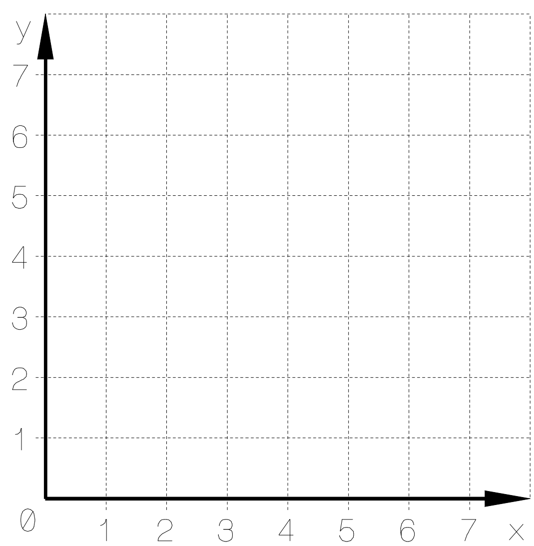
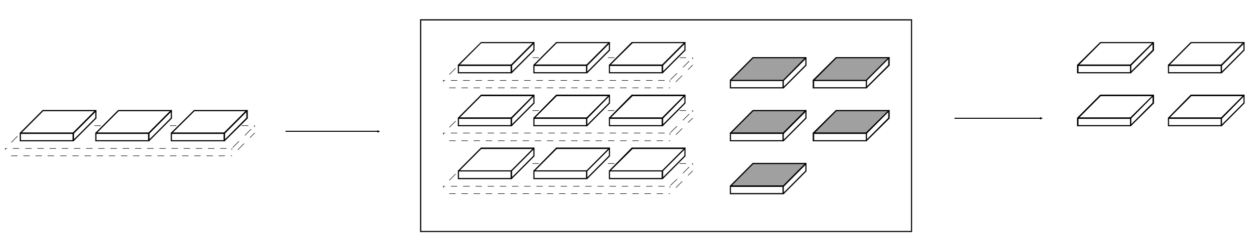
Chapter 12
Rules and Graphs

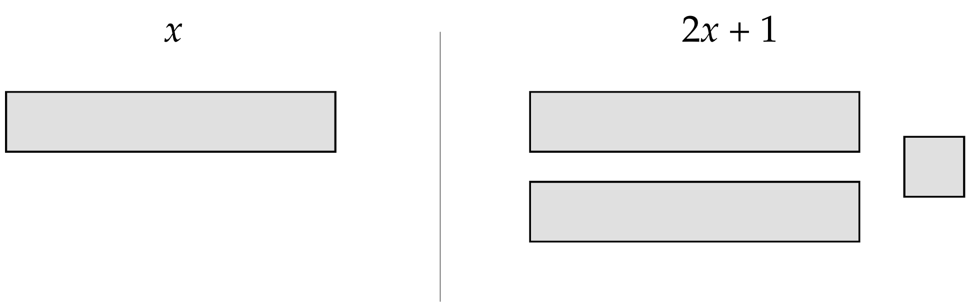




Section 1
Related Numbers

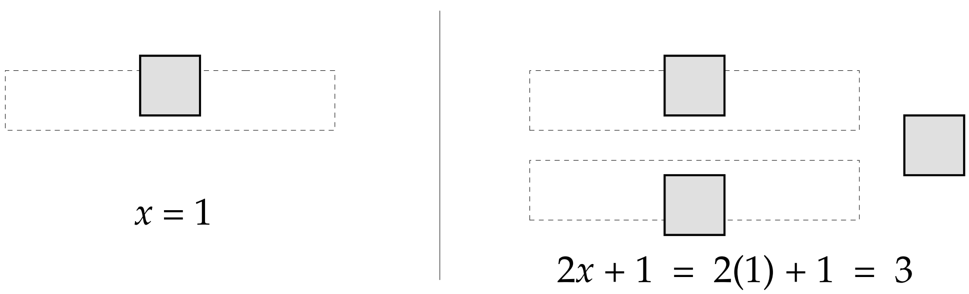
Related Pairs

In this chapter and the next we are going to discuss pairs of variables which are related to each other. For example, consider two variables where the first variable is x and the second variable is $2x + 1$.

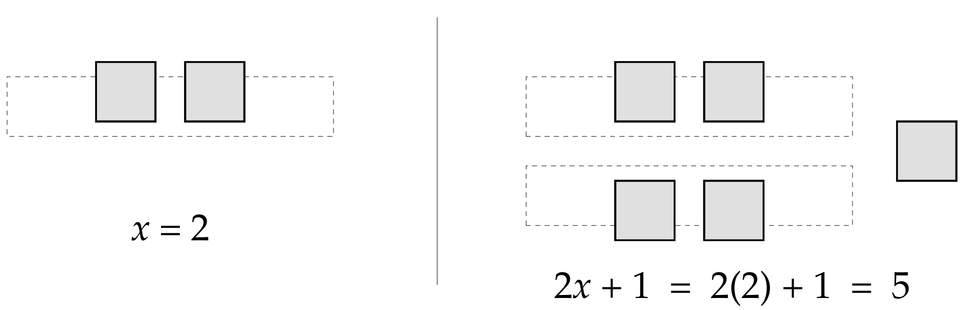


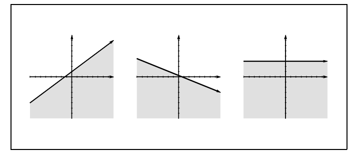
If we agree that the x-bar from the variable on the left has the same value as the x-bar found in the variable on the right, we can see that the values of these two variables will be different, but related to each other.

If the x has a value of 1, then $2x + 1$ will have a value of 3:

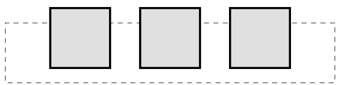


If $x = 2$, then $2x + 1 = 5$.

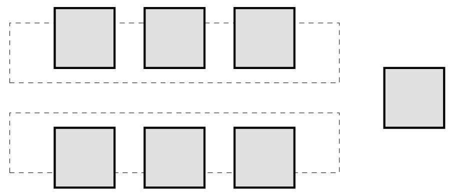




Every time we replace the x-bar with a certain number of chips, each of the other x-bars will also represent that same number of chips, and the two variables will take on values which are specifically related to each other. Using the above example again, if $x = 3$ then $2x + 1 = 7$.

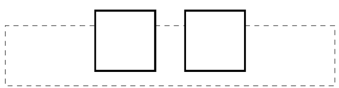


$x = 3$

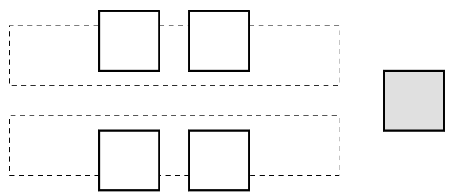


$2x + 1 = 2(3) + 1 = 7$

If $x = {}^{-}2$ then $2x + 1 = {}^{-}3$.



$x = {}^{-}2$

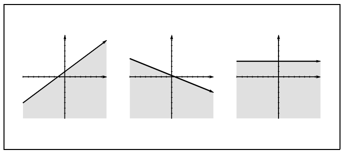


$2x + 1 = 2({}^{-}2) + 1 = {}^{-}3$

We often show the related values of two related variables by making a **table of values**. To make the table, we set up two columns: one representing the values of x, and the other representing the related values for $2x + 1$.

x	$2x + 1$
1	3
2	5
3	7
$^{-}2$	$^{-}3$

When filling in the table, we can choose any values we wish for x, but once an x-value is selected we must use that same x-value in calculating the related value for $2x + 1$. (Normally when choosing x-values we choose simple ones, like 1, 2, 3, 0, $^{-}1$, $^{-}2$, but we could choose any other values we desire). We call x the **independent variable** and $2x + 1$ the **dependent variable**; the first number can be chosen *independently*, but the second value *depends* on the first one.



In the expanded table below, some new values for x have been suggested. How can we calculate the related values of $2x + 1$?

x	**$2x + 1$**
1	3
2	5
3	7
$^-2$	$^-3$
$^-3$	
0	
5	

To find the missing values, we substitute the value of x into the expression $2x + 1$ as we did in the chapter on EXPRESSIONS.

