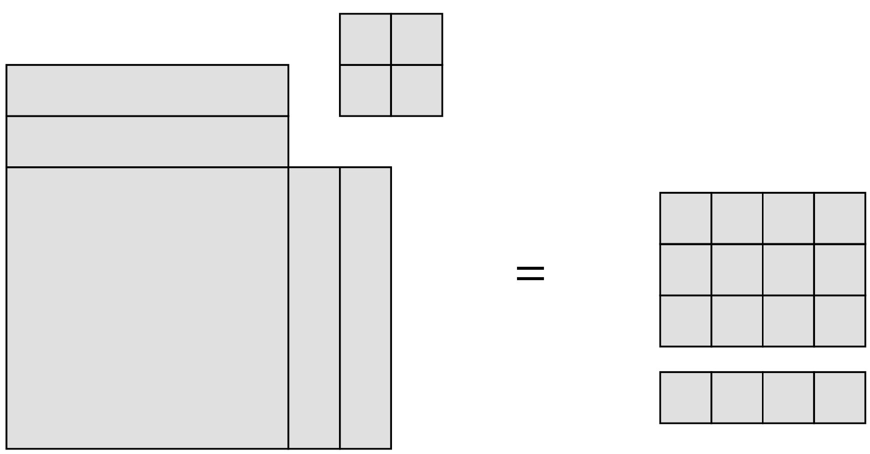
Chapter 11
Quadratic Equations



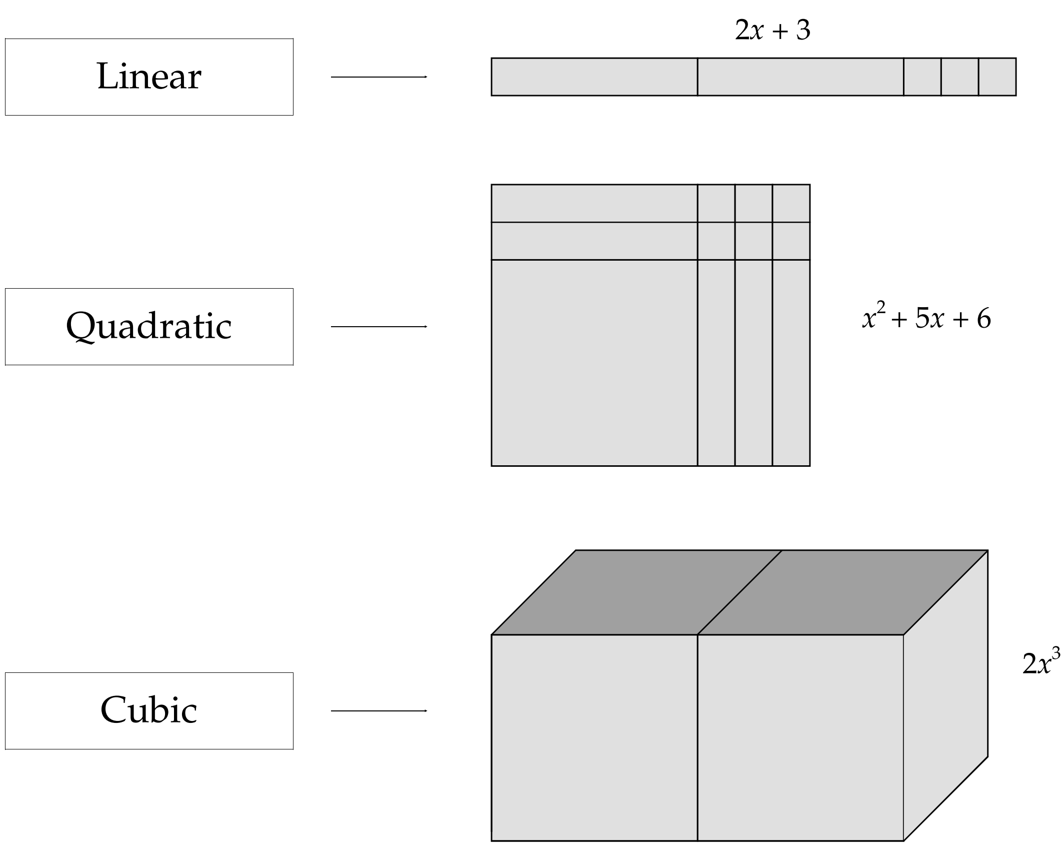
Section 1
Introduction

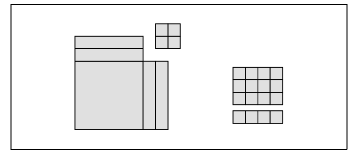
Polynomials and Quadratic Expressions

In the last two chapters we have worked with polynomials. We have learned about terms, combining polynomials, and factoring. The terms in any polynomial expression can include units (small squares), x's (bars), x^2 pieces (large squares), x^3 pieces (cubes, which we don't have in our kit), and higher powers of x (like x^4, x^5, …).

When a polynomial expression contains x^2 pieces (large squares), but does *not* contain any higher powers of x, we call the expression a **quadratic expression**. Expressions having combinations of squares, bars and chips are called *quadratic expressions* is because they can be represented using flat *four*-sided figures—squares or rectangles. The prefix "Quadri" means having *four* parts, in this case four sides.

In a similar way, expressions having only x's and units are called **linear**, because they can be represented using *lines*; and expressions having x^3's as their highest term (biggest piece) are called **cubic**, because they can be represented using *cubes*.





Just as a quadratic expression is an expression having x^2-squares as its biggest piece (highest term), so **quadratic equations** are *equations* having x^2-squares as their biggest piece, along with some combination of x-bars and unit-chips. Some examples of quadratic equations are

$$\begin{aligned} x^2 + 2 &= 3x + 6 \\ x^2 - 6x + 8 &= 0 \\ 2x^2 - 7 &= x^2 - 7x + 1 \end{aligned}$$

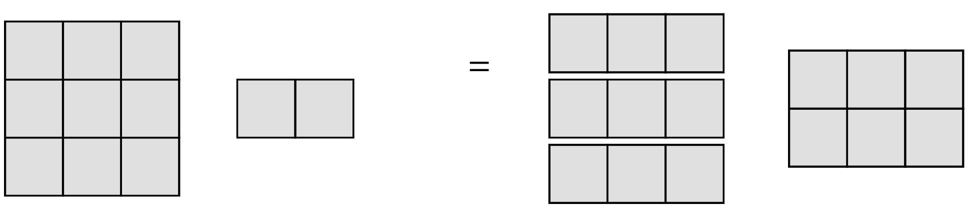
These are all equations having x^2 as the highest term. As with all equations, each statement says that the quantities on the left and the right of the = (equal) sign have the same value. However these statements of equality will only actually be true for *certain values* of the unknown (values of x). To be sure you understand this idea you should try guessing a value for x which will make one of the given equations into a true statement. For example, look at the first equation:

$$x^2 + 2 = 3x + 6$$

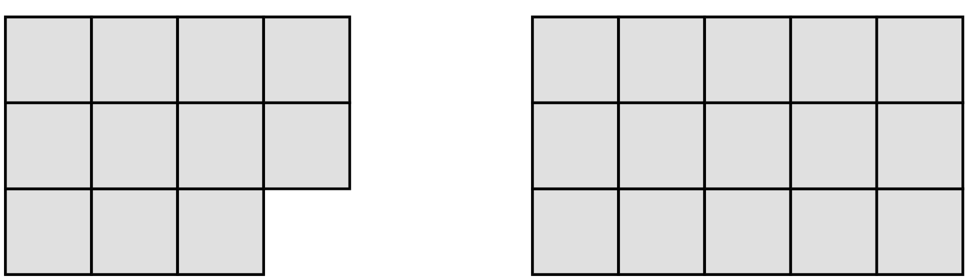
If we guessed that x might be 3, we would get

$$\begin{aligned} (3)^2 + 2 &= 3(3) + 6 \\ 9 + 2 &= 9 + 6 \\ 11 &= 15 \quad \text{(Not true)} \end{aligned}$$

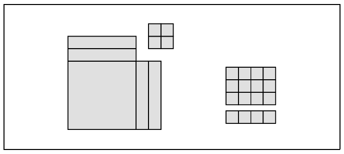
$$(3)^2 \quad + \quad 2 \quad = \quad 3(3) \quad + \quad 6$$



$$11 \neq 15$$

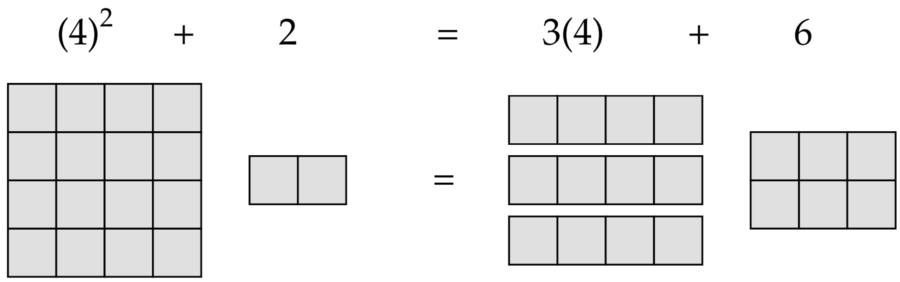


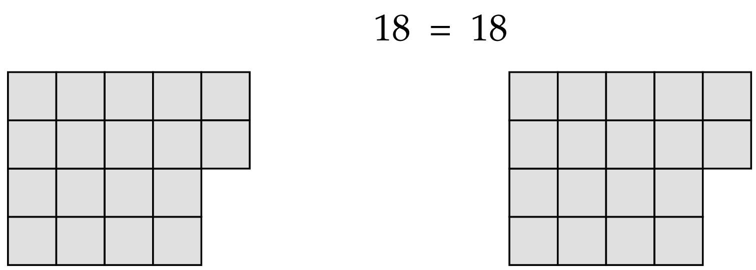
This is obviously not true because there are 11 units one the left and 15 units on the right side.



However, next we might choose to try $x = 4$; this would give us

$$\begin{aligned}(4)^2 + 2 &= 3(4) + 6\\ 16 + 2 &= 12 + 6\\ 18 &= 18\end{aligned}$$





This *is* true. Since $x = 4$ makes this equation true, we say that one **solution** for the equation is

$$x = 4$$

The purpose of this chapter is to learn how to find the correct solutions (values of the unknown) to quadratic equations *without guessing !* It's like a puzzle, where the real values required for the unknown are hidden within every quadratic equation, and your job is to solve the puzzle and find the true solution—the value of x. The puzzle-solving process isn't very hard; it is based upon ideas that you already know from previous chapters.

Exercises

Decide if the following items are expressions or equations. Then decide if they are linear, quadratic, or cubic:

1. $1{,}000{,}000x + 17$

2. $x^2 = 25$

3. $x^2 + 3x + 2$

4. $23 + 32x + 2x^2 + 3x^3 = 27$

Section 2
The Zero Product Rule

Multiplying and Zero

There is a very simple fact which is used with astonishing power for solving quadratic equations. This fact, which you already understand, is that *when two numbers are multiplied together the answer is never zero unless one of the numbers being multiplied is zero.* Let us illustrate this. Look at the following products of two numbers:

$$(3)(2) = 6$$

$$\frac{1}{2} \cdot 4 = 2$$

$$(^{-}3)(^{+}5) = {}^{-}15$$

$$\left(-\frac{2}{3}\right) \cdot \left(-\frac{3}{5}\right) = +\frac{2}{5}$$

$$(0)(7) = 0$$

If we multiply any two positive numbers, negative numbers, whole numbers, or fractions, we always get a positive or negative whole number or fraction for an answer, unless we multiply by zero. If we multiply by zero we always get zero for an answer; and *any time we get zero as the answer to a multiplication, then one of the multipliers must be zero.* If you know that:

$$(x)(3) = 0$$

then x must be zero (since 3 can't be zero). If you know that

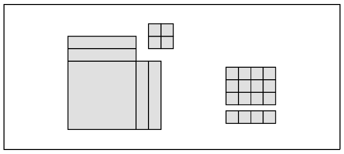
$$(x)(x-4) = 0$$

then either one or the other of the two parentheses must be zero. Either

$$x = 0 \quad \text{or} \quad x - 4 = 0$$

In such a situation, there could be two possible values for x:

$$\boxed{x = 0} \quad \textit{or} \quad x - 4 = 0 \;\rightarrow\; \boxed{x = 4}$$



Perhaps you have already recognized how this relates to quadratic equations. If we have a quadratic equation like

$$x^2 - 4x = 0$$

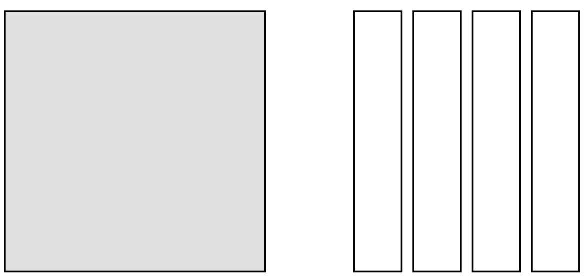
rather than guessing what values of x will make this a true statement, we can get the correct solutions for x very quickly if we can *factor* the expression on the left side of the equation:

$$x^2 - 4x = 0$$
$$(x)(x - 4) = 0$$

So either:

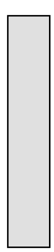
$x = 0$		$x - 4 = 0$
$\mathbf{x = 0}$	*or*	$\mathbf{x = 4}$

Using chips, this equation would look like:



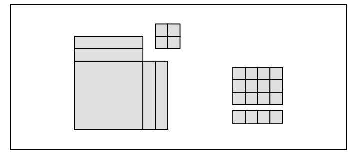
$= \quad 0$

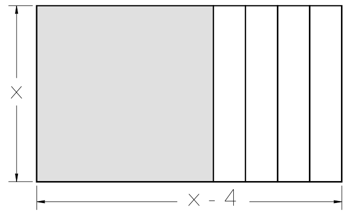
Obviously, if the areas of the pieces on the left combine to make zero, then the positive area (the square), must be canceled out by the negative area (the bars). The question is, what number must x (the length of the bar and the side of square) be for these areas to really cancel?



$= \quad ?$

We find out by factoring, or making a rectangle.

