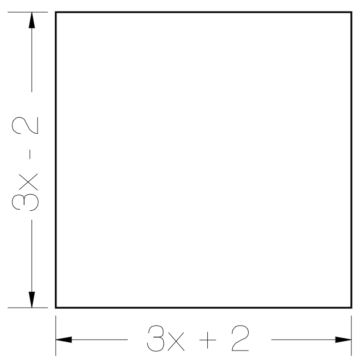
$$\begin{aligned}(x+4)(x-4) &= (x)(x)+(x)(-4)+(4)(x)+(4)(-4)\\ &= x^2-4x+4x-16\\ &= x^2+0x-16\\ &= x^2-16\end{aligned}$$

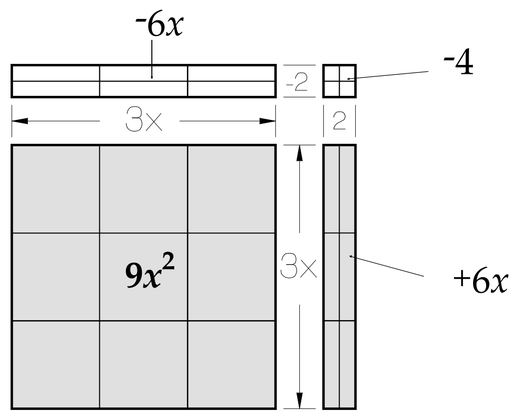
In the result, each of the two terms is a perfect square and the negative sign means to take the *difference*, or subtract, one perfect square from the other.

Here is a second example of a product which will generate the difference of two perfect squares:

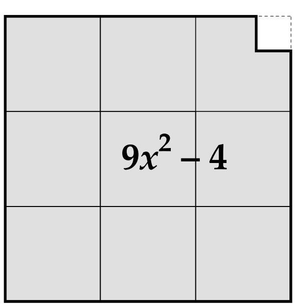
$$(3x+2)(3x-2)$$

Again the two binomials in the product differ only in the sign on the second term.



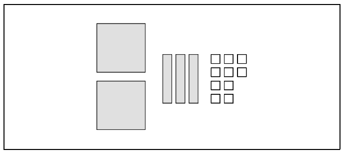


We can use both a diagram and the FOIL method to obtain the results of the product.



$$\begin{aligned}(3x+2)(3x-2) &= (3x)(3x)+(3x)(-2)+(2)(3x)+(2)(-2)\\ &= 9x^2-6x+6x-4\\ &= 9x^2-4\end{aligned}$$

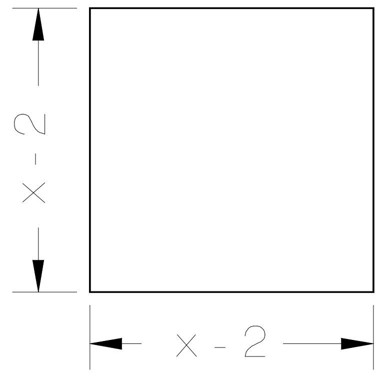
In both results we are subtracting one perfect square from another; *the difference of two perfect squares*.



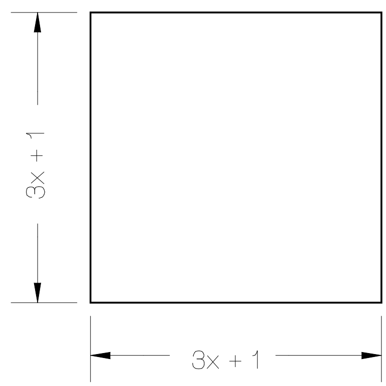
Exercises

Fill in the four smaller rectangles included in these perfect squares and then use the FOIL method to get the same results using algebraic symbols.

1. $(x-2)^2 =$



2. $(3x+1)^2 =$



Multiply out the following perfect squares; verify using a sketch.

3. $(7)^2$

4. $(2x-7)^2$

5. $(3x)^2$

6. $(3x+2)^2$

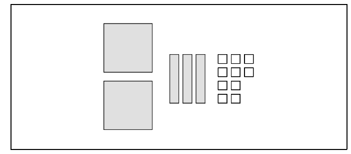
7. $(x+4)^2$

8. $(2x-1)^2$

9. $(x-9)^2$

10. $(5x+3)^2$

Choose only the products which will generate the difference of two perfect squares, and work out only those products both in a diagram and using the FOIL method to verify your results.



11. $(2x + 5)(2x - 3)$

12. $(x + 2)(x - 2)$

13. $(3x + 4)(3x + 4)$

14. $(3x + 5)(5x - 3)$

15. $(3x + 5)(3x - 5)$

16. $(x - 7)(x + 7)$

17. $(2x - 1)(2x + 3)$

18. $(2x - 1)(x + 1)$

19. $(2x + 1)(2x - 1)$

20. $(3x - 2)(2x + 3)$

21. $(5x - 6)(5x + 6)$

22. $(7x - 1)(7x + 1)$

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