For $x = {^-3}$:

$$2x + 1 \;=\; 2({^-3}) + 1 \;=\; {^-6} + 1 \;=\; {^-5}$$

For $x = 0$:

$$2x + 1 \;=\; 2(0) + 1 \;=\; 0 + 1 \;=\; 1$$

For $x = 5$:

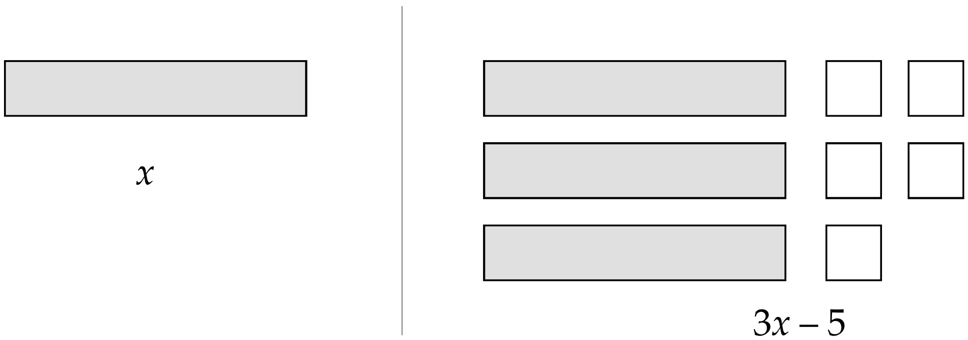
$$2x + 1 \;=\; 2(5) + 1 \;=\; 10 + 1 \;=\; 11$$

The completed table looks like this:

x	**$2x + 1$**
1	3
2	5
3	7
$^-2$	$^-3$
$^-3$	**$^-5$**
0	**1**
5	**11**

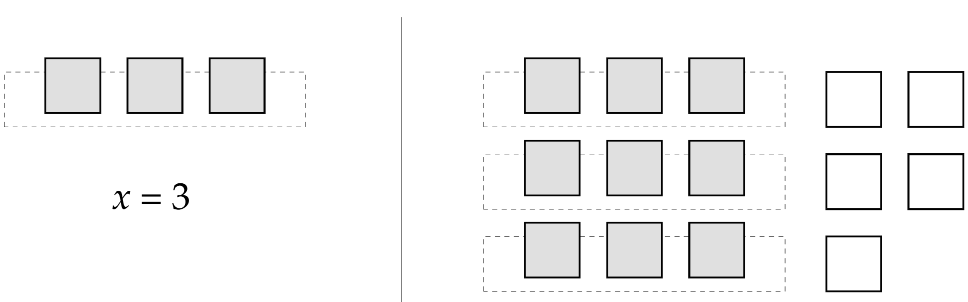
You can see that there will be no end to the number of values we might pick for x; the table could go on forever. Each line of the table represents one entry—one pair of related numbers which are specific values of the two related variables. Each line of the table may be referred to as an **ordered pair**, or a pair of numbers where the first number refers to the value of the independent variable, and the second number represents the related value of the dependent variable.

Now let's use what we have discussed to make a table of values for a different pair of related variables. We will always start with x as the independent variable. This time let's consider the dependent variable $3x - 5$. Use your chips and substitution techniques to complete the following table of values.

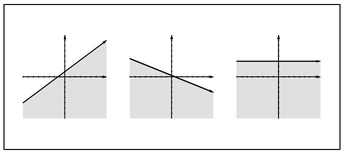


x	$3x - 5$
1	-2
2	1
3	
4	
0	
-1	
-2	

When $x = 3$ the picture will look like this:



$$3x - 5 = 3(3) - 5 = 4$$

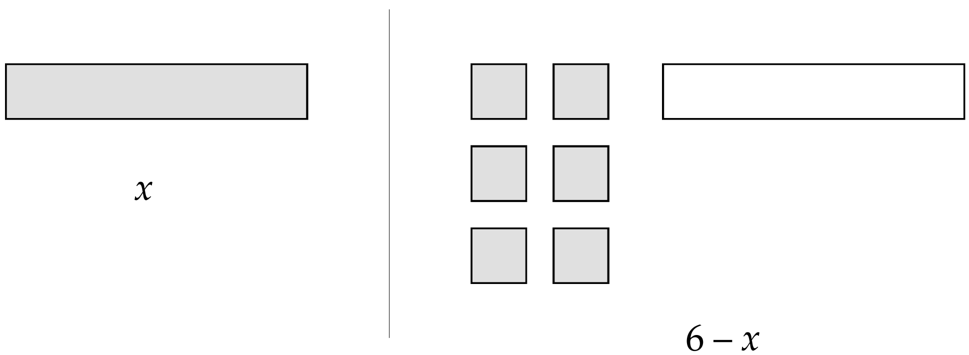


The completed table will be:

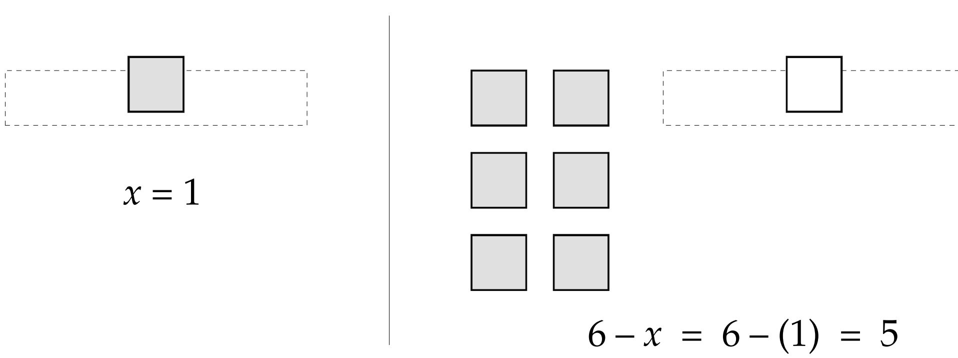
x	**$3x - 5$**
1	$^{-}2$
2	1
3	**4**
4	**7**
0	**$^{-}5$**
$^{-}1$	**$^{-}8$**
$^{-}2$	**$^{-}11$**

In the picture you can easily see that every time we add one more unit chip to the value of the independent variable x, the dependent variable $3x - 5$ will increase by 3 unit chips: one for each of its x-bars. This is an important idea which we will use later for graphing.

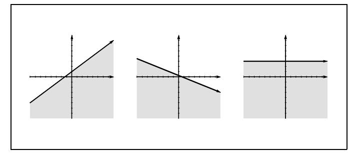
Let's make one more table where the dependent variable will have a negative x-bar.



In this case, when we let $x = 1$, the $-x$ in the dependent expression will be flipped over to $^{-}1$.



When substituting, remember to put the value for x inside a parentheses with the negative sign out in front of the parentheses. This will help you to calculate the correct values for $6 - x$.



Complete this table.

x	$6 - x$
1	
2	
3	
0	
-1	
-2	8
-3	
-4	
-5	
-6	

The completed table looks like this.

x	$6 - x$
1	**5**
2	**4**
3	**3**
0	**6**
-1	**7**
-2	8
-3	9
-4	**10**
-5	**11**
-6	**12**

Exercises

Complete the following tables: When x-values aren't listed, choose your own values for x.

1.

x	$3x - 4$
1	-1
2	
3	
0	
-1	
-2	

2.

x	$2x + 3$
1	
2	
3	
0	
-1	
-2	

3.

x	$5 - x$
1	
2	3
3	
0	
-1	
-2	

4.

x	$3 - 2x$
1	
2	-1
3	
0	
-1	
-2	

5.

x	$-3x$
1	
2	-6
?	
0	0
?	
?	

6.

x	$5x - 3$
?	
?	
?	
?	
?	
?	

Make your own table of values for the following related variables. (Use x for the independent variable.) Choose at least five values for x.

7. x and $4x - 5$

8. x and $-3x + 2$

9. x and $-5 - x$

10. x and $7 - 2x$

11. x and $3x + 1$

12. x and $-4x$

Section 2
Rules, Machines, and a Second Variable

Using *x* and *y*

In the previous section, two related variables were both represented using the same letter; one variable was x, the other was represented in terms of x. In this section we will expand the idea slightly by calling the two variables by different letters — x and y — and by using a rule. The rule will be an equation showing the relationship between y and x.