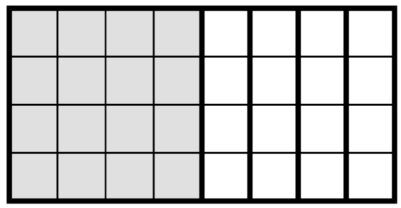




$= 0$

The area of this rectangle will only be zero if either the height is zero or the width is zero. The height will be zero when $x = 0$. The width will be zero when $x = 4$. Substituting these values into our picture we have:

$= 0$ *or*



$= 0$

Both are true, and no other value for x will work. (Try some.) Substituting these values numerically into the original equation gives:

$$(0)^2 - 4(0) = 0$$

$$\mathbf{0 = 0}$$

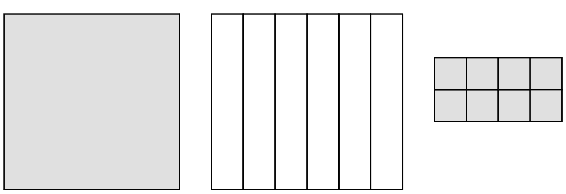
and

$$(4)^2 - 4(4) = 0$$

$$\mathbf{0 = 0}$$

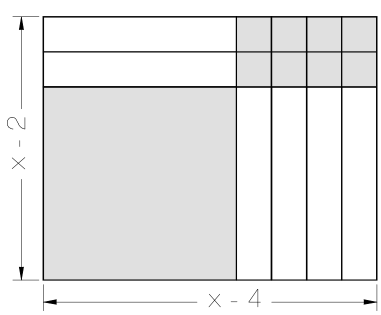
Here is a second example:

$$x^2 - 6x + 8 = 0$$

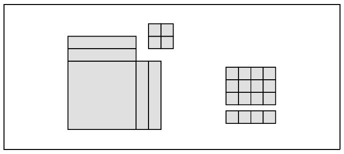


$= 0$

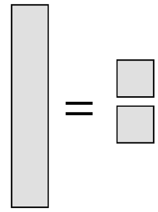
If the total area is zero, then the white pieces must cancel out the colored pieces. What size must x be to make this happen? We factor (make a rectangle):



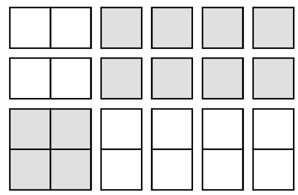
$= 0$



Now we can see that the height would be zero if $x = 2$:

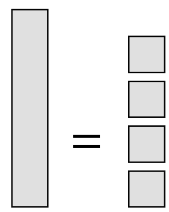


This would give:

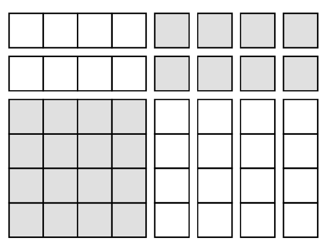


$= \quad 0$

Similarly the length would be zero if $x - 4 = 0$, which means that $x = 4$:



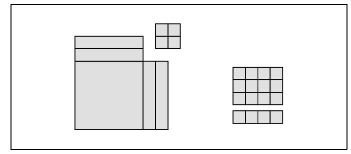
The picture would now be:



$= \quad 0$

which is also true. Here are the steps with algebra symbols alone:

The equation	$x^2 - 6x + 8 = 0$
Factors into	$(x-2)(x-4) = 0$
Using the zero product rule:	$x - 2 = 0 \quad or \quad x - 4 = 0$
Gives two solutions:	$x = 2 \quad or \quad x = 4$



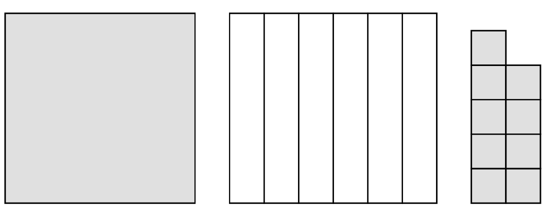
Numerical substitution of these results into the original equation gives:

$(2)^2 - 6(2) + 8 = 0$	$(4)^2 - 6(4) + 8 = 0$
$4 - 12 + 8 = 0$	$16 - 24 + 8 = 0$
$0 = 0$	$0 = 0$

Quadratic equations have at most two solutions for the value of the unknown. When the pieces are factored into a rectangle, either the height or the width of the rectangle can be zero. We get *two* answers because our figures (rectangles) have just *two* dimensions (height and width). The highest term of a quadratic equation is x^2, where the exponent 2 literally means *two dimensions*, giving two possible solutions. (Similarly, linear equations, where the highest term is x, have at most one solution, as seen in the equations chapter. Cubic equations, which have an x^3 term, have up to 3 solutions.)

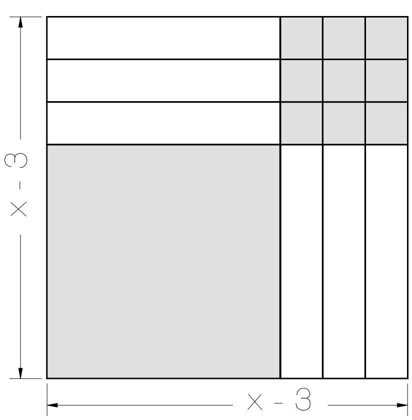
It is possible that the two solutions to a quadratic equation will be the same. For example, find the solutions for

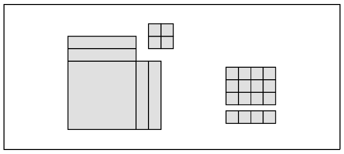
$$x^2 - 6x + 9 = 0$$



$$x^2 \quad - \quad 6x \quad + \quad 9 \quad = \quad 0$$

Factoring, we find that we have a perfect square:

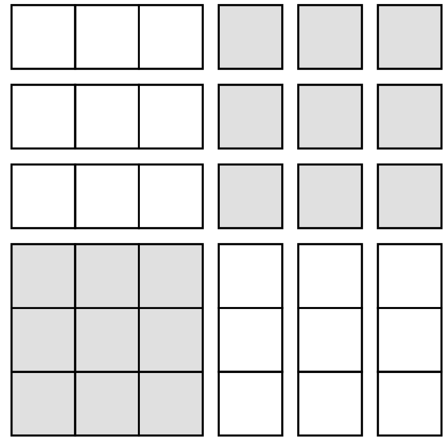




Either the height or the width can be zero, but since the height and the width are equal, our two resulting solutions are both the same:

$x - 3 = 0$ $x - 3 = 0$

$x = 3$ $x = 3$

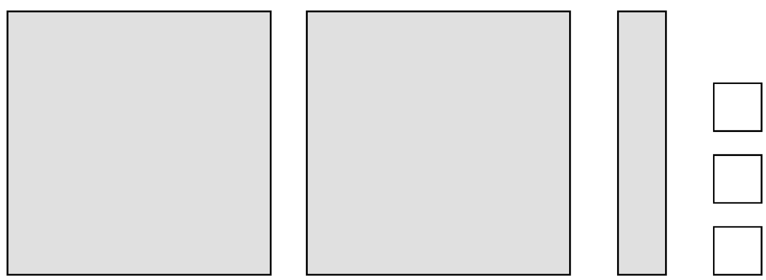


Numerical substitution gives

$$(3)^2 - 6(3) + 9 = 0$$
$$9 - 18 + 9$$
$$0 = 0$$

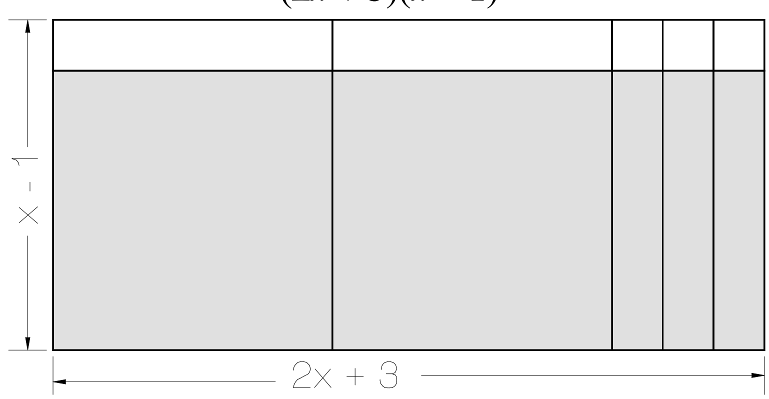
Here is an example with more than one x^2:

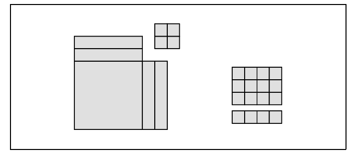
$$2x^2 + x - 3 = 0$$



This factors (with the addition of ^+2x and ^-2x) into

$$(2x + 3)(x - 1)$$





In this case the zero product rule gives us:

$$\begin{aligned} 2x + 3 &= 0 \\ 2x &= {}^{-}3 \\ \boldsymbol{x} &= \mathbf{-\frac{3}{2}} \end{aligned}$$

or

$$\begin{aligned} x - 1 &= 0 \\ \boldsymbol{x} &= \mathbf{1} \end{aligned}$$

We can check these results by substituting into the original equation:

$2\left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right) - 3 = 0$	$2(1) + (1) - 3 = 0$
$2\left(\frac{9}{4}\right)^2 - \left(\frac{3}{2}\right) - 3 = 0$	$2 + 1 - 3 = 0$
$\frac{9}{2} - \frac{3}{2} - 3 = 0$	$0 = 0$
$0 = 0$	

Exercises

Solve these quadratic equations:

1. $x^2 - 6x + 8 = 0$

2. $x^2 - 8x + 16 = 0$

3. $x^{2-}8x + 12 = 0$

4. $x^2 - 7x + 12 = 0$

5. $2x^2 - 9x + 9 = 0$

6. $3x^2 - 8x + 5 = 0$

7. $3x^2 - 16x + 5 = 0$

8. $2x^2 - 11x + 12 = 0$

9. $3x^2 - 13x + 12 = 0$

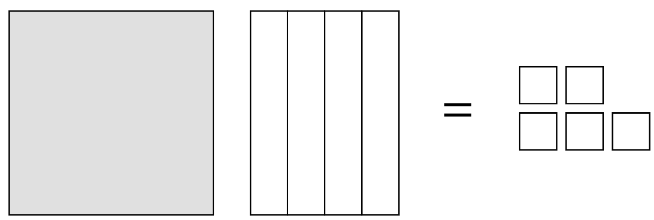
10. $x^2 - 10x + 21 = 0$

Section 3
Standard Form

Changing the Form of the Equation

If we are going to use the zero product rule to help solve a quadratic equation, then the first thing we must do is to be sure that one side of the equation is zero. This will allow us to factor the other side of the equation, and then to set each of the factors equal to zero. For example, if we have an equation which starts out as

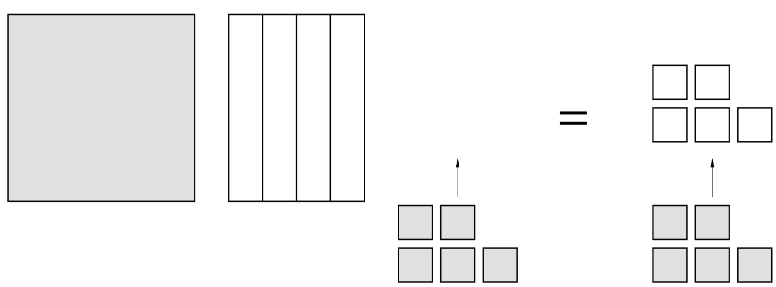
$$x^2 - 4x = {}^{-}5$$



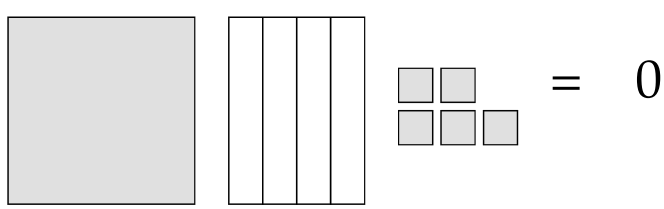
we *must* get one side of the equation to be zero. We can do this easily by adding 5 to both sides of the equation:

$$x^2 - 4x = -5$$

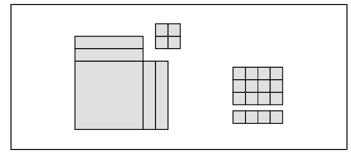
$$+5 \qquad +5$$



$$\boxed{x^2 - 4x + 5 = 0}$$



Now we are ready to factor the left side of the equation and to set the factors equal to zero, as demonstrated in the last section.



When we write a quadratic equation showing the x^2-term first, followed by the x-term and the units-term, all equaling zero, we say that the quadratic equation is in **standard form**. Standard form is shown as

$$Ax^2 + Bx + C = 0$$

where A, B and C represent numbers, and x represents the unknown. (A and B are called **coefficients** of x^2 and x respectively; *coefficient* means the number multiplying an unknown.) So if we have an equation like

$$2x^2 + 7x + 5 = 0$$

we would say $A = 2$, $B = 7$, and $C = 5$.

In the equation:

$$x^2 - 4x + 5 = 0$$

$$A = 1,\ B = {}^{-}4,\ \text{and } C = 5.$$

we can see that the coefficient may be implied (1 in $1x^2$) or negative (${}^{-}4$ in $-4x$). B is ${}^{-}4$ because we can write $-4x$ as $+({}^{-}4)(x)$.

It is important to note that in standard form all of the terms of a quadratic equation are in a particular order, with the highest term (x^2) first on the left, the x-term in the middle, and the units-term last, followed by the equal sign (=) and the zero (0).

This order of terms is called **descending order** because the size of the pieces (the power of x) starts with the largest term (highest power) first on the left, and then decreases (descends) as we move to the right. It is standard practice to arrange all expressions in descending order. Keeping this order consistent makes it easier to recognize and combine like terms; it also makes it easier to factor.

If a quadratic equation starts out written in some form other than standard form, we must first rearrange the terms until we have the standard form before we can proceed to the solution (find the value of x). So equations like

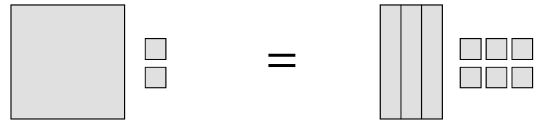
$$x^2 + 2 = 3x + 6$$

and

$$2x^2 - 7 = x^2 - 7x + 1$$

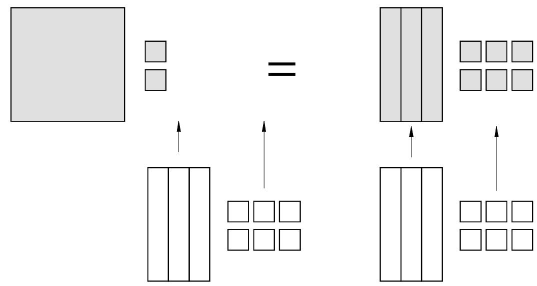
need to be rearranged and put into standard form before they are factored and solved. As mentioned earlier, this is done by adding to both sides until one side (usually the right side) equals zero, while combining like terms and arranging the other side in descending order. For the above examples the process looks like this:

Beginning $x^2 + 2 = 3x + 6$

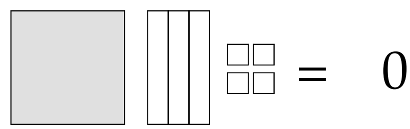


Add Opposites

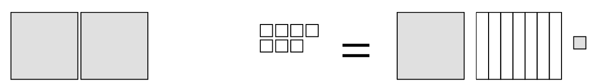
$$\begin{aligned} x^2 + 2 &= 3x + 6 \\ -3x - 6 &\quad -3x - 6 \end{aligned}$$



Standard Form $x^2 - 3x - 4 = 0$

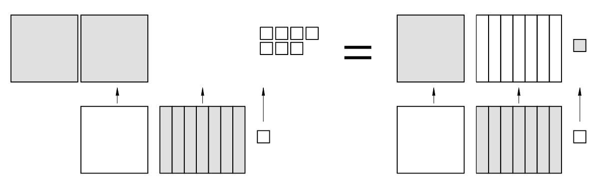


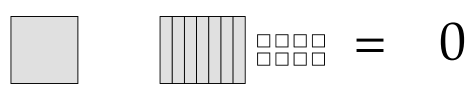
Beginning $2x^2 - 7 = x^2 - 7x + 1$



Add Opposites

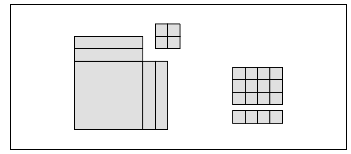
$$\begin{aligned} 2x^2 - 7 &= x^2 - 7x + 1 \\ -x^2 + 7x - 1 &\quad -x^2 + 7x - 1 \end{aligned}$$



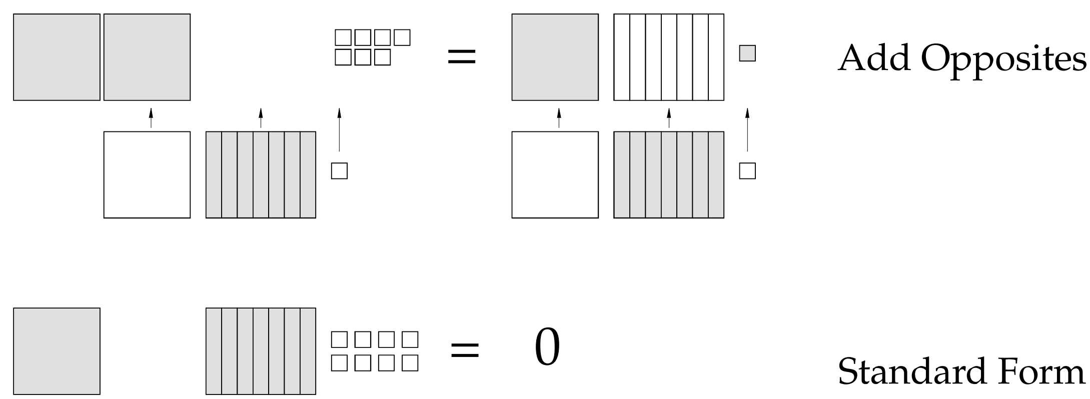


Standard Form $x^2 + 7x - 8 = 0$

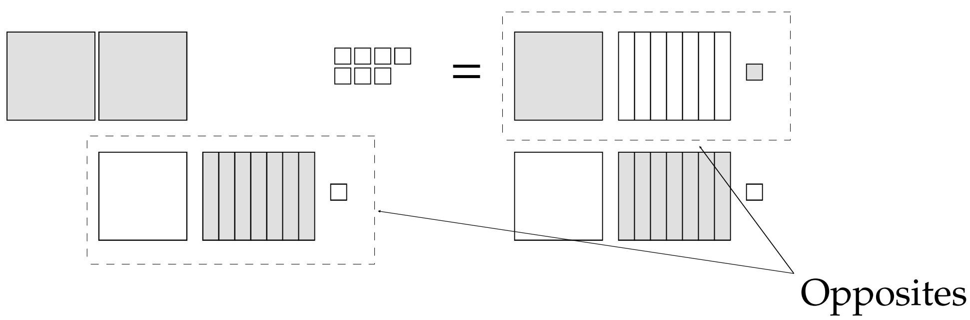
The Flip-Chip Short Cut



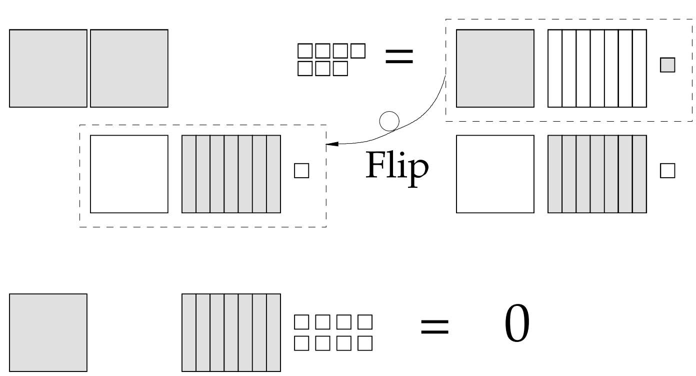
When rearranging an equation into standard form we get rid of all of the pieces (terms) on one side of the equal sign (=) by adding their opposites to both sides:

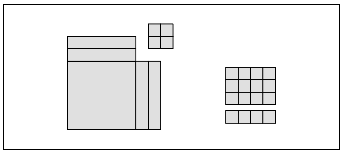


In this process of adding opposites, the equal sign is like a dividing line between the two sides of a balance, and adding the same amount to both sides gets one side to equal zero while maintaining the balance. Notice that when one side gets canceled out, the opposite of each of its terms appears on the other side (across the =).



So as the pieces disappear from one side of the equation, the same pieces appear in flipped over form (with opposite signs) across the equal sign, on the other side of the equation. But this is just the same as taking the pieces we are canceling and, while moving them across the equal sign, flipping them over.





You can move pieces (terms) across the equal sign as long as you flip them over (change their sign) when they cross over from one side of the equal (=) to the other. This is equivalent to adding the opposite of the term to both sides.

Like all short cuts, you must be very careful with this one when you use it. In particular, be sure not to flip terms without reason. Pieces flip (change sign) only when

- **They are multiplied by a negative.**
- **They move across the equal sign.**

Exercises

Arrange these quadratic equations in standard form. Do not solve.

1. $6 = 5x - x^2$

2. $3x^2 - 2x = 5x + 2x^2 - 12$

3. $x^2 + 12 = 11x - x^2$

4. $2x^2 - 6x = x^2 - 8$

5. $3 + 3x^2 = 10 - 5x + x^2$

6. $2 + x^2 - 2x = 20 + 5x$

7. $8x - 1 = -3x^2 - 6$

8. $3x^2 - 6x + 2 = 2x^2 + 4x - 19$

9. $x^2 - 5 = {}^-2x^2 + 16x$

10. $2x^2 + 5x = 14x - 9$

11. $x(2x - 3) = 5 - x$

12. $x^2 - 5 = 3(2x + 1)$

13. $x + 8 = 7 - 2x^2$

14. $(x + 2)(x - 5) = 3 - 2x^2$

15. $2x(x - 2) = 17$

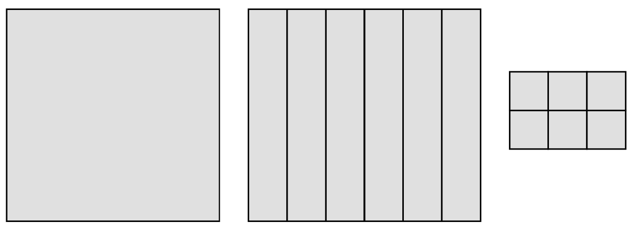
Section 4
Factoring Quadratic Equations

Quadratic Equations Having Positive *x*-bars

In section 2 of this chapter we gave several examples of quadratic equations which had negative x-bars when they were written in standard form. Now we will look at other types of quadratic equations.

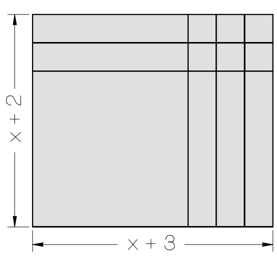
For example, let's factor and solve this quadratic equation:

$$x^2 \quad + \quad 5x \quad + \quad 6 \quad = \quad 0$$



$= \quad 0$

All the chips are colored side up, so how can we get zero for their sum? We know the factors are $(x + 2)$ and $(x + 3)$:

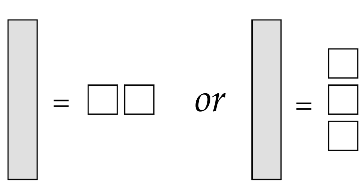


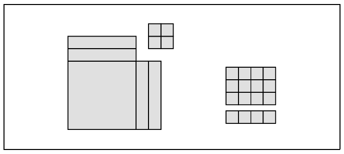
$= \quad 0$

This means the height and width will be zero if

$$\begin{array}{rcl} x + 2 &=& 0 \\ ^-2 && ^-2 \end{array} \quad or \quad \begin{array}{rcl} x + 3 &=& 0 \\ ^-3 && ^-3 \end{array}$$

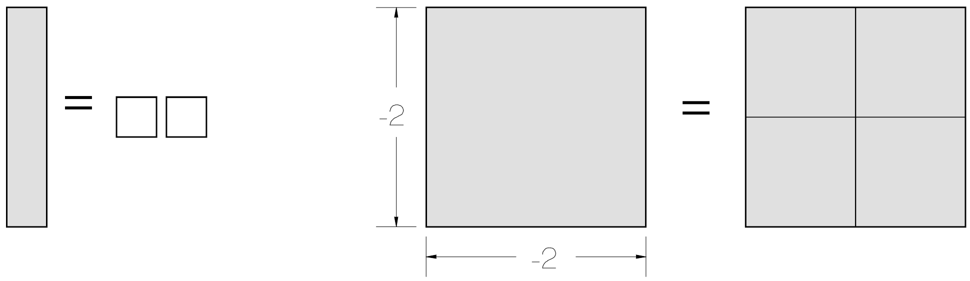
$$\mathbf{x = {}^-2} \qquad\qquad \mathbf{x = {}^-3}$$



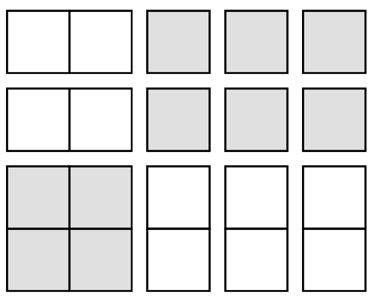


When the x-bar is replaced by any number of negative chips, the x^2 square will still be positive. This is because *both* dimensions of the square will be negative (two flips):

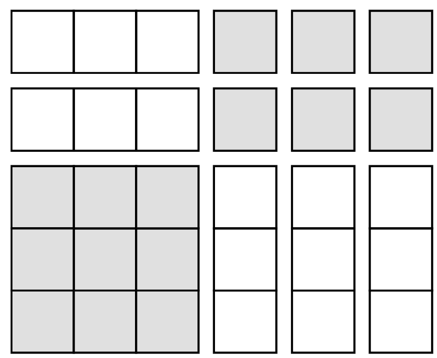
$$(^{-}2)\cdot(^{-}2) = {}^{+}4$$



So we can replace the x-bar with $^{-}2$ and the x^2 with $^{+}4$, or we can let the x-bar be $^{-}3$ and the x^2 be $^{+}9$, giving:



$x = {}^{-}2$



$x = {}^{-}3$

These are both equal to zero.

Substituting these results numerically into the original equation gives:

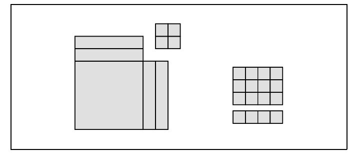
$$(-2)^2 + 5(-2) + 6 = 0 \qquad (-3)^2 + 5(-3) + 6 = 0$$

$$4 - 10 + 6 = 0 \qquad 9 - 15 + 6 = 0$$

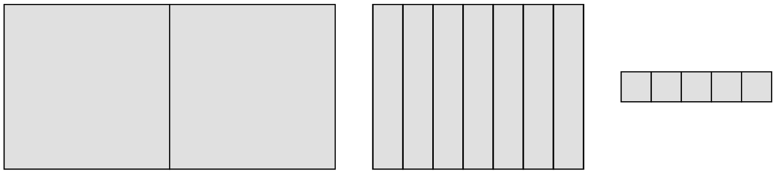
$$0 = 0 \qquad 0 = 0$$

Both answers are correct: $x = {}^{-}2$ and $x = {}^{-}3$.

A second example is



$$2x^2 + 7x + 5 = 0$$

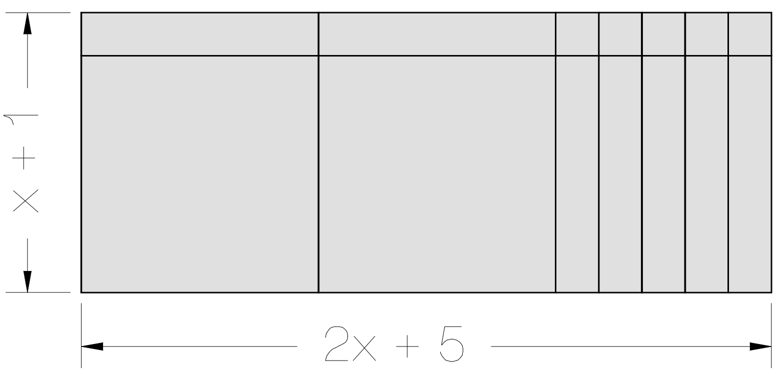


$= \quad 0$

$$2x^2 \quad + \quad 7x \quad + \quad 5 \quad = \quad 0$$

We can factor this, with the result

$$(x + 1)(2x + 5) = 0$$



$= \quad 0$

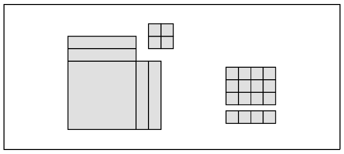
This time:

Either $\quad x + 1 = 0 \quad$ *or* $\quad 2x + 5 = 0$

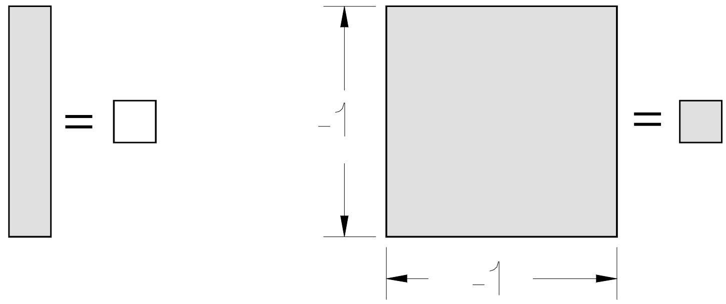
$$\boldsymbol{x = -1}$$

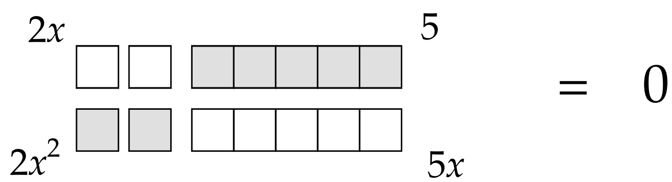
$$2x = -5$$

$$\boldsymbol{x = -\frac{5}{2}}$$

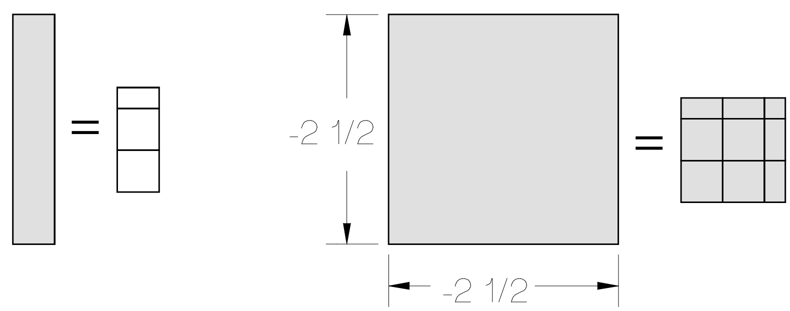


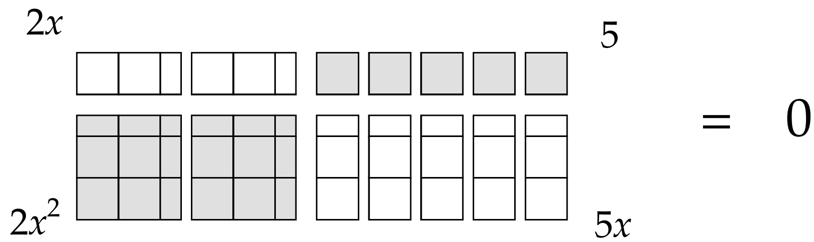
We can picture these results in the following way:



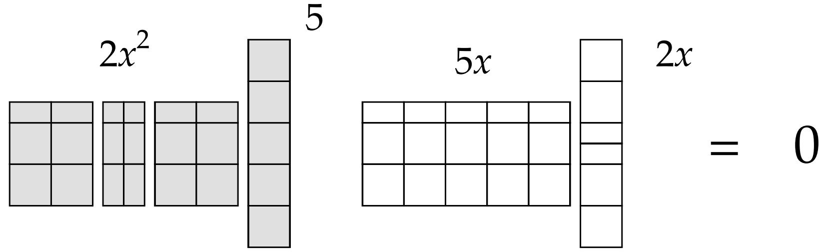


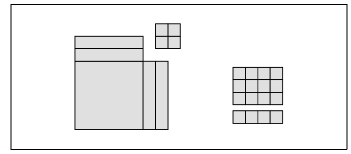
The fractional result can also be demonstrated, but it requires care. Remember that each x is - $5/2$:





Rearranging shows the positives match the negatives, so the total is zero:





Numerically this result can be checked by substitution:

$$2\left(-2\,\frac{1}{2}\right)^2 + 7\left(-2\,\frac{1}{2}\right) + 5 \;=\; 0$$

$$2\left(\frac{25}{4}\right) + 7\left(-\frac{5}{2}\right) + 5 \;=\; 0$$

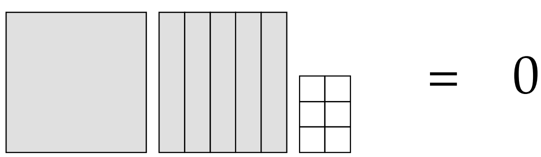
$$12\,\frac{1}{2} - 17\,\frac{1}{2} + 5 \;=\; 0$$

$$0 \;=\; 0$$

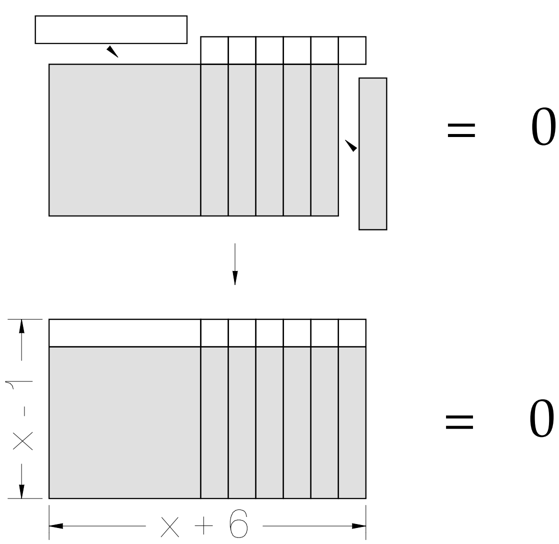
Quadratic Equations Having Negative Units

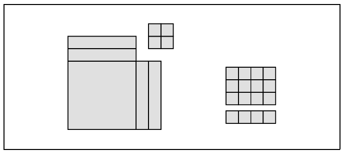
When a quadratic equation in standard form has negative units, we handle it in a way similar to that shown above. For example, consider the equation:

$$x^2 + 5x - 6 \;=\; 0$$



To solve this equation we factor by adding one positive and one negative x-bar, as shown.