For example, rather than saying that our two related variables are x and $3x + 5$, we now will say that the two variables are x and y, and the rule relating them is

$$y = 3x + 5$$

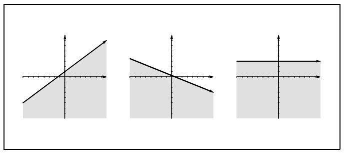
A table of values expressing this would be
Sometimes, rather than listing a rule and a whole table of values for x and y, we may want to talk about just the pairs of values. If we list only the pairs, *it is convenient to agree that we will list the pair in parentheses with x first and y second, separated by a comma.*

Each pair of x and y is called an **ordered pair**, and when we see an ordered pair written in this way we know that the two values are usually connected by some rule.
The rule above can now be shown as a list of pairs:

$$y = 3x + 5: \quad (0, 5), (1, 8), (2, 11), (^{-}1, 2)$$

Machines and Rules

Another way to illustrate a rule is to imagine a factory or machine that takes x's and manufactures y's. The individual machine is the rule; the input is the x and the output is the y.
The machine uses the rule as a pattern. Each y is set up as an expression using x-bars and unit squares. If x is 2, then 2 units are put in the place of each x-bar in the pattern and the total is sent out as y:
If x is $^{-}3$, then $^{-}3$ chips are put on each x in the pattern:
If the rule is $y = 2x - 3$ then the machine would take each x and pair it with three less than twice x. The machine would look like this:
If y = $-x + 4$ then the machine would perform the operation of taking the opposite of x and adding four to it. The machine would look like this:
Note that each of these machines involves only two basic operations—multiplying x by a quantity and then adding another quantity to the result.



Some rules can even be constants which don't depend on x. An example would be $y = 3$. The machine would be:

The value of y would be 3 for any choice of x. Here is a list of ordered pairs:

$$(1, 3), (2, 3), (0,3), ({}^{-}1, 3), ({}^{-}2,3)$$

Working Backwards to Find *x*

If we are given a rule like $y = 2x - 5$, it is possible to take any given value of y and use the rule to calculate the corresponding related value for x.

For example, using $y = 2x - 5$, we can find the x-value corresponding to $y = 9$ by substituting 9 for y and then solving for x in the remaining equation:

$$y = 2x - 5$$

$$\begin{aligned} \text{Since } y = 9, \text{ then } \quad (9) &= 2x - 5 \\ 9 + 5 &= 2x - 5 + 5 \\ 14 &= 2x \\ \frac{14}{2} &= \frac{2x}{2} \\ 7 &= x \end{aligned}$$

Summary

Two related variables and their values can be illustrated in several ways:

- **An equation defining y in terms of x using a rule:**

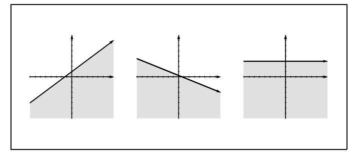
$$\mathbf{y = 3x + 5}$$

- **A *table of values*:**

Rule: $\mathbf{y = 3x + 5}$

x	y
0	5
1	8

2	11
-1	2



- **A list of *ordered pairs*:**

(0, 5), (1,8), (2, 11), (-1, 2)

- **A *machine* with a clear rule to determine y from x:**

Exercises

Finish the tables:

1. $y = 3x$

x	y
-5	-15
5	?
10	?
3	?
1	?
-1	?

2. $y = -2x + 1$

x	y
-1	3
0	1
1	?
3	?
2	?
-5	?

3. $y = -x$

x	y
6	-6
-4	4
0	?
2	?
-3	?
17	?

Using the given rule, complete the list of ordered pairs by filling in the missing y's:

4. $y = x - 17$: (2, _), (5, _), (-5, _), (17, _)

5. $y = 3x + 1$: (2, _), (5, _), (-5, _), (17, _)

6. $y = 17 - x$: (2, _), (5, _), (-5, _), (17, _)

Using the given rules, choose four x's and then calculate the matching y's:

7. $y = 17x - 17$ (_ , _), (_ , _), (_ , _), (_ , _)

8. $y = x + 17$ (_ , _), (_ , _), (_ , _), (_ , _)

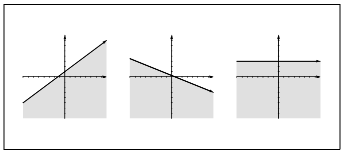
9. $y = 17 - x$ (_ , _), (_ , _), (_ , _), (_ , _)

10. $y = 3x - 3$ (_ , _), (_ , _), (_ , _), (_ , _)

11. $y = 3x - 4$ (_ , _), (_ , _), (_ , _), (_ , _)

12. $y = 4x - 5$ (_ , _), (_ , _), (_ , _), (_ , _)

13. $y = -x - 1$ (_ , _), (_ , _), (_ , _), (_ , _)



14. $y = -x + 1$ (_ , _), (_ , _), (_ , _), (_ , _)

15. $y = 7$ (_ , _), (_ , _), (_ , _), (_ , _)

Take the given rule and the given value of y and work backwards to find the related value of x:

16. $y = 17x - 17$ $y = 17$

17. $y = x + 17$ $y = 21$

18. $y = 17 - x$ $y = 0$

19. $y = 3x - 3$ $y = {}^{-}9$

20. $y = 3x - 4$ $y = {}^{-}10$

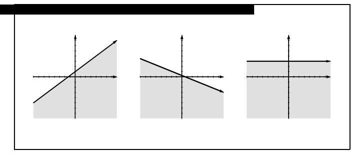
21. $y = 4x - 5$ $y = 15$

22. $y = -x - 1$ $y = {}^{-}6$

23. $y = -x + 1$ $y = {}^{-}4$

24. $y = {}^{-}3x + 1$ $y = 1$

25. $y = {}^{-}3x - 1$ $y = 2$



Section 3
Graphs and Coordinates

Chips and Ordered Pairs

One value from an ordered pair can easily be represented by a column of unit chips. For the pair (1, 3), we line up a column of 3 chips to represent y: If we want to show a list of pairs from an equation, we first prepare a column for the y -value in each ordered pair:

$$y = 2x + 1: \quad (1, 3), (2, 5), (3, 7)$$

We then take a horizontal number line and arrange the columns (y) at the place on the line which matches that pair's x. The position on the line represents x and the height of the column represents y for each ordered pair.

If negative numbers are included, we expand the x number line to the left to represent negative values of x; if the y value is negative, we arrange the chips down instead of up.

In summary, we can show a list of pairs as a group of columns:

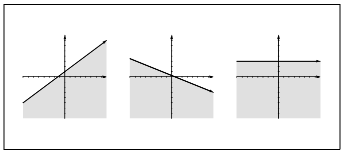
From Chips to Points

To simplify the picture, we can use the position of the center of the end of the column (top or bottom) to represent the related values of both x and y; the chips are no longer needed. This means that an ordered pair can be shown as a single point:

To represent several pairs at once, we simply draw each point separately in the same picture:

The Coordinate System

The idea we have just developed is called the **coordinate system**. The number lines showing the values for x and y are called **axes**; individually they are called the ***x* axis** and the ***y* axis**. An ordered pair now represents two



numbers—the first is the x **coordinate** and the second is the y **coordinate**. The point where x and y are both zero — (0, 0) — is known as the origin.

Each ordered pair can be shown as a single point. When we illustrate a list of ordered pairs as points on the coordinate system, we call this a **graph**.

Here are two examples of graphs:

Graphing Points

To graph a point, we start at the origin. We then move x units to the right (if x is positive) or to the left (if x is negative). The second step is to move up y units (if y is positive) or down (if y is negative). The ending point represents that ordered pair (x, y):

Did you notice that x and y can be both positive, both negative, or one positive and the other negative?

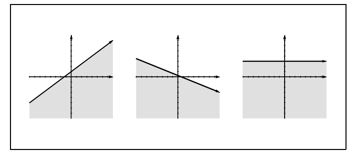
Alternative Methods

An ordered pair can be graphed in several different ways; the result is of course the same. The procedure shown above is to begin at the origin, move horizontally (right or left) for x, then vertically (up or down) for y:
It is obvious that we can accomplish the same thing by first locating the y by moving up or down and then locating the x by moving right or left:
Finally, we can locate the x and y separately; the point is graphed at the intersection of the lines on the grid:

These methods are essentially the same—the choice is yours. You must keep our agreements:

- **In an ordered pair, the x is listed first and the y second.**
- **On the graph, the x axis is horizontal (left to right) and the y axis is vertical (up and down).**
- **On the coordinate system, we count from the origin (0, 0).**
- **On the x axis, positive numbers are to the right of the center and negative numbers are to the left of the center.**
- **On the y axis, positive numbers are up from the center and negative numbers are down from the center.**

These agreements are a sensible method of ensuring that we are all talking about the same things; the choice of positive and negative directions is natural. If we were inventing mathematics ourselves, we could alter the order of x and y or the direction of the axes. But since this system is already in use, we will abide by the rules so that we are all speaking the same language.



Coordinates that are Fractions or Irrationals

The coordinates of a point can be fractions or irrational numbers. The method of finding the proper position of such a point is the same, but you must determine the approximate locations on the axes.

Fractions such as $^3/_2$ should be written as mixed numbers ($1\,^1/_2$) to help you find the locations. Square roots should computed on a calculator or approximated (as in the chapter on POWERS AND ROOTS).

Exercises

Set up each pair as a column of chips at the correct x position:

1. $(5, {}^-1)$

2. $({}^-1, 5)$

3. $(3, 1)$

4. $(0, 3)$

5. $({}^-5, 0)$

Graph each ordered pair:

6. $(1, 6)$

7. $(^7/_2, 3)$

8. $({}^-3, {}^7/_2)$

9. $({}^-3, -\,^7/_2)$

10. $({}^-5, 4)$

Graph each list of ordered pairs on the same coordinate graph.

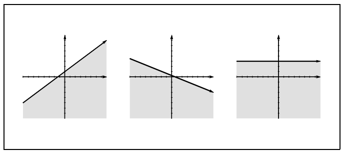
11. $(0, 0), (3, 3), (5, 5), ({}^-1, {}^-1), ({}^-3, {}^-3)$

12. $(0, 7), ({}^-1, 8), (2, {}^-5), (3, 4), (4, 3)$

13. $(2, 1), (1, 2), (2, 6), (6, 2), (3, 4), (4, 3), (5, {}^{12}/_5)$

14. $(0, 1), (1, 8), ({}^-1, 8), (2, 6), ({}^-2, 6), (3, 1), (4, {}^-6), ({}^-3, 1), ({}^-4, {}^-6)$

15. $(0, 0), (1, {}^-1), ({}^-2, 2), (2, {}^-2), ({}^-5, 5), (7, {}^-7), ({}^-4, 4)$



Section 4
Graphs of Lines

Lines on the Coordinate Graph

In the previous section, we graphed individual points and lists of points. Now we will learn to graph lists of points from a rule. Each rule will have a specific shape.

In this chapter, we will limit ourselves to rules that include *x*'s, numbers, adding/subtracting, and multiplying/dividing. We will *not* consider rules or equations that include x^2, x^3, $1/x$, and other more complicated formulas.

To graph a rule, follow these simple steps:

- **Choose several *x* values. Zero and small numbers are usually best.**
- **Calculate the matching *y* for each *x*, making an ordered pair (*x*, *y*).**
- **Put each ordered pair onto the graph.**
- **Connect the points.**

Here are the graphs of two examples:

Lines and Rules

The graphs shown above are in the form of a **line**. A rule that is a line on the graph is called **linear**. The following rules are linear:

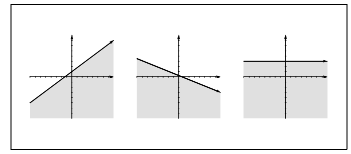
$$y = 3x + 4$$
$$y = {}^{-}2x + 5$$
$$y = 10 - x$$
$$y = 2x$$

These rules have an *x* term and/or a number term and are in the form

$$y = mx + b$$

where *m* and *b* are any numbers. Such a rule will be a line. As you might expect, the reverse is also true—any graph that is a line has a rule which can be written in the form of $y = mx + b$.

Not all equations graph as lines. The following equations are *not* lines; they involve higher powers of *x* (x^2, x^3), division by *x*, or products of *x* and *y*:



$$y = x^2 + \frac{1}{x}$$

$$xy + 3 = x$$

To graph a line, we can follow the steps in Section 3, but since we know the graph will be a line, we will only need three (x, y) pairs. Here are the steps:

- **Confirm the rule is a line and has the form $y = mx + b$.**
- **Choose any three x values.**
- **Calculate the y values.**
- **Graph each ordered pair as a point.**
- **Draw a line through the three points.**

If we never made errors, two points would be enough. Using three points helps to ensure that we graph the correct line; if the three points are not in a line, than it is obvious that we have made a mistake in calculating the y values or in placing the values on the graph.
Here are two more examples of graphing a linear rule:

A Special Case

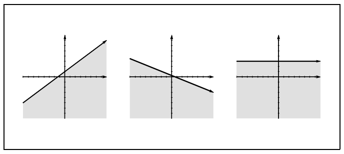
What is the meaning of the following rule?

$$y = 5$$

If x is 2, what is y? If x is $^{-}4$, what is y? The answer is very simple—*whatever x you choose, the answer is that y is 5*! As a table, it looks like this:

As a machine, it looks like this:
The graph of $y = 5$ is a simple horizontal line. We choose several ordered pairs where the y-value is 5:
Notice that this is a rule—it is just a very simple one. As a machine, we can see that we can send in anything that we want, but it always sends out three chips.

As a graph, $y = 5$ is a horizontal line because as x changes, y stays level at exactly five. The line travels along from left to right, but y remains constant.



The y bar.

We will now create a chip for y. Since y represents some unknown number of unit squares, we will show it as a bar that is 1 unit wide and unknown number of units long:
The bars for y and x are similar but stand for different unknown quantities:
We will use the symbol **$-y$** to stand for the opposite of y. The other (white) side of the bar will represent $-y$.
The following cautions may be helpful:

- **x is *shown a*s longer than y to help us tell the two apart; x is not necessarily greater than y.**

- **x may be greater, less than or equal to y in any given problem.**

- **x and y are shown with their shaded side up, but they may stand for numbers that are negative, zero, or positive.**

- **The symbols $-x$ and $-y$ represent the opposites of x and y.**

- **The symbols $-x$ and $-y$ may stand for numbers that are negative, zero, or positive.**

Equations and Rules

We will often encounter an equation involving both x and y. An **equation** is defined as any two expressions that are given as equal to each other. This is not quite the same as a rule because a rule gives y as some expression including numbers and x's.

Some equations can be rewritten to represent rules by the use of our previous techniques for solving equations. Our goal is to rearrange the equation to "isolate" y. For example, begin with:

$$x + y = 12$$

Isolate y:

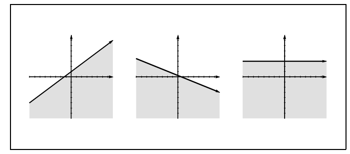
$$x - x + y = -x + 12$$
$$y = -x + 12$$

Notice that we are only rearranging the equation to show the rule. In general, *you cannot solve the equation to determine one x or y.* (You can find the rule for y in terms of x.) Here are several more examples:

The same problem shown with symbols only:

$$4x + 2y = 6$$
$$4x - 4x + 2y = {}^{-}4x + 6$$

$$
\begin{aligned}
2y &= {}^-4x + 6 \\
\frac{1}{2}(2y) &= \frac{1}{2}({}^-4x + 6) \\
y &= {}^-2x + 3
\end{aligned}
$$



A more difficult example:

The same problem shown with symbols only:

$$
\begin{aligned}
3x + 6 - 2y &= y + 3 + 2(x - 6) \\
3x + 6 - 2y &= y + 3 + 2x + {}^-12 \\
3x + 6 - 2y &= y + 2x + {}^-9 \\
3x + 6 - 2y - y &= y - y + 2x + {}^-9 \\
3x + 6 - 3y &= 2x + {}^-9 \\
3x + {}^-3x + 6 + {}^-6 + {}^-3y &= 2x + {}^-3x + {}^-9 + {}^-6 \\
{}^-3y &= -x + {}^-15 \\
-\frac{1}{3}({}^-3y) &= -\frac{1}{3}(-x + {}^-15) \\
y &= \frac{1}{3}x + 5
\end{aligned}
$$

Summary

If a rule is given as an equation in x and y, here are the steps to graph the line:

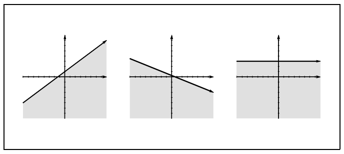
- **Solve the equation for y:**
  Multiply out any quantities in parentheses.
  Add y's to both sides to "isolate" the y's on one side.
  Add x's to both sides to leave x's only on the other side.
  Add units to both sides to leave units and x's on the other side.
  Multiply (or divide) both sides to leave only one y.