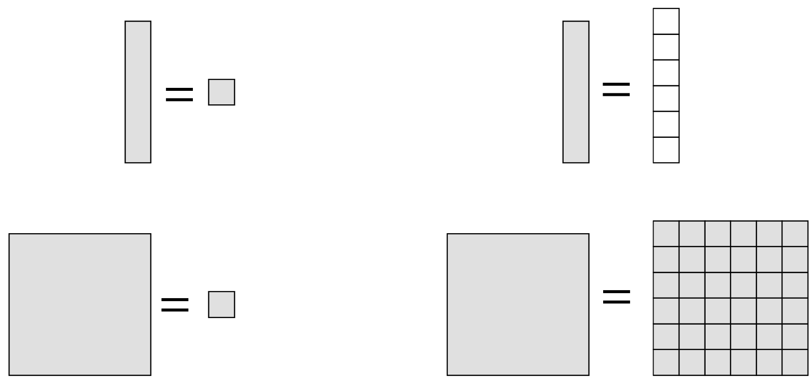


This gives

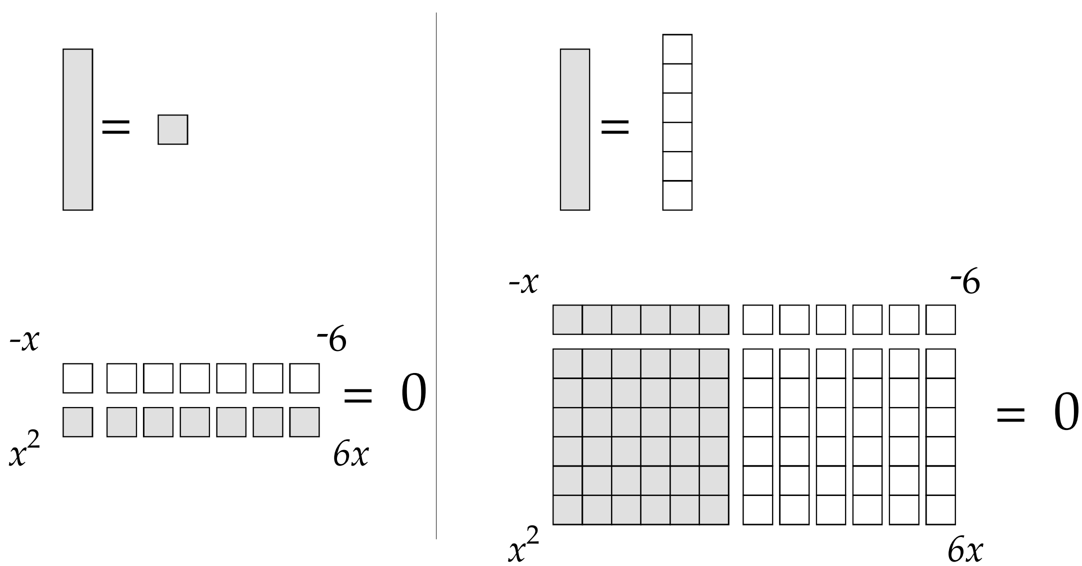
$$(x - 1)(x + 6) = 0.$$

As before, the quadratic expression on the left will equal zero if either the height or the width of the rectangle is zero, which requires that

Either	$x - 1 = 0$	*or*	$x + 6 = 0$
	$+1 \quad +1$		$-6 \quad -6$
	$\mathbf{x = 1}$		$\mathbf{x = -6}$

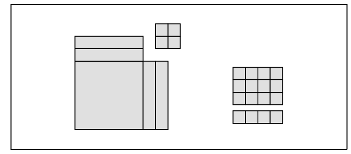


Substituting these values into our diagram we get



If you count carefully, you will see that both of the solutions do give zero for their results.

Numerically we can check our solutions by substituting the values for x into the original equation:



$x^2 + 5x - 6 = 0$	$x^2 + 5x - 6 = 0$
$x = 1$	$x = -6$
$(1)^2 + 5(1) - 6$	$(-6)^2 + 5(-6) - 6$
$1 + 5 - 6$	$36 - 30 - 6$
0	0

Both solutions make the original equation true. (Will any other solutions work? If you think they will work, try checking your suggested solution using chips and using numerical substitution. Work carefully.)

So now you know how to solve any quadratic equation which is factorable when put into standard form. A summary of the steps required is:

- **Put the quadratic equation into standard form:**

  $Ax^2 + Bx + C = 0$

- **Factor the quadratic expression on the left side of the equation.**
- **Use the zero product rule to set each of the factors equal to zero, giving two linear equations.**
- **Solve each of these linear equations for the unknown, giving two possible solutions.**
- **Check these two solutions by substituting each value for the unknown in the original equation .**

Exercises

Solve each of these quadratic equations, giving both solutions.

Check your work.

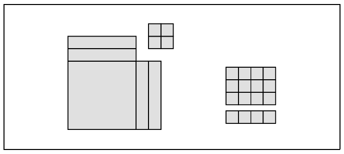
1. $x^2 - 5x + 6 = 0$

2. $2x^2 + 5x - 7 = 0$

3. $x^2 - 6x + 8 = 0$

4. $3x^2 + 8x + 5 = 0$

5. $x^2 = 2x + 8$



6. $2x^2 + 9 = -2x^2 + 12x$

7. $3x^2 + 6x = 2x^2 - 3x - 8$

8. $x^2 - 4 = -x^2 - 7x - 10$

9. $x^2 + 2x - 15 = 0$

10. $2x^2 - 3x - 2 = 0$

11. $x^2 + 3x = 5 - x$

12. $6x(x + 2) = x + 10$

13. $x^2 + 15x = 3(x - 9)$

14. $2x^2 - 6x = 5(x - 3)$

15. $x^2 - 1 = 16(x - 4)$

Section 5
Completing the Square

Quadratic Equations that Won't Factor

Now that you can solve any quadratic equation which can be factored, you might not wish to know that there are more quadratic expressions which *cannot* be factored than there are expressions which *can* be factored.

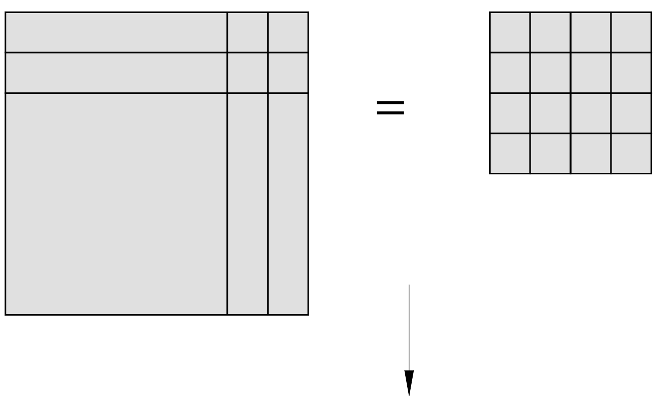
But do not fear; in this section and the next we are going to learn a method which will allow us to solve nearly any quadratic equation, whether we can factor it or not. To do this we will use another obvious and simple idea:

- **If two squares have equal area, then their sides are also equal. (The length of the side is the square root of the area.)**

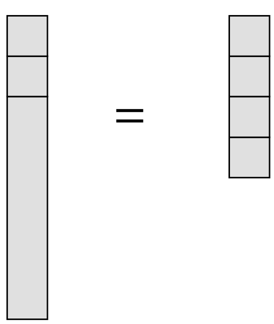
This method is called **completing the square.**

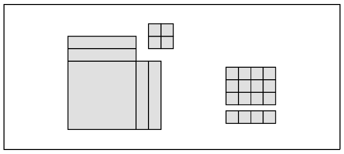
Translated into an example using chips, this means:

If 2 squares are equal in area,



Then their sides are equal in length.





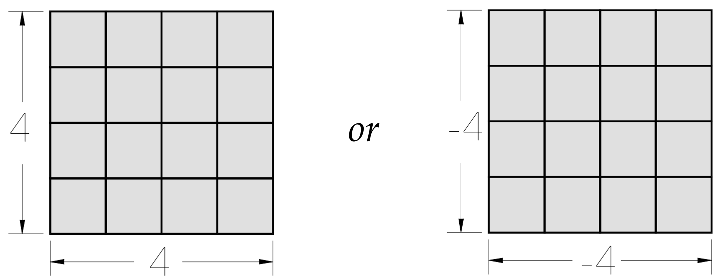
In symbols, if

$$(x+2)^2 = (4)^2$$

then

$$x+2 = 4$$

This idea shouldn't seem too hard. There is one extra twist: if the square on the right has 16 positive units in it, there is the possibility that its sides can be either $^{+}4$ or $^{-}4$, since either possibility will give the same number of positive units in the square.



So we include both possibilities by saying that if

$$(x+2)^2 = (4)^2$$

then

$$x+2 = {}^{+}4 \text{ or } {}^{-}4$$

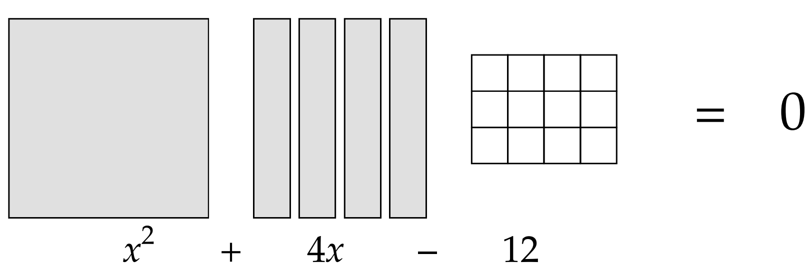
The two options $^{+}4$ or $^{-}4$ are usually written in a shorthand form as **± 4**:

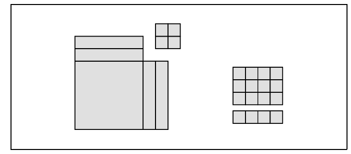
$$x+2 = \pm 4$$

where the symbol ± is read **plus or minus**; it means that there are two answers, both of which are equally valid.

Now that we see how the situation works with the equal squares, let's use this idea to solve a quadratic equation. We'll begin with this equation:

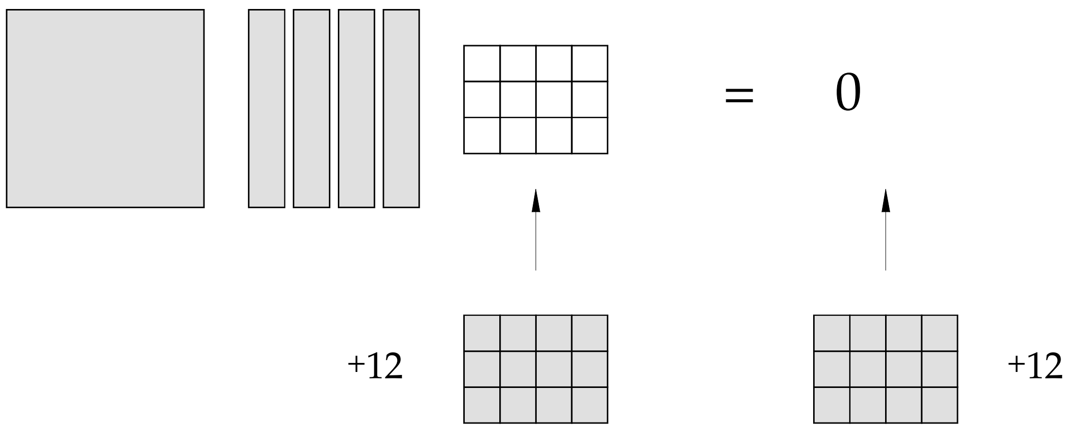
$$x^2 + 4x - 12 = 0$$



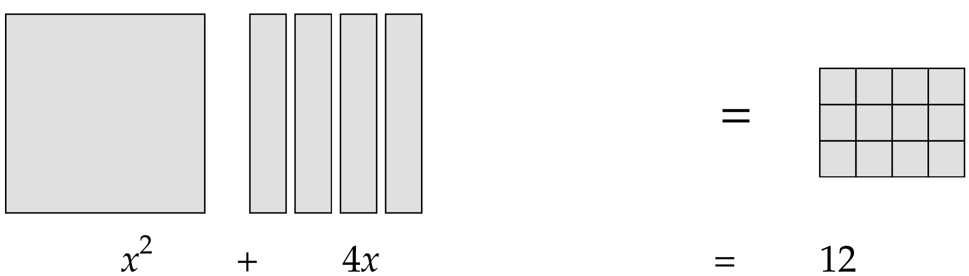


Even though this equation is factorable, let's pretend that we can't see the factors, which means that we can't use any of the methods we have learned so far to solve for the unknown (x). We are going to follow a different approach; we are going to rearrange things instead of factoring.

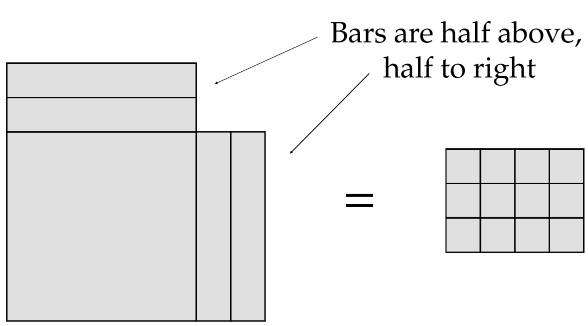
First we are going to get all of the units away from the x^2 and x terms on the left side of the equation by adding 12 units to each side.

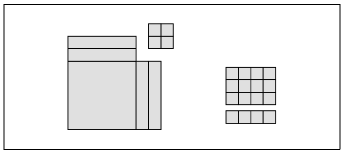


This gives

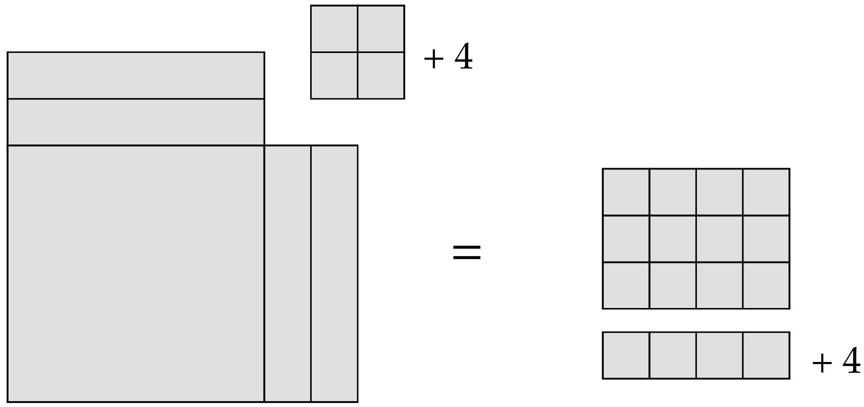


Next we arrange the x^2 and the x-bars on the left side of the equation into a form which is as close to a square as we can get without having any units. To do this we put half of the bars above the square, and half of the bars beside it on the right:

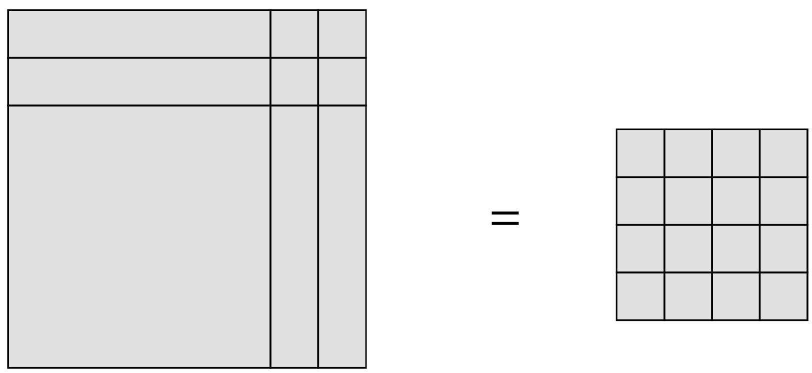




Finally we add enough unit chips to each side to **complete the square** on the left; we can see that four (4) chips will be required.

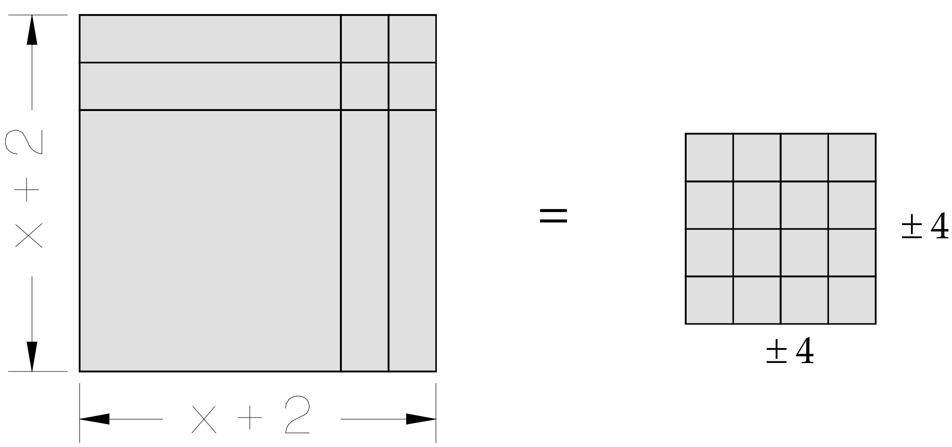


This gives us a picture we have seen before.



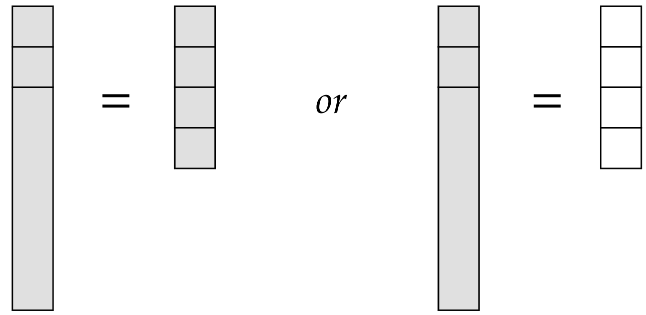
$$x^2 \quad + \quad 4x \quad + 4 \quad = \quad 16$$

We have two squares that we know are equal, so their sides (square roots) must also be equal.



$$x \quad + \quad 2 \quad = \quad \pm 4$$

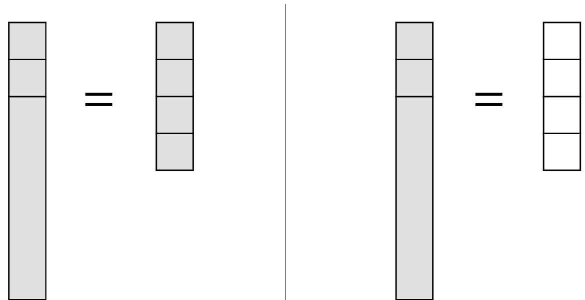
Now we see that if



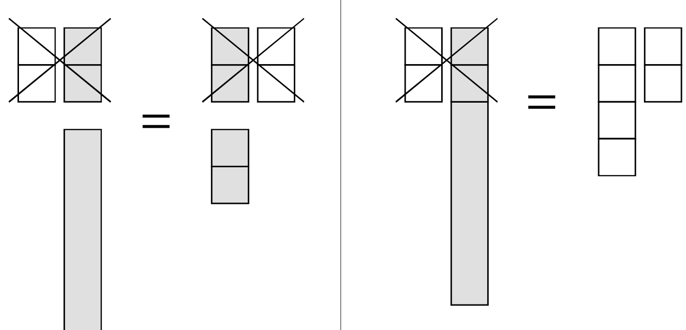
$x + 2 = {}^{+}4$ *or* $x + 2 = {}^{-}4$

then, adding ${}^{-}2$ to both sides gives us:

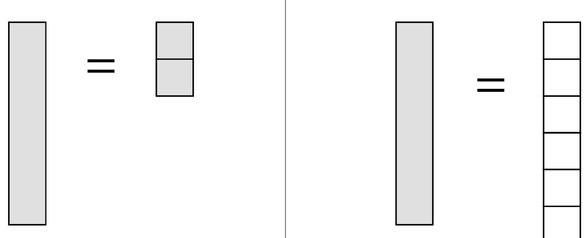
or



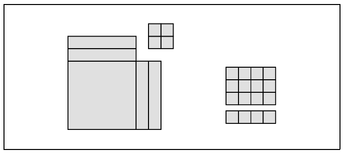
$x + 2 = {}^{+}4$ | $x + 2 = {}^{-}4$



$x + 2 - 2 = 4 - 2$ | $x + 2 - 2 = {}^{-}4 - 2$

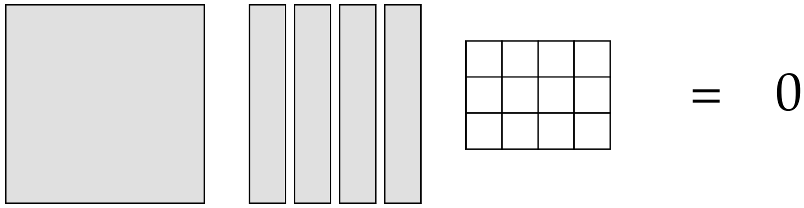


$x = 2$ *or* $x = {}^{-}6$



Here we have the two possible solutions for our original quadratic equation. To check them we can substitute these solutions into the original equation, either using chips or using numbers.

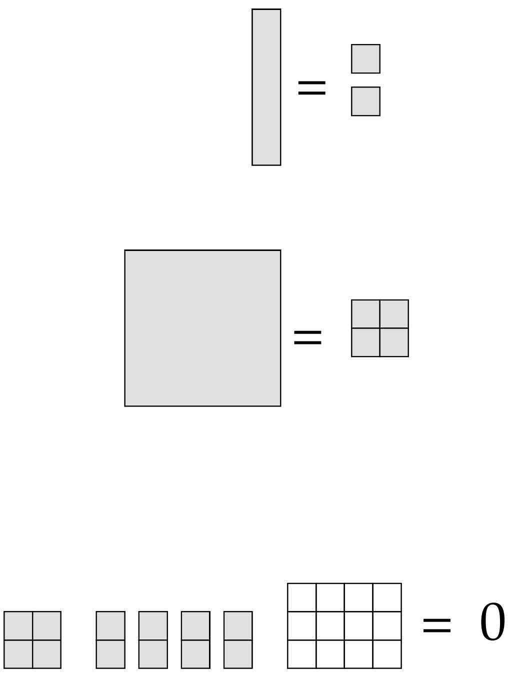
We started with:



$$x^2 + 4x - 12 = 0$$

Now we know that either

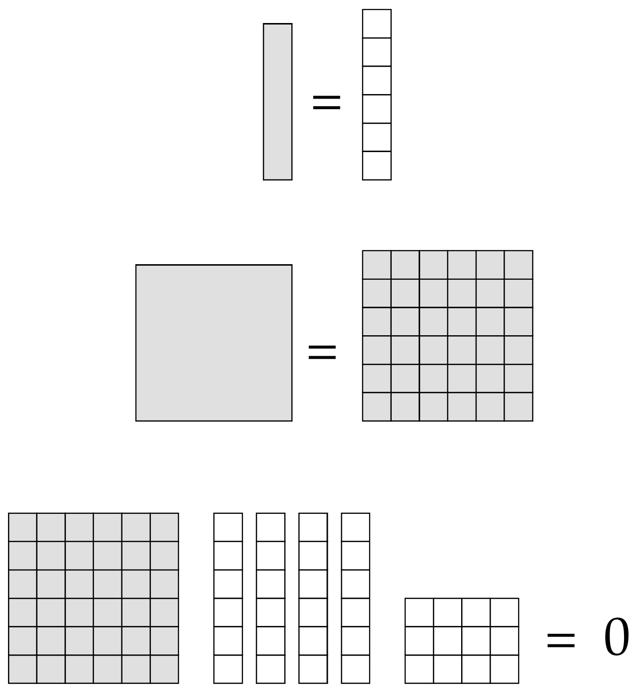
$x = 2$ *or* $x = {}^{-}6$



$$(2)^2 + 4(2) - 12 = 0$$

$$4 + 8 - 12$$

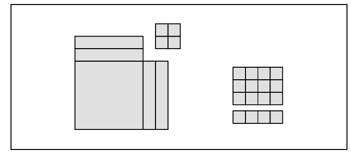
$$0$$



$$(-6)^2 + 4(-6) - 12 = 0$$

$$36 - 24 - 12$$

$$0$$



Both solutions check. (If we had done this problem by factoring would we have gotten the same result? Try it and see.)

Summary of Steps

The method of *completing the square* which we used to solve this example has several specific steps. So far these steps are:

- **Arrange the quadratic equation in standard form.**

- **Move the units (chips) across the equal sign by adding their opposite to both sides.**

- **Now that the x^2-square and the x-bars are isolated, use them together to make a figure as close to a square as possible. Put half of the bars above the square and half of the bars beside the square. Because you have no units, there will be a square hole in the corner.**

- **Add enough units to both sides of the equation to *complete the square* begun by the x^2 and the x-bars. Across the equal sign, also make the units into a square, or as close to a square as possible.**

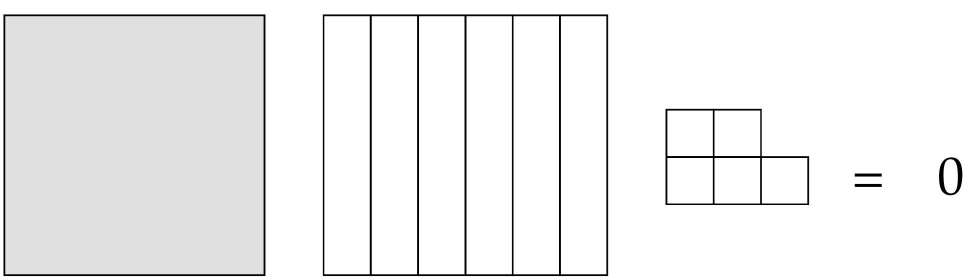
- **Take the square root of both squares by noting the length of their sides. Remember that the side length of the units square can be either + or – its square root.**

- **Set the square roots of the expressions equal to each other.**

- **Isolate the unknown (x) on the left side of the equation by adding the necessary positive or negative units to each side of the equation. This gives the two solutions for x.**

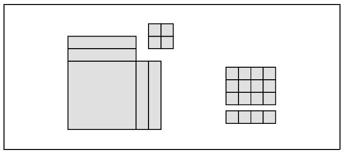
Another Example

Let's work through a second example. (This time we will choose one that



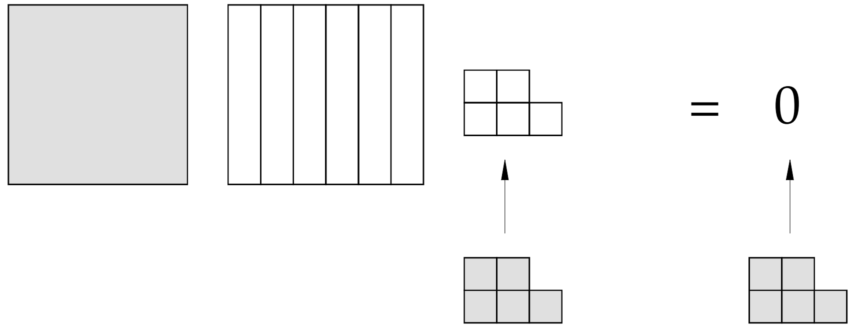
$$x^2 - 6x - 5 = 0$$

really can't be factored.)

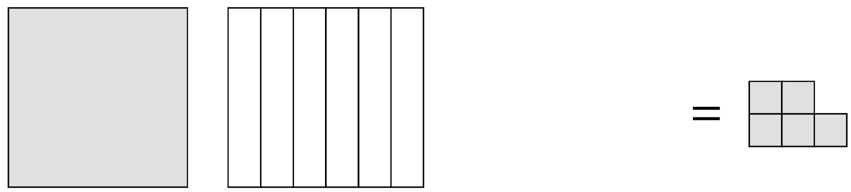


This expression can't be factored, so we work through the steps outlined above:

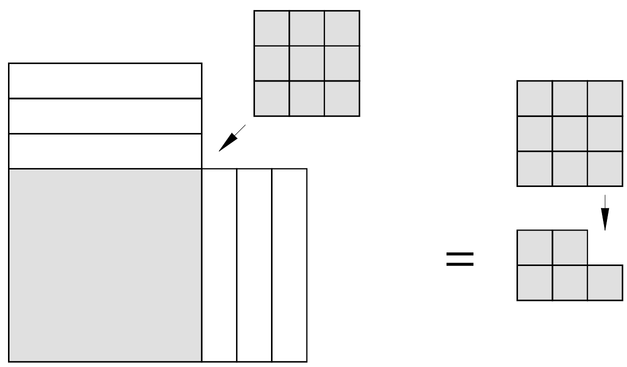
$$\begin{aligned} x^2 - 6x - 5 &= 0 \\ +5 \quad &\;\; +5 \end{aligned}$$



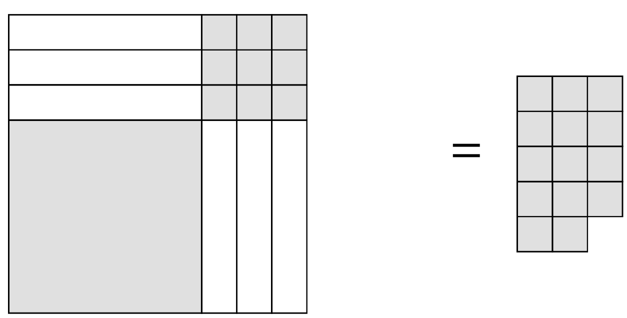
$$x^2 - 6x \quad = 5$$



$$x^2 - 6x + 9 = 5 + 9$$

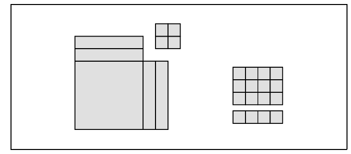


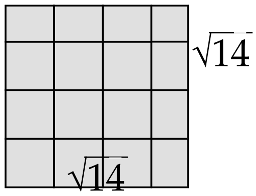
$$(x-3)^2 = 14$$



$(x-3)$ on each side

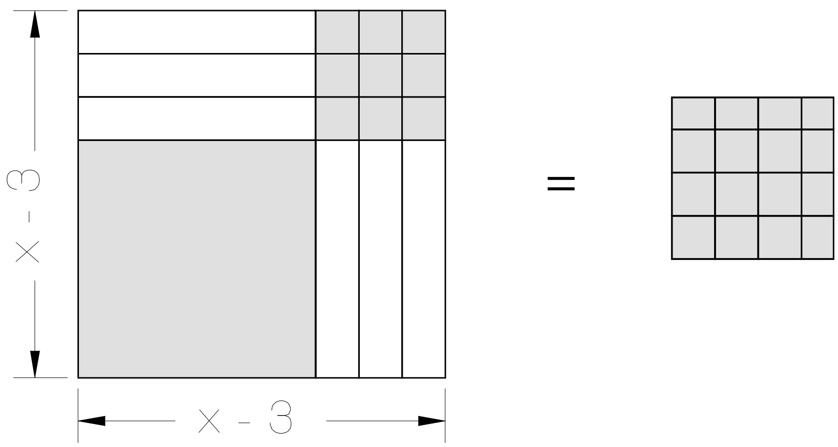
The units on the right cannot be arranged into a perfect square without cutting some of them into smaller pieces. But if we imagine trimming some of the unit squares so that we can make a nice square having exactly 14 units in it, we already know how to express the length of the side of that square—the side will be $\sqrt{14}$.