- **Pick three values of x. Calculate the matching y values to give ordered pairs (x, y).**

- **Plot the points on the graph.**

- **Draw a line through the points.**

It should be obvious that the first steps are almost identical to the steps for solving equations with only x. The difference is that we solve for y as equal to an expression containing x's and numbers. As before, it does not matter which side you pick for y; you may choose, however, to write the final rule with the y on the left.



Exercises

Which of the following rules are linear? (a linear rule can be written in the form $y = mx + b$)

1. $y = 999x + 1234$

2. $y = 125 - .532x$

3. $y = x$

4. $y = 0$

5. $y = 2x^2 + 1$

6. $y = 0x^2 + 2x + 1$

Graph the following rules:

7. $y = x + 1$

8. $y = x - 3$

9. $y = {}^{-}3x + 7$

10. $y = 2x - 5$

11. $y = {}^{-}2x - 5$

12. $y = 6$

Change the following equations to rules by solving for y. Graph the result by plotting at least three points for each equation:

13. $x + y = 3x + 2y + 3$

14. $2x + y = 3x - 5$

15. $2x + y = {}^{-}3x + 7$

16. $3y + x = 2x - 2y + 15$

17. $2x + y - 6 = 3x - 5$

18. $3x + 2y + 5 = -x + 9$

19. $2(x - 3) = -x + y$

20. $x + 3(2 - x) = x + 2(y + 1) - y$

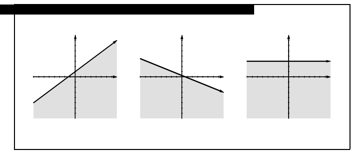
21. $2(y + x) = 2x + y + 6$

22. $3[2 - 4(y + 3)] = 6(x - 1)$

23. $y + \frac{1}{2} = 2x - \frac{3}{2}$

24. $5y + 6 = 3x + 1$

25. ${}^{-}1[{}^{-}1 - 1(y - 1)] = -x - 1$



Section **5**
Slopes and Intercepts

Uphill and Downhill

Consider the following lines as side views of roads in the real world. We will imagine traveling the roads from *left to right*.
Like a road, a line on the graph is either uphill, downhill, or flat. We say the line has **slope**; as we move to the right we say that an uphill line has positive slope, a downhill line has negative slope, and a flat line has zero slope.

If we want to assign a number to represent the slope, it makes sense to call a flat line "0" and to give higher numbers to steeper uphill slopes. It also makes sense to give increasing downhill slopes values which are increasingly negative ($^{-}1$, $^{-}2$, $^{-}3$, …):
The formal definition is as follows:
On the graph, you can see that the slope measures how far the line moves up for each unit that it moves to the right.

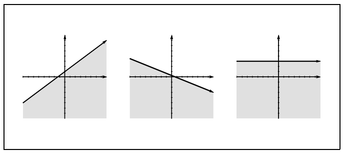
The slope can be measured at any point. It is useful to think of a line as a series of stair steps that move 1 to the right, then up or down, 1 more to the right, then up or down again, etc.
Here are some other examples of slopes:

Another Definition

We can also define the slope as a fraction or ratio. In the line shown on the next page, the slope is clearly 2 when we measure over 1 and up 2. On the same line, if we measure over 2 or 3 on the x axis, the y value increases at a rate that is in the same *proportions.* This means that the y increase *relative* to the x increase is still the same:
We write this as a fraction with the change in y (rise) on top and the change in x (run) on the bottom. *The slope can also be defined as the rise over the run.*
Here are some other examples:

Two Definitions Compared

We now have two seemingly different definitions of slope:
It would not be useful if the definitions were actually different; in fact they are really the same:
The first definition is in fact a ratio of the distance up relative to 1, while the



second definition is the ratio of rise to run over any distance. The two results are identical, so in each problem we can use the definition that is more convenient or more comfortable.

The Slope Between Two Points

It takes time to graph points and lines. If we wish to know the slope of the line between two points, we can calculate the slope *without* making the graph. First, look at the slope of the line between (2, 1) and (5,7):

The rise is 6 and the run is 3. We can find this by taking the difference of the y coordinates (rise) and the difference of the x coordinates (run):
Because the individual x or y coordinate represents the distance of a given point from one axis of the graph, subtracting the x or y coordinates of two points gives the horizontal or vertical difference between the two points. The slope is the vertical difference divided by the horizontal difference:
As seen above, you can use either point as the first point, as long as you use the same point first when subtracting for both x and y; you do not need to memorize this formula; if you understand the concept of a slope, you will be able to construct the formula any time you need it.

Intercepts

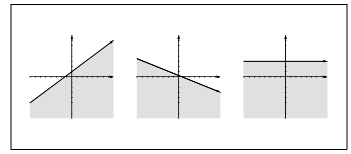
A place where a line hits the y axis or x axis is called an **intercept**. As you might expect, if the line hits the x axis we call the point an ***x* intercept** and if it hits the y axis we call the point a ***y* intercept**.
In this book, we will use the term **intercept** loosely; it will mean either the ordered pair (point) at the intersection or the coordinate on the axis. For example:
We can see that x intercepts always have coordinates of (_ , 0) and y intercepts always have coordinates of (0, _). In addition, we can see that a linear rule may have a y intercept or it may have both a y intercept and an x intercept. (Can it have only an x intercept?). The intercepts may be positive, negative, or zero.

Finding Intercepts

Intercepts can be found by graphing the rule and then examining the graph. This can be slow; if the intersection points are not exact integers, it can also be inaccurate:

If we use the idea that the x intercept is a point where $y = 0$, then we can solve the equation algebraically as follows:

$$y = 2x - 5$$



$$\text{for } x \text{ intercept, } y = 0$$
$$(0) = 2x - 5$$
$$5 = 2x$$
$$\frac{5}{2} = x$$

To find the y intercept, we use the idea that points on the y axis have $x = 0$:

$$y = 2x - 5$$

Since $x = 0$

$$y = 2(0) - 5$$
$$y = 0 - 5$$
$$y = {}^{-}5$$

To find the intercepts:

- **For the x intercept:**
  1. On the x axis, y is 0.
  2. Substitute $y = 0$ in the formula for the line.
  3. Solve for x. This is the value of the x intercept.

- **For the y intercept:**
  1. On the y axis, x is 0.
  2. Substitute $x = 0$ in the formula for the line.
  3. Solve for y. This is the value of the y intercept.

It is not necessary (and not helpful) to attempt to memorize this process. For each intercept, decide from your understanding of graphing which coordinate is 0, then substitute and solve the equation.

Exercises

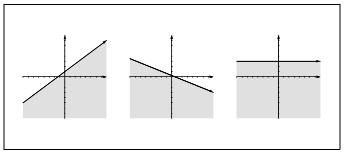
Find the slopes of these lines by examining the graphs:

Find the slope of the line between each pair of points without graphing. Then graph the line to see if you are correct:

3. (2, 1), ($^-2$, 5)

4. ($^-1$, $^-1$), (5, 6)

5. (1, 1), (4, 4)



6. (2, 1), (7, 1)

7. (12, 1), (1, 12)

8. ($^{-}1$, $^{-}2$), (4, $^{-}5$)

Find the slope and intercepts by graphing:

9. $y = -x - 7$

10. $y = {}^{-}3x - 12$

11. $2y + 3 = 4x + 9$

Find the intercepts by using the formula alone, without graphing:

12. $y = x - 6$

13. $y = {}^{-}2x - 2$

14. $3y + 3 = -x + 9$

Find the slope between the two intercepts. Remember that each intercept is an ordered pair.

Example: *y* intercept is 5, *x* intercept is 2:

Solution: Points are (0, 5) and (2, 0). Slope is $\frac{0-5}{2-0} = \frac{{}^{-}5}{2}$

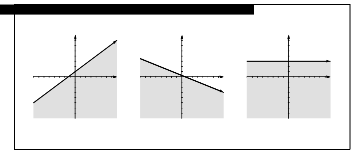
15. The *y* intercept is (0, 3) and the *x* intercept is (1, 0).