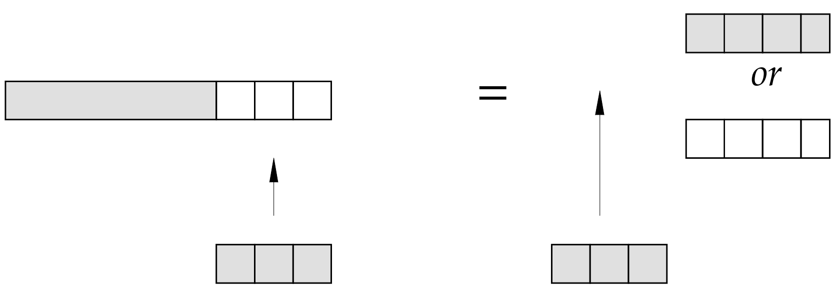


From the picture, we can estimate that the $\sqrt{14}$ is between $3\frac{5}{6}$ and $3\frac{5}{7}$.

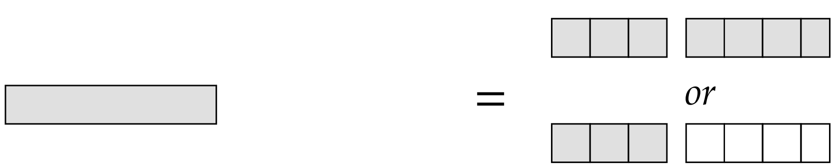
We continue with completing the square:



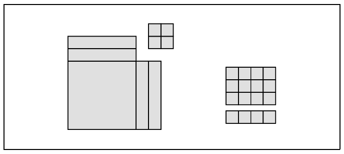
$$(x-3)^2 = 14$$



$$\begin{aligned} x - 3 &= \pm\sqrt{14} \\ +3 \quad & \quad +3 \end{aligned}$$

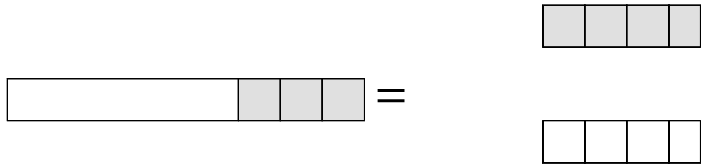


$$x = 3 \pm \sqrt{14}$$



Note that the diagram might be interpreted in a different way:

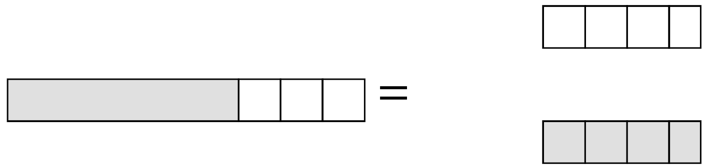
$$-x + 3 = \pm\sqrt{14}$$



Flipping both sides over (multiplying by $^-1$) gives:

$$^-1(-x + 3) = {}^-1(\pm\sqrt{14})$$

$$x - 3 = \pm\sqrt{14}$$



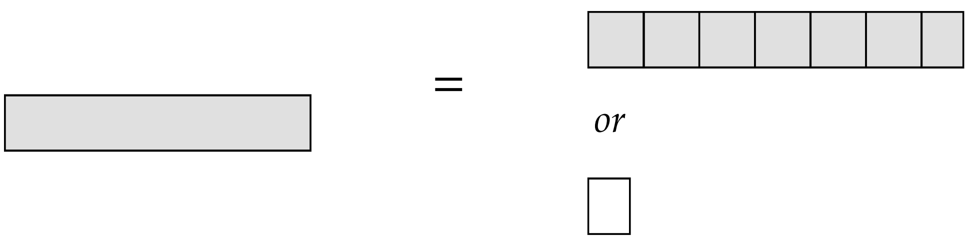
This will obviously have the same result as before.

It may not feel totally comfortable to you to have answers like

$$x = 3 \pm \sqrt{14}$$

From the pictures we can see that this means approximately

x is about $6\frac{5}{7}$ or $-\frac{5}{7}$



For more exact values we can use a calculator to get a decimal value for 14:

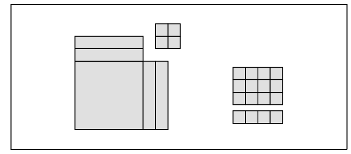
$$\sqrt{14} = 3.7417\ldots$$

This means that

$$x = 3 + \sqrt{14} = 3 + 3.7417 = 6.7417$$

or

$$x = 3 - \sqrt{14} = 3 - 3.7417 = {}^-0.7417$$



These answers are not nice round numbers. This shows that the method of completing the square can be used to find solutions for quadratic equations which don't have simple integral factors. To check our answers for this example, it is easiest to use a pencil and paper and a *calculator* to substitute these values into the original equation. (Be sure to write down everything you are doing as you are entering numbers into your calculator so that you don't get lost half-way through the operation.)

$x^2 - 6x - 5 = 0$	
$x = 6.7417$ *or*	$x = {}^{-}0.7417$
$(6.7417)^2 - 6(6.7417) - 5 = 0$	$({}^{-}0.7417)^2 - 6({}^{-}0.7417) - 5 = 0$
0.0003	0.0003

If we use even more accurate decimal values for $3 \pm \sqrt{14}$ we will get results which come out even closer to being exactly zero.

You have learned enough of the process of completing the square to have a good start. Try your hand at using this new method to do the following problems on your own. Use your chips and a scratch pad to do these exercises. Check your results, using a calculator if necessary.

Exercises

1. $x^2 + 6x - 7 = 0$

2. $x^2 - 4x - 5 = 0$

3. $x^2 + 8x = 9$

4. $x^2 = 10x + 16$

5. $x^2 + 15x = 3x - 20$

6. $x^2 + 10x + 18 = 0$

7. $x^2 + 6x - 9 = 0$

8. $x^2 + 14x = 2x - 11$

9. $x^2 - 5x = 3x + 3$

10. $x^2 - 10x = {}^{-}17$

11. $x^2 - 3x - 5 = 0$

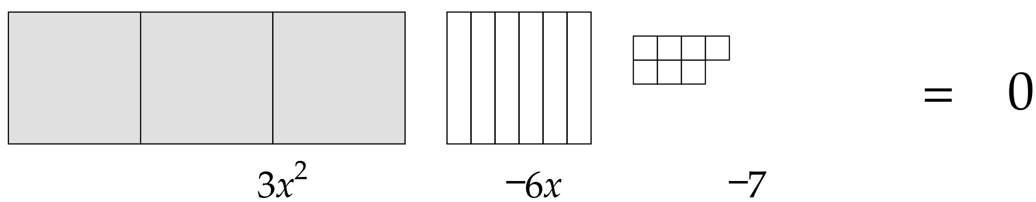
12. $x^2 - 5x + 2 = 0$

Section 6
Equations with More than One x^2

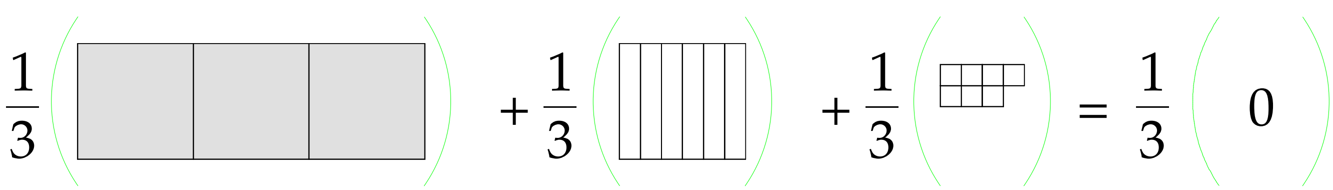
Beginning With More Than One x^2

When we begin with a quadratic equation in standard form which has more than one x^2 we must add one additional step to our methods for completing the square. Let's begin with

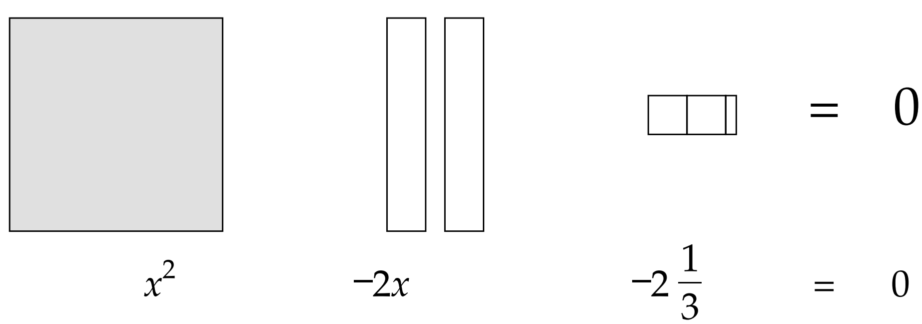
$$3x^2 - 6x - 7 = 0$$



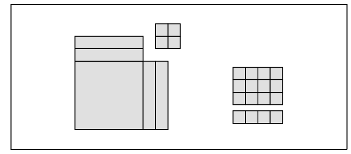
It will not be possible to make a perfect square using the x^2's and the x-bars, since there is no way to arrange three x^2's into a square. To deal with this we take ⅓rd of all the terms on both sides of the equation, so that we are left with only one x^2. (One way to describe this is to say that we *divide by the coefficient of* x^2.)

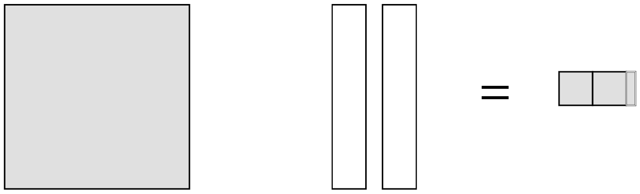


This gives:



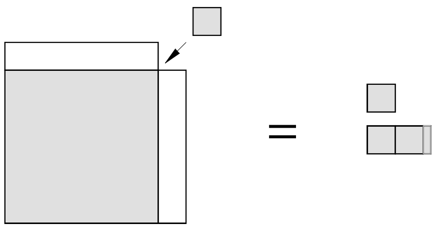
Don't let the fraction scare you; from here we proceed just as we did in the last example. Since we can't factor, we move the unit chips to the other side of the equation (remembering to change their sign when they cross the equal sign).

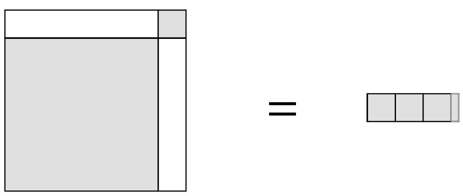




$$x^2 - 2x = 2\frac{1}{3}$$

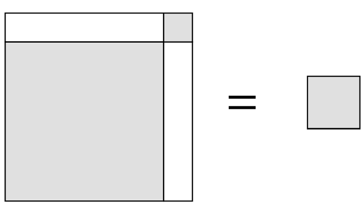
Now make the square





$$(x - 1)^2 = 3\frac{1}{3}$$

Taking the square root of both sides we get:



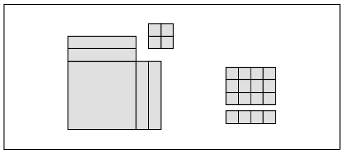
$$x - 1 = \pm\sqrt{3\frac{1}{3}}$$

This is more commonly written as

$$x - 1 = \pm\sqrt{\frac{10}{3}}$$

(It is hard to draw a picture of the square root of $^{10}/_{3}$, but it is just the side of a square having $3\frac{1}{3}$ units of area inside it.) Using a calculator we find that

$$\sqrt{10/3} = 1.8257$$



so

$$x - 1 = \pm 1.8257$$

Adding one to both sides gives

$$x = 1 + 1.8257$$
or
$$x = 1 - 1.8257$$

$$x = 2.8257 \quad \textit{or} \quad -0.8257$$

Use your calculator and check these results in the original equation.

Summary of Steps

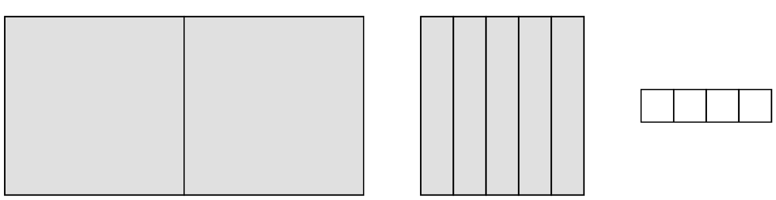
To accommodate equations which begin with more than one x^2 when in standard form, we must add one more step to our procedure for completing the square. Now the procedure will read:

- **Arrange the quadratic equation in standard form.**
- **If the first term has more than one x^2, divide all terms on both sides of the equation by the coefficient of x^2, leaving only one x^2.**
- **Move the units (chips) across the equal sign by adding their opposite to both sides.**
- **Proceed as before.**

A Final Example

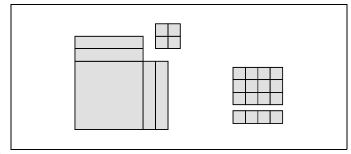
Here is one final example:

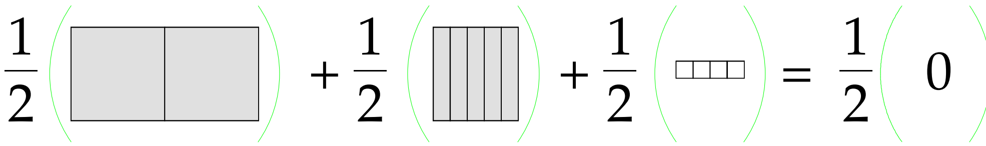
$$2x^2 + 5x - 4 = 0$$



$= \quad 0$

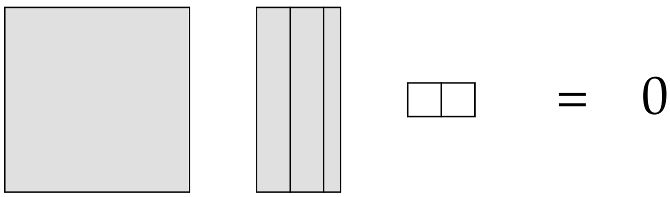
First divide each term by 2 (multiply by ½), to leave just a single x^2.





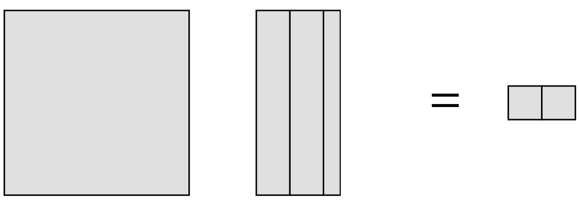
$$\frac{1}{2}(2x^2) + \frac{1}{2}(5x) + \frac{1}{2}(-4) = \frac{1}{2}(0)$$

This leaves:



$$x^2 + \frac{5}{2}x - 2 = 0$$

Moving the units across the equal sign we have:

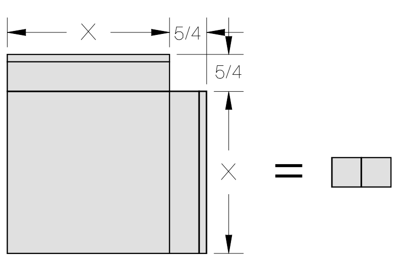


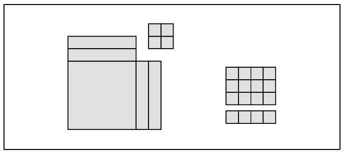
$$x^2 + \frac{5}{2}x = 2$$

To make a square from the left side of the equation, we have to cut the $(\frac{5}{2})x$ into two equal pieces, so that one can go above the large square and the other can go beside the large square. Again, don't let the fractions scare you. Half of $(\frac{5}{2})x$ is easy to figure out:

$$\frac{1}{2}\left(\frac{5}{2}x\right) = \frac{5}{4}x$$

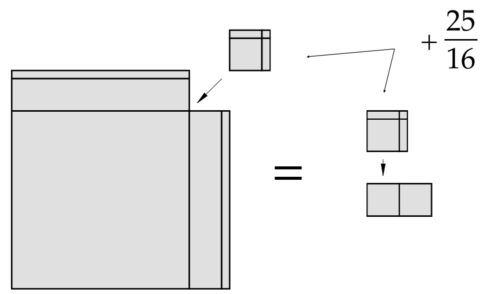
The figure which results looks like this:





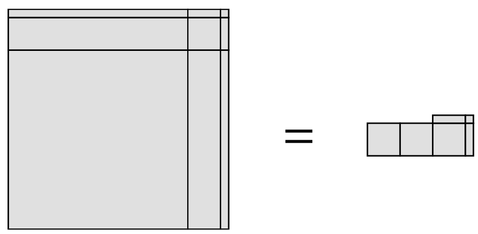
This time the number of unit chips required to fill in the corner of the square will be a fraction. The square corner to be filled in is ($\frac{5}{4} \cdot \frac{5}{4}$), so the required amount will be

$$\frac{5}{4} \cdot \frac{5}{4} = \frac{25}{16}$$



This gives

$$\left(x + \frac{5}{4}\right)^2 = 3\,\frac{9}{16}$$



Although our results will be fractions, the process of taking the square root is no different than before.

$$\left(x + \frac{5}{4}\right)^2 = 3\,\frac{9}{16} = \frac{57}{16}$$

$$x + \frac{5}{4} = \pm\sqrt{\frac{57}{16}} = \pm\frac{\sqrt{57}}{4}$$

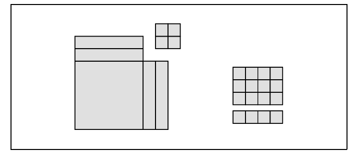
Adding -$\frac{5}{4}$ to both sides gives

$$x = -\frac{5}{4} \pm \frac{\sqrt{57}}{4}$$

which can also be written (since the two fractions have the same denominator) as:

$$x = \frac{-5 \pm \sqrt{57}}{4}$$

Using a calculator this comes out to be



$$x = \frac{-5 \pm 7.5498}{4}$$

$$x = +0.6375$$

or

$$-3.1375$$

Use your calculator to check these results as shown below, and then congratulate yourself for working through such a challenging problem.

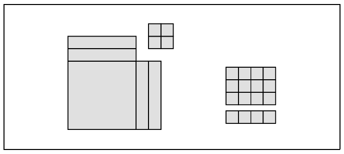
Check:

$2x^2 + 5x - 4 = 0$	
$x = 0.6375$ *or*	$x = -3.1375$
$2(0.6375)^2 + 5(0.6375) - 4 = 0$	$2(-3.1375)^2 + 5(-3.1375) - 4 = 0$
0.0003	0.0003

Does it check out?

Now we will review the steps in this process, then you can try some more problems on your own. Have chips, a scratch pad and your calculator close at hand and work carefully.

- **Arrange the quadratic equation in standard form.**
- **If the first term has more than one x^2, divide all terms on both sides of the equation by the coefficient of x^2 which will leave only one x^2.**
- **Move the units (chips) across the equal sign by adding their opposite to both sides.**
- **Now that the x^2 piece and the x-bars are isolated, use them together to make a figure as close to a square as possible while having no units, by putting half of the bars above the square and half of the bars beside the square.**



- **Add enough units to both sides of the equation to *complete the square* begun by the x^2 and the x-bars. Across the equal sign also make the units into a square, or as close to a square as possible.**

- **Take the square root of both squares by noting the length of their sides. Remember that the side length of the units square can be either + or – its square root.**

- **Set the square roots of the expressions equal to each other.**

- **Isolate the unknown (x) on the left side of the equation by adding the necessary positive or negative units to each side of the equation. This gives the two solutions for x.**

Exercises

Solve for x by completing the square:

1. $x^2 + 10x - 24 = 0$

2. $x^2 + 6x - 4 = 0$

3. $2x^2 - 20 = x^2 - 8x$

4. $2x^2 - 6x = 2x - 11$

5. $2x^2 - 8x - 6 = 0$

6. $3x^2 - 12x + 9 = 0$

7. $3x^2 - 11x = x - 5$

8. $4x^2 - 9x = x^2 + 6x - 6$

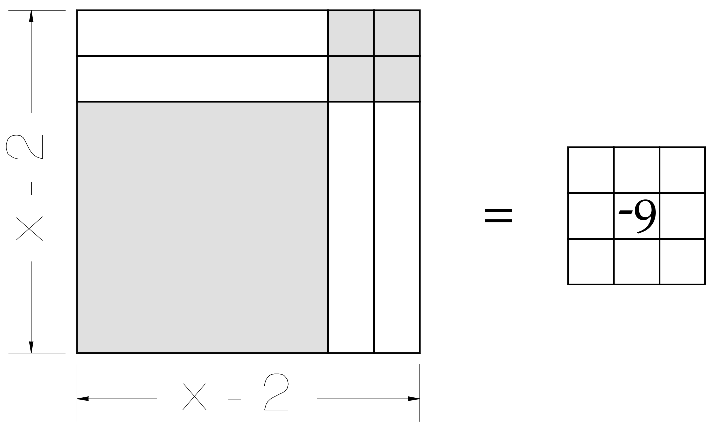
9. $2x^2 + 7x - 8 = 0$

10. $x^2 - 8x - 10 = x - x^2$

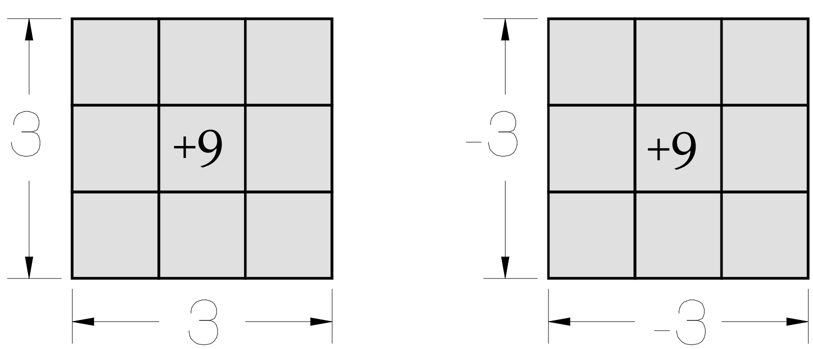
Section 7
Imaginary Solutions

The Square Root of a Negative Number

There are still some quadratic equations which we cannot solve, even when using our new methods. These equations involve situations where we have a perfect square equaling a negative number. For example:

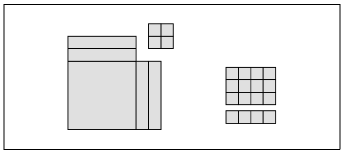


The reason we cannot solve equations like these is that we have not defined any square roots of negative numbers. Square roots (the sides of squares) must be equal, by definition, and there are no two equal numbers that multiply together to give a negative area.

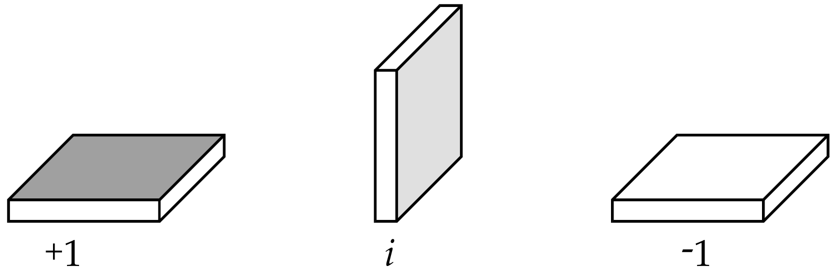


$$(+3) \cdot (+3) = +9$$
$$(-3) \cdot (-3) = +9$$

Because of this problem, we are going to invent a new type of number which we will only use for the very special purpose of solving these equations. (Later math courses will have more uses for these special numbers.) We will call these new numbers **imaginary**, because they won't be positive and they won't be negative and they won't be zero; in fact they may not seem to really exist at all except within our imaginations, hence the name.

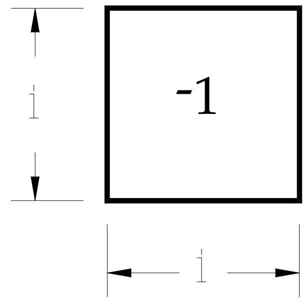


Imaginary numbers will be like chips flipped only half of the way over, or standing on edge; they aren't plus and they aren't minus.



We call the unit imaginary number ***i***, and we define ***i*** by the equation

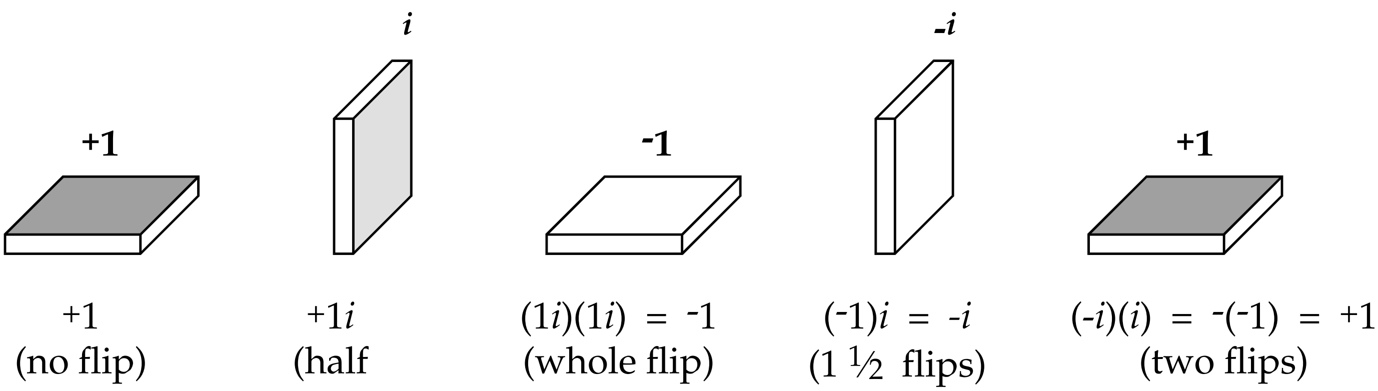
$$i \cdot i = -1$$
$$or$$
$$i^2 = -1$$



Another way of saying this is

$$\boldsymbol{i = \sqrt{-1}}$$

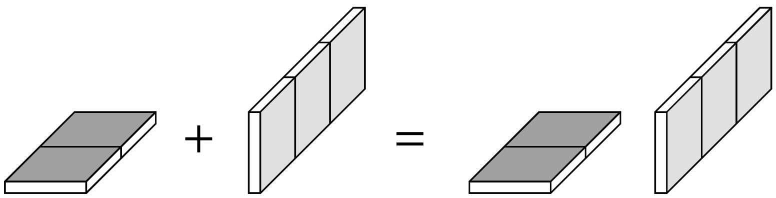
Two imaginaries multiplied together will give two half flips, or one whole flip, which is the same as a minus (-) sign.



Compared to imaginaries, the other regular numbers (which we have been working with up until now) are called **real numbers**. Real numbers and imaginary numbers can only interact in certain ways.

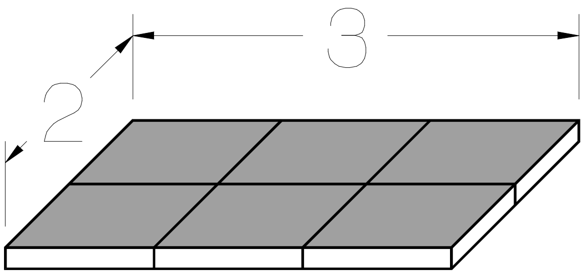
Adding and Subtracting with Imaginaries

We cannot combine real and imaginary numbers because they are different kinds of chips (in geometry, we would say that they are on different planes). *A real number and an imaginary number cannot cancel or add together. They must stay as separate terms.*



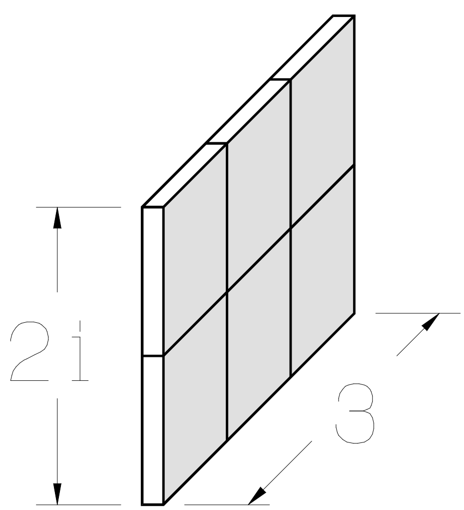
$2 + 3i$

If we multiply real and imaginary numbers, the unit imaginary (i) acts very much like a sign (plus or minus) rather than like a number. The numbers multiply together, making a rectangle as usual; and the imaginaries (i) and negative signs (–) tell how many half flips or whole flips the rectangle goes through. *Each **i** makes one half flip; each negative sign makes one whole flip.*



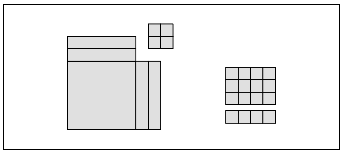
(2)(3) = 6

No flips



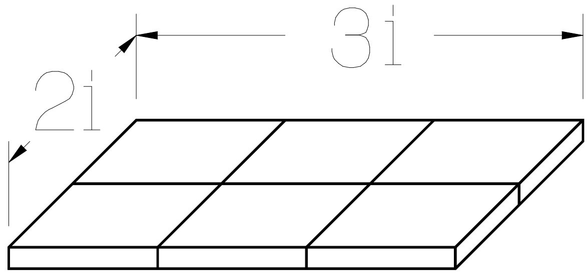
$(2i)(3) = 6i$

(half flip) (no flip) → (half flip)



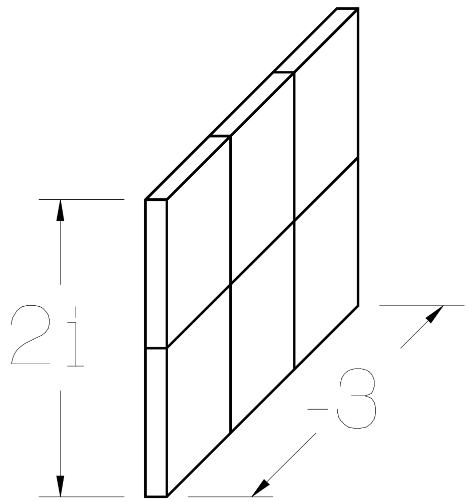
$$(2i)(3i) = -6$$

$$\begin{pmatrix}\text{half}\\\text{flip}\end{pmatrix}\begin{pmatrix}\text{half}\\\text{flip}\end{pmatrix} \rightarrow \begin{pmatrix}\text{whole}\\\text{flip}\end{pmatrix}$$



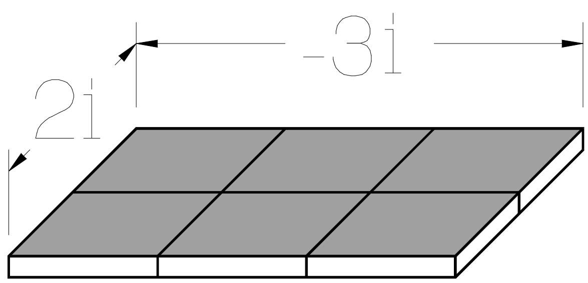
$$(2i)(-3) = -6i$$

$$\begin{pmatrix}\text{half}\\\text{flip}\end{pmatrix}\begin{pmatrix}\text{whole}\\\text{flip}\end{pmatrix} \rightarrow \begin{pmatrix}1\tfrac{1}{2}\\\text{flips}\end{pmatrix}$$

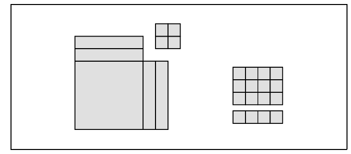


$$(2i)(-3i) = -(-6) = +6$$

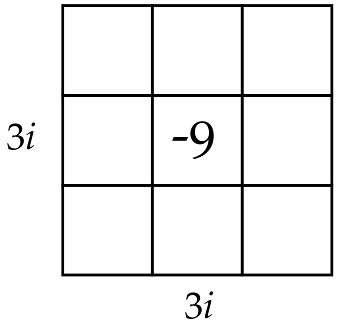
$$\begin{pmatrix}\text{1 half}\\\text{flip}\end{pmatrix}\begin{pmatrix}\text{1 whole}\\+\text{ 1 half flip}\end{pmatrix} \rightarrow \begin{pmatrix}\text{2 whole}\\\text{flips}\end{pmatrix}$$



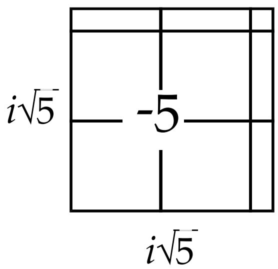
At this stage we will just introduce the idea of imaginary numbers, but you will learn more about them and work with them more in Intermediate Algebra.