16. The *y* intercept is (0, $^{-}3$) and the *x* intercept is (6, 0).

17. The *y* intercept is (0, 6) and the *x* intercept is ($^{-}3$, 0).

18. The *y* intercept is $^{-}1$ and the *x* intercept is $^{-}1$.

19. The *y* intercept is $^{-}5$ and the *x* intercept is $\frac{3}{2}$.



Section 6
Graphing with Slopes and Intercepts

The Slope-Intercept Method

Graphing a line by making a table of points can be lengthy and inefficient. This section will cover a better method that allows you to merely *look* at the rule and then immediately graph it. In addition, you will be able to look at any graph of a line and then immediately know the rule that it represents. To accomplish these feats, we will use the ideas of slope and intercept from the previous section.

First, examine the following graphs, their slopes, and their y intercepts:

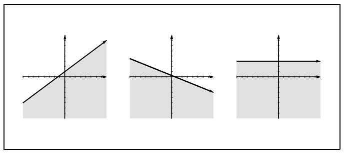
We can see that when the rule is written in the form of $y = mx + b$, m is the slope and b is the y intercept. This is true for all linear rules, even if there is a negative slope or a zero slope.

You may want to rearrange the terms to put the rule in the form of $y = mx + b$. You can add missing ones and zeros to make the equation fit the format:

$y = 2x$	becomes	$y = 2x + 0$
$y = 10 - x$	becomes	$y = {}^{-}1x + 10$
$y = 5$	becomes	$y = 0x + 5$
$y = \frac{x}{2}$	becomes	$y = \frac{1}{2}x + 0$
$y = 3x - 5$	becomes	$y = 3x + {}^{-}5$
$y = \frac{{}^{-}2x}{5}$	becomes	$y = \frac{{}^{-}2}{5}x + 0$

To graph a rule or equation using this method:

- **If necessary, solve the given equation for y.**



- □ **Rewrite the rule in the form of $y = mx + b$. Write all terms as adding negatives rather than subtracting.**

- **Determine the slope (m) and the y intercept (b).**

- **On the graph, start at the y intercept and go to the right at the correct slope.**

- **Draw the line.**

You have two methods to draw the line when the slope is a fraction such as $\frac{2}{3}$ — either go over one and then up $\frac{2}{3}$, or go over 3 (run) and up 2 (rise).

Here is how to graph

$$y = \frac{1}{2}x + 3$$

Here is how to graph

$$y = {}^{-}2x \qquad (y = {}^{-}2x + 0)$$

The Slope-Intercept Method—Why it Works

The slope-intercept method is not just a way to memorize how to make graphs quickly—there is a simple reason *why* the m turns out to be the slope and the b turns out to be the y intercept. As we can see from the next two diagrams, m is the number of x's; it controls how much y goes up each time we increase x by 1. If there are 3 x's, then y goes up 3 each time x goes up 1.

We can also see that b is the number of unit chips we have when x is zero. This is the "starting point" on the y axis or the y intercept.

Below, we can see a machine for $y = 3x + 5$. If we think of x as first 0, then 1, then 2, we are moving along the x axis from the origin to the right. Because the machine has 5 units ($b = 5$), we start at 5 when x is 0. Because the machine has 3 x's ($m = 3$), y goes up 3 each time x goes up 1. This is a slope of 3.

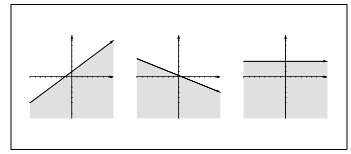
Special Lines

We have already looked at lines such as

$$y = 3 \quad \text{or} \quad y = {}^{-}2$$

If you use the slope-intercept method, first rewrite the equation in the $y = mx + b$ form as:

$$y = 0x + 3 \quad \text{and} \quad y = 0x + {}^{-}2$$



We now can see that these lines have a slope of 0 (they are flat) and have y intercepts of 3 and $^{-}2$:
What about an equation such as

$$x = 5\,?$$

First, this is *not* really a rule because it gives no instructions to determine y from x. We can graph all the ordered pairs with $x = 5$ and we see that we get a vertical line. Using (5, 1), (5, 2), and (5, $^{-}1$):

The slope of this line is also hard to define because it is infinitely steep. By our previous definition, it has a run of zero and division by zero is not defined. We must then agree that the slope is not defined.

Exercises

Graph the following lines by using the slope and y intercept:

1. $y = 2x + 1$

2. $y = 2x + 3$

3. $y = {}^{-}2x - 1$

4. $y = -\frac{1}{2}x + 3$

5. $y = x - 2$

Write the rule for the following lines. (Hint: Determine the slope and y intercept, then write the equation in the format of $y = mx + b$ by filling in m and b.:

Solve the following equations for y and then graph using the slope-intercept method:

8. $y - 2 = x + 1$

9. $y + 5 = 2x + 4$

10. $2y - 1 = x - 7$

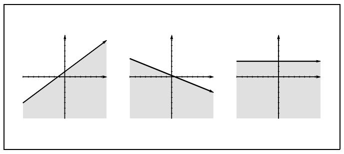
11. $2y + 6x = 10$

12. $3y + 2x = 12$

13. $2y - 2x = x - 4$

14. $y + 4 = -x - 2$

15. $2y = {}^{-}6 + y$



16. $x + y + 11 = 5 + 3x + 6$

17. $3y - 5 = 7 + 2x$

Which of the following equations have undefined slopes?:

18. $0 = y$

19. $y + x = y + 2x + 3$

20. $3(x + 2y) - 13 = 2x + 6(y - 2)$

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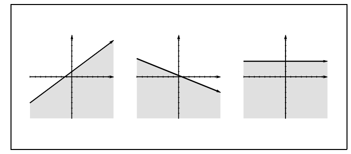
ing With Two Intercepts



Standard Form

Equations involving x and y do not have to be written in the slope-intercept form to generate a straight line graph. Any equation which has both x and y to the first power will generate a straight line. More specifically, for the graph to be a straight line, neither variable can have exponents higher than one (like x^2 or y^3); no two variables can be multiplied together (like xy) and neither variable can appear in the denominator of a fraction (like $3/x$ or x/y). These examples will help to illustrate.

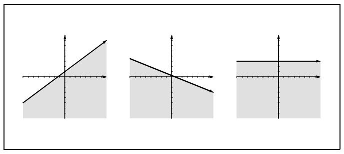
$3x + 2y = 6$	Straightline graph
$y = 2x - 3$	Straightline graph
$x^2 + 2 = 5y$	NOT a straight line
$2xy - 5x = 10$	NOT a straight line
$\frac{3}{x} + 5 = 2y$	NOT a straight line

Any equation which does generate a straight line graph can be written in many ways. Other than the slope-intercept form, another common way of writing linear equations is called **standard form**. Standard form always shows the x-term added to the y-term equaling a constant. For example, below are two linear equations written in standard form.

$$3x + 2y = 6$$

$$x - 5y = 10$$

To graph linear equations which are written in standard form we could rearrange each equation, solving it for y and thus putting it into slope-intercept form before graphing it. This method would work, but there is another way that is usually easier.



Two-Intercept Graphing

We know that we can find the y-intercept of a graph (where the graph crosses the y-axis) by letting $x = 0$ in the equation and solving for y. Similarly, we can find where the graph crosses the x-axis (the x-intercept), by letting $y = 0$ and solving for x in the equation.

Finding both the x and y intercepts of a graph is particularly easy when the equation is written in standard form, as these examples will illustrate:

Start with the equation:

$$3x + 2y = 6$$

To find the y-intercept, let $x = 0$:

$$\begin{aligned} 3(0) + 2y &= 6 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

The y-intercept is (0,3)

Now to find the x-intercept, let $y = 0$.

$$\begin{aligned} 3x + 2(0) &= 6 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

The x-intercept is (2,0)

Now that we know that where the graph crosses each axis, we can mark the two intercepts and draw the graph.
A second example would be:

$$2x - 6y = 12$$

To find the y-intercept let $x = 0$.

$$\begin{aligned} 2(0) - 6y &= 12 \\ {}^{-}6y &= 12 \\ y &= {}^{-}2 \end{aligned}$$

The y-intercept is $(0, {}^{-}2)$.

To find the x-intercept let $y = 0$:

$$\begin{aligned} 2x - 6(0) &= 12 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

The x-intercept is (6, 0)

The graph is:

When an equation is written in standard form, finding each of the intercepts is really a one step process requiring dividing the constant term by the coefficient of the letter whose intercept is being calculated.

Difficulties With the Two Intercept Method

Although the intercepts for an equation may be easy to calculate, they may not come out to be whole numbers. This may make the graph harder to draw accurately. For example, consider the equation

$$5x - 2y = 8$$

We can find the intercepts:

Let $x = 0$

$$\begin{aligned} 5(0) - 2y &= 8 \\ {}^{-}2y &= 8 \\ y &= {}^{-}4 \end{aligned}$$

The y-intercept is $(0,\ {}^{-}4)$

Let $y = 0$

$$\begin{aligned} 5x - 2(0) &= 8 \\ 5x &= 8 \\ x &= \frac{8}{5} \end{aligned}$$

The x-intercept is $(\frac{8}{5}, 0)$

Exactly graphing a value like $\frac{8}{5}$ is more difficult than graphing a whole number. It may be difficult to make the graph accurate.

The problem of accuracy becomes particularly pronounced when one of the intercepts is a fraction and both intercepts are very near the origin (0, 0) of the graph.

We can deal with these difficult cases by solving the equation for y and thus converting the equation into slope-intercept form.

$$
\begin{aligned}
5x - 2y &= 8 && \text{Add } {}^{-}5x \\
{}^{-}2y &= {}^{-}5x + 8 && \text{Divide by } {}^{-}2 \\
y &= \frac{5}{2}x - 4
\end{aligned}
$$

The graph has a y-intercept of (0, $^{-}4$) and a slope of $5/2$. This can now be graphed more accurately:

Exercises

Graph these equations using the two-intercept method.

1. $2x - 3y = 6$

2. $x + 2y = 8$

3. $2x - y = 4$

4. $3x + 4y = 12$

5. $x - 5y = 5$

6. $^{-}3x + 2y = {}^{-}12$

7. $5x - 3y = 15$

8. $-x - 3y = 8$

9. $6x - 2y = 6$

10. $4x - y = 8$

Convert these equations to slope-intercept form by solving for y, and then graph.

11. $^{-}3x + 2y = 8$

12. $2x - y = 5$

13. $4x - 2y = {}^{-}2$

14. $5x + y = 4$

15. $^{-}4x + 3y = {}^{-}6$

Section **8**
Summary

Related Variables

We now have many different meanings for the idea of two related variables. This concept is one of the most powerful and useful ideas in mathematics. Important ideas can usually be shown in many ways; we have presented more than one illustration in order to help you to understand, not to confuse you.

The ideas discussed in this chapter are tables, rules, lists of ordered pairs, machines, and graphs. Each is useful in different areas of mathematics, but each is essentially describing the same concept. To review, the different methods of showing two related variables are:

- **A *machine*:**

- **A *graph*:**

Note: See the APPENDIX for more information on the idea of a rule and the new concept of a **function.**

Exercises

Illustrate each of the following rules with a table, a list of ordered pairs, a machine, and a graph:

1. $y = -x + 6$

2. $y = 2x - 1$

3. $y = 2x - 3$

4. $y = {}^{-}3x + 5$

5. $y = \frac{1}{2}x - 3$

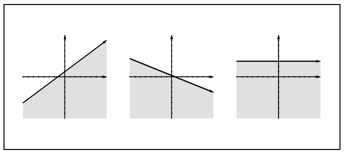
6. $y = 3x - 3$

7. $y = 0$

8. $y = -x$

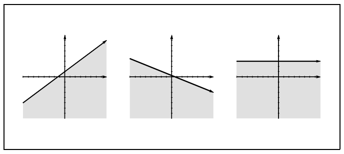
9. $y = 3$

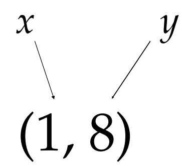
10. $y = x$

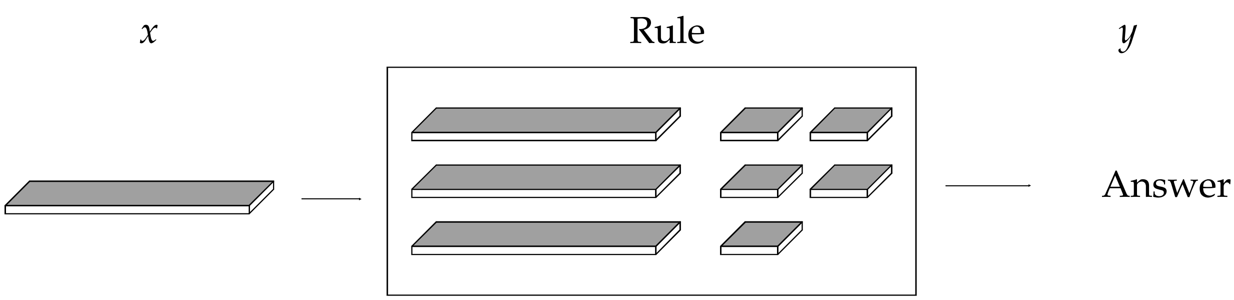


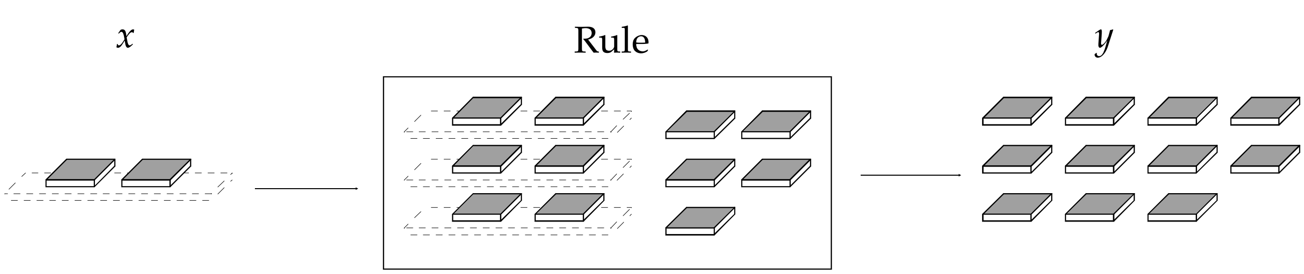
Rule: $y = 3x + 5$

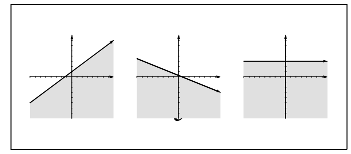
x	y
0	5
1	8
2	11
-1	2
-2	-1
-3	-4
-4	-7



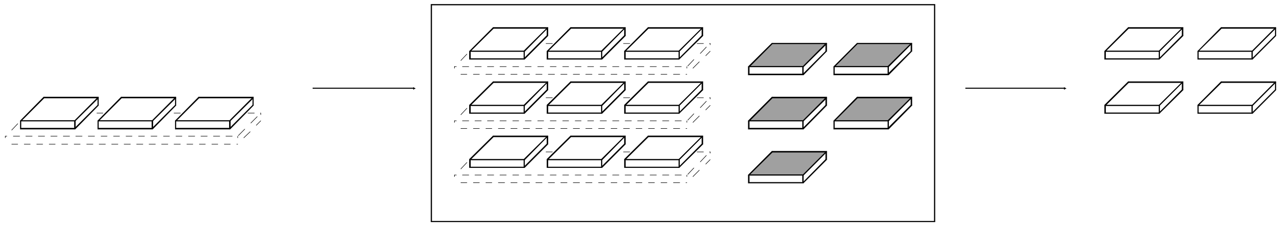

x
y
(1, 8)

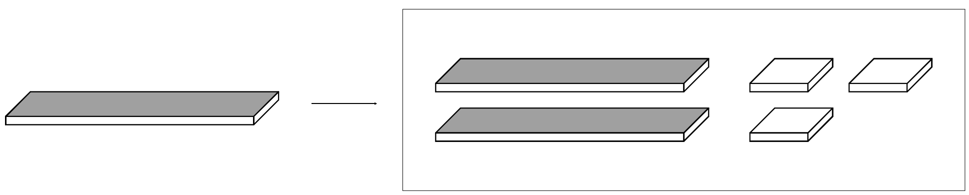

x
Rule
y
Answer

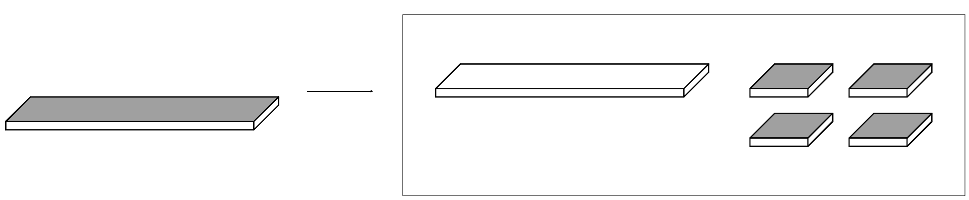

x
Rule
y

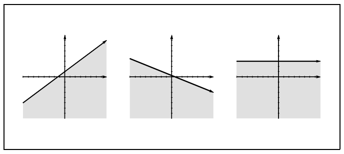


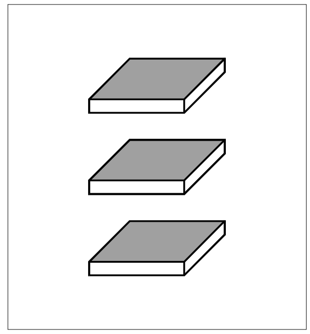
x Rule

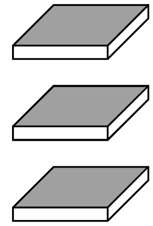


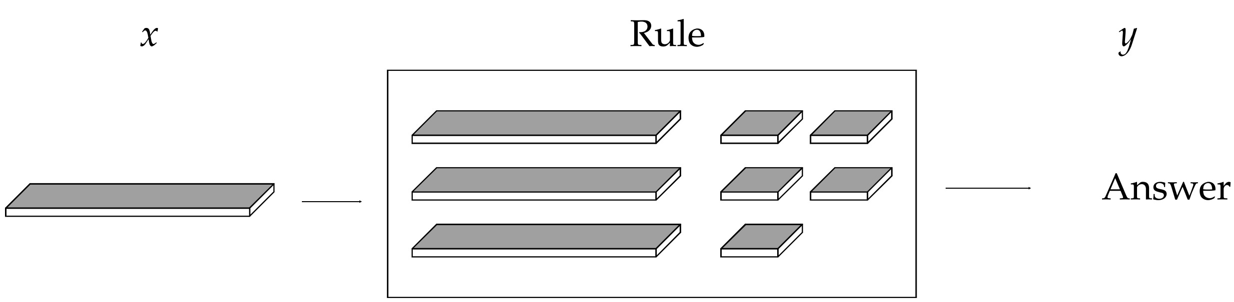


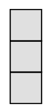


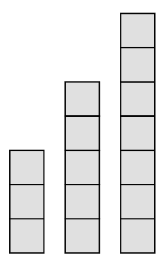


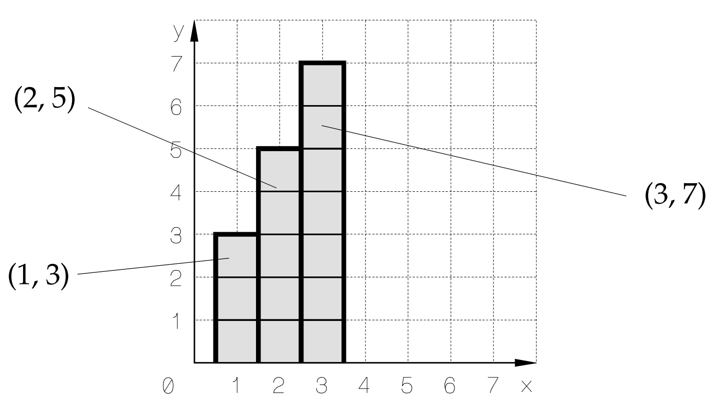


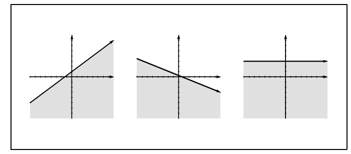


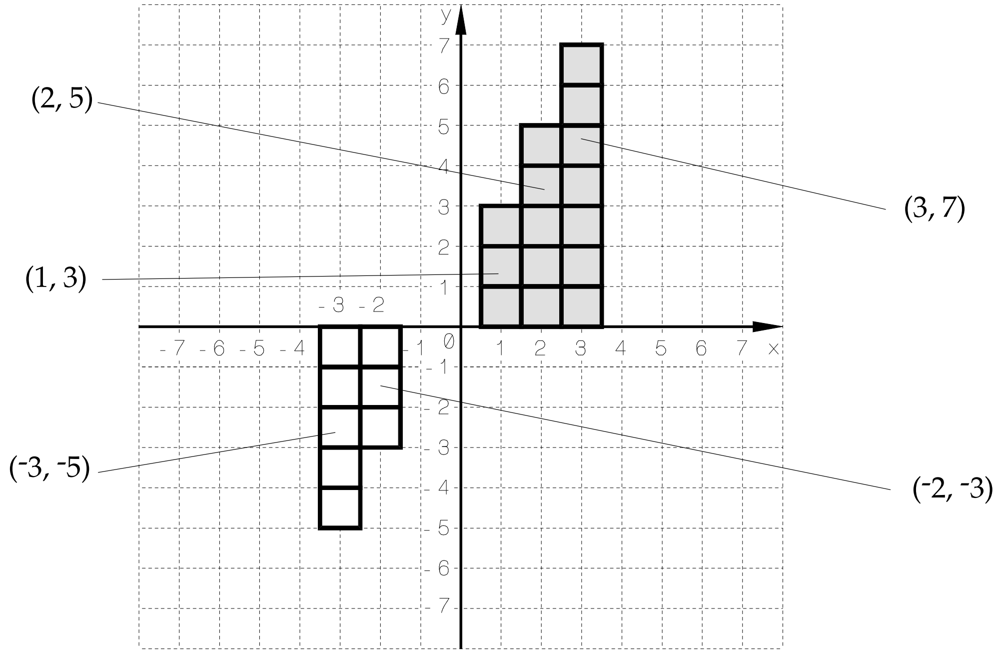

x
Rule
y
Answer

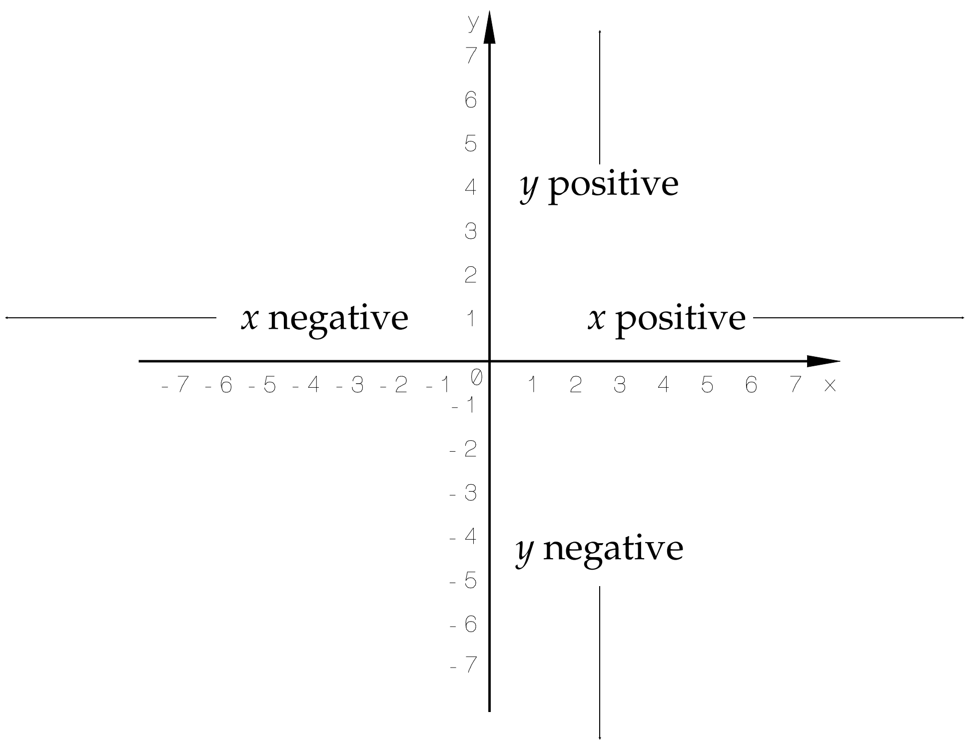