For now we will use imaginary numbers only to describe the square root of negative numbers. Such results will be written as shown below:



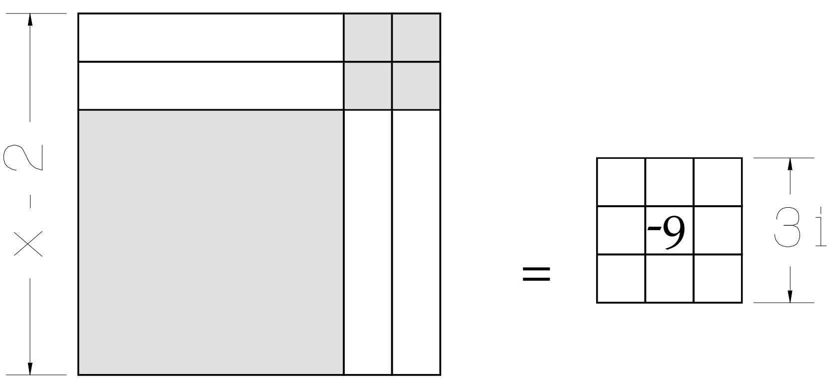
$$\sqrt{-9} = 3i$$



$$\sqrt{-5} = i\sqrt{5}$$

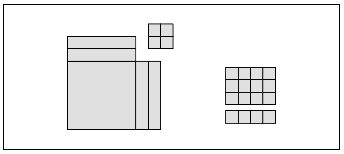
Notice that we write the symbol *i* after a number but before a root.

This notation will allow us to give solutions in situations such as the example shown earlier:



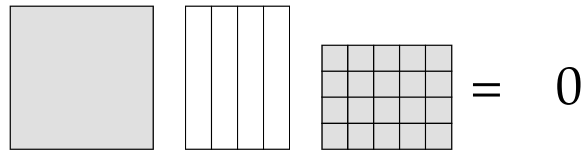
The solution will look like this:

$$
\begin{aligned}
(x+2)^2 &= -9 \\
x-2 &= \pm\sqrt{-9} \\
x-2 &= \pm 3i \\
x &= 2 \pm 3i
\end{aligned}
$$



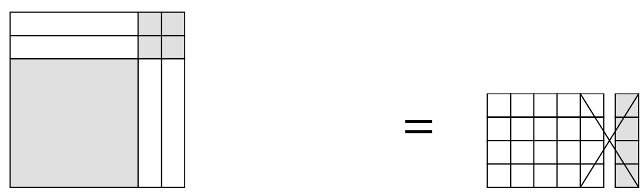
Worked out from the beginning, a quadratic equation of this type will look something like this:

$$x^2 - 4x + 20 = 0$$

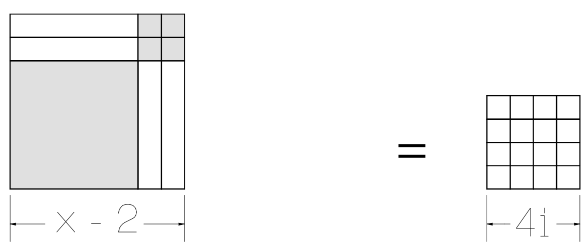


$$x^2 - 4x = -20$$

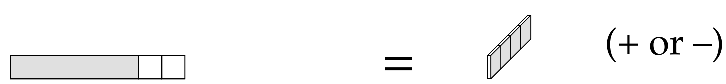
$$x^2 - 4x + 4 = -20 + 4$$



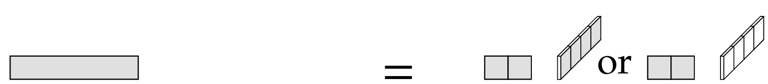
$$(x - 2)^2 = -16$$



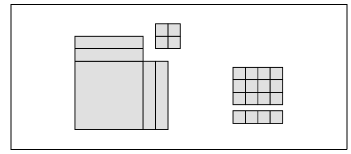
$$x - 2 = \pm 4i$$



$$\boldsymbol{x = 2 \pm 4i}$$



It is often more of a challenge to check a solution having real and imaginary terms than it is to find the solution. One method of checking your results is to work carefully backwards through the problem. To check by substitution,



the procedure is the same as we have outlined before; just substitute the solution values one at a time into the original equation and simplify. Here we will show the check for one solution of this example. You may wish to try checking the other solution.

$$x^2 - 4x + 20 = 0 \qquad\qquad x = 2 - 4i$$

$$(2-4i)^2 - 4(2-4i) + 20 =$$

$$(2-4i)(2-4i) + (-4)(2-4i) + 20 =$$

$$4 - 8i - 8i + (-4i)(-4i) - 8 + (-4)(-4i) + 20 =$$

$$4 - 16i - 16 - 8 + 16i + 20 =$$

$$4 - 16 - 8 + 20 - 16i + 16i =$$

$$-20 + 20 + 0i = 0$$

We will avoid more difficult examples using imaginary numbers in this text. If you continue with further studies of mathematics, you will work with imaginaries again and this introduction will help guide you then.

Exercises

Solve for x:

1. $x^2 - 6x + 25 = 0$

2. $x^2 - 10x + 34 = 0$

3. $x^2 + 4x + 5 = 0$

4. $x^2 + 8x + 20 = 0$

5. $x^2 + 6x + 15 = 0$

6. $x^2 - 2x + 12 = 0$

7. $2x^2 + 6x = x^2 - 6x - 37$

8. $23 - x^2 = 8x - 2x^2$

9. $2x^2 = x^2 + 4x - 11$

10. $x^2 + 6x + 12 = 0$

11. $x^2 - 5x = 5x - 33$

12. $x^2 + 3x + 1 = {}^{-}5(x + 4)$

Section 8
The Quadratic Formula

Introduction

We have learned to use the method of **completing the square** to solve *any* quadratic equation. Once the equations are written in standard form we simply follow the list of specified steps until we reach the solution. Since these steps are always the same, we can use them to write a *formula* for the solution to any quadratic equation written in standard form. We can use this formula, called **the quadratic formula**, to solve any quadratic equation without going through all the steps of completing the square every time.

Deriving the Formula

To derive the **quadratic formula** we will begin with the standard form for all quadratic equations:

$$Ax^2 + Bx + C = 0$$

A, B, and C represent the number or coefficients in the equation; we will perform the steps of completing the square with these variables instead of with the numbers. Reviewing the steps for completing the square, the first thing we do with an equation in standard form is divide through all terms by the coefficient of x^2; so in our formula we first divide each term by A.

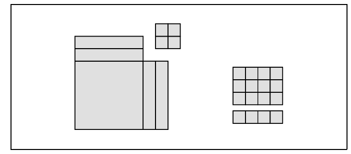
$$x^2 + \left(\frac{B}{A}\right)x + \left(\frac{C}{A}\right) = 0$$

We will call this form of the quadratic equation the **simplified standard form**. Equations which begin with only a single x^2 start out in this form, and all other quadratic equations can be put into this form as soon as we begin working toward the solution. To make the simplified standard form easier to read, we will substitute new letters for the fractions:

$$\frac{B}{A} = D$$

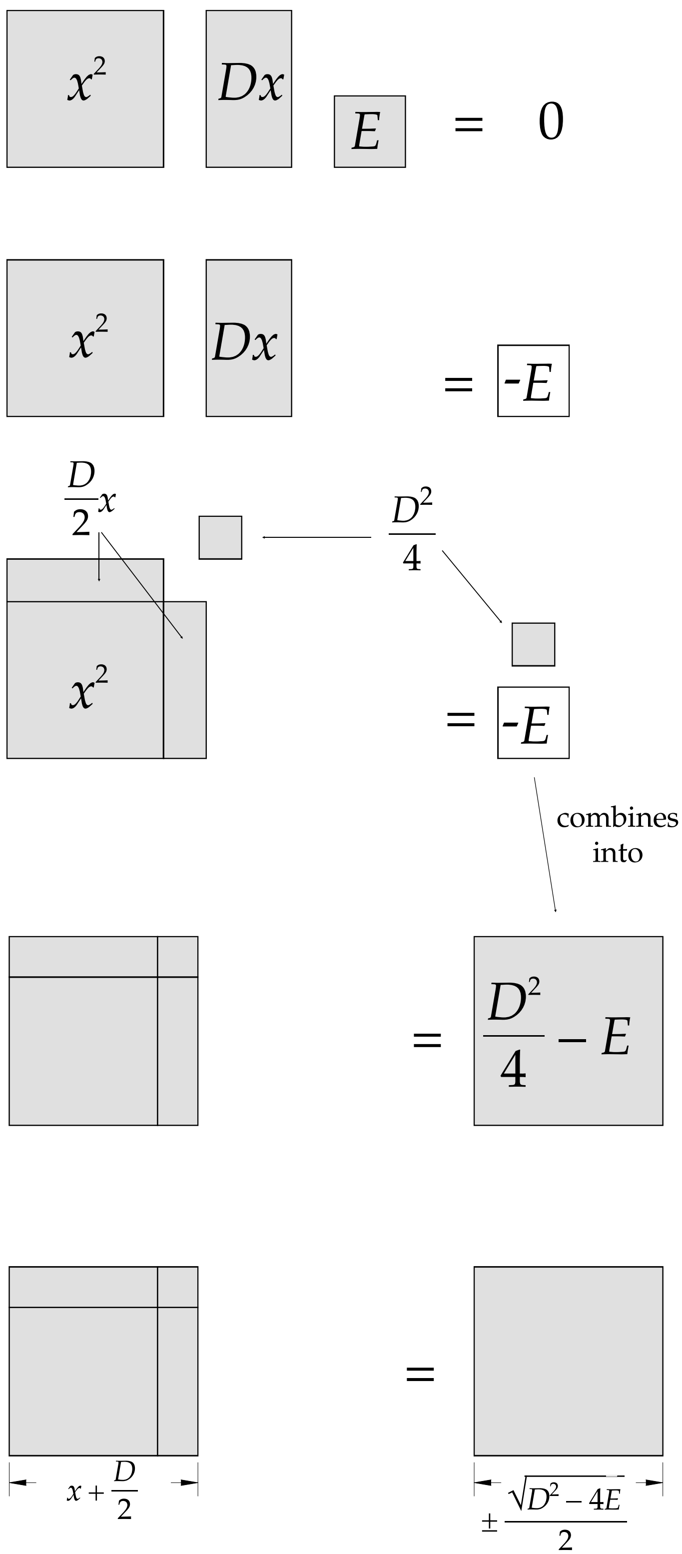
$$\frac{C}{A} = E$$

Now the simplified standard form of the equation reads:



$$x^2 + Dx + E = 0$$

Now we can easily use both pictures and symbols to derive the *quadratic formula*.



$$x^2 + Dx + E = 0$$

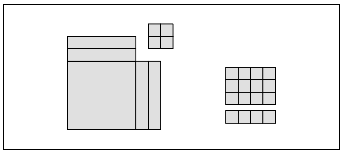
$$x^2 + Dx = -E$$

$$x^2 + Dx + \frac{D^2}{4} = \frac{D^2}{4} - E$$

$$\left(x + \frac{D}{2}\right)^2 = \frac{D^2}{4} - E$$

$$= \frac{D^2 - 4E}{4}$$

$$x + \frac{D}{2} = \pm\frac{\sqrt{D^2 - 4E}}{2}$$



Isolating x, we get our solution in its general form.

$$x = \frac{-D}{2} \pm \frac{\sqrt{D^2 - 4E}}{2}$$

Since both fractions have the same denominator, we can combine the fractions into one, as follows:

$$x = \frac{-D \pm \sqrt{D^2 - 4E}}{2}$$

$$\text{where } D = \frac{B}{A}, \text{ and } E = \frac{C}{A}$$

This formula will give the solution to **any** quadratic equation if the equation is written in simplified standard form or standard form, and then the values of D and E (or A, B and C) are substituted into the solution shown here.

Let's use the formula to do a few examples. Solve the quadratic equation

$$x^2 + 6x + 5 = 0$$

In this equation $D = 6$ and $E = 5$. Substituting these values into the quadratic formula, we find

$$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(5)}}{2}$$

$$x = \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$x = \frac{-6 \pm \sqrt{16}}{2}$$

$$x = \frac{-6 \pm \sqrt{16}}{2}$$

$$x = \frac{-6 \pm 4}{2}$$

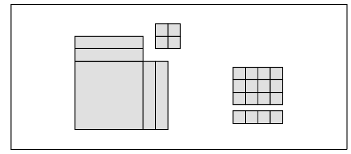
$$x = \frac{-6 + 4}{2} \quad or \quad x = \frac{-6 - 4}{2}$$

$$x = \frac{-2}{2} \quad or \quad x = \frac{-10}{2}$$

$$\mathbf{x = -1} \quad \mathbf{or} \quad \mathbf{x} = -5$$

Check these solutions for yourself to verify that they are accurate.

For quadratic equations where the solutions are integers, the quadratic formula may not save much time in getting to the answer, but for more difficult solutions the formula can save a lot of time.



Solve for the unknown in this equation. Use your calculator to reduce your answers to decimal numerical form:

$$x^2 - 5x - 3 = 0$$

This time $D = {}^{-}5$ and $E = {}^{-}3$.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(-3)}}{2}$$

$$x = \frac{5 \pm \sqrt{25 + 12}}{2}$$

$$\boldsymbol{x = \frac{5 \pm \sqrt{37}}{2}}$$

$$\boldsymbol{x = 5.54138 \quad \textit{or} \quad x = -0.54138}$$

Here is one final example:

$$2x^2 + 3x + 8 = 0$$

This time $D = 3/2 = 1.5$, and $E = 8/2 = 4$.

$$x = \frac{-(1.5) \pm \sqrt{(1.5)^2 - 4(4)}}{2}$$

$$x = \frac{-1.5 \pm \sqrt{2.25 - 16}}{2}$$

$$x = \frac{-1.5 \pm \sqrt{-13.75}}{2}$$

$$\boldsymbol{x = \frac{-1.5 \pm i\sqrt{13.75}}{2}}$$

Exercises

Solve the following equations using the quadratic formula. Do some of the problems by factoring or completing the square and compare your results.

1. $x^2 - 5x - 3 = 0$

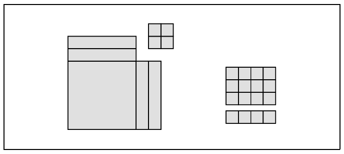
2. $2x^2 - 7x = x^2 + 6$

3. $2x^2 + 6x + 3 = 0$

4. $x^2 - 5x - 2 = -x^2 + 3x - 1$

5. $x^2 - 3x + 5 = 0$

6. $x^2 - x + 6 = 0$



7. $3x^2 - 5x - 2 = 0$

8. $5x^2 - 6x - 4 = 0$

9. $3x^2 - 8x + 5 = 0$

10. $2x^2 - 7x + 6 = 0$

11. $(x + 3)^2 = 12$

12. $x^2 + 7x + 3 = x(2 - x)$

13. $x^2 - 4x = {}^{-}8$

14. $3x^2 + 2x - 1 = 0$

15. $x^2 - 5x = x - 13$

16. $4x(x + 2) = {}^{-}1$

17. $2x(x - 1) = 5$

18. $2x^2 + x = 7$

19. $x^2 - 8x = {}^{-}25$

20. $x^2 - 4x + 5 = 0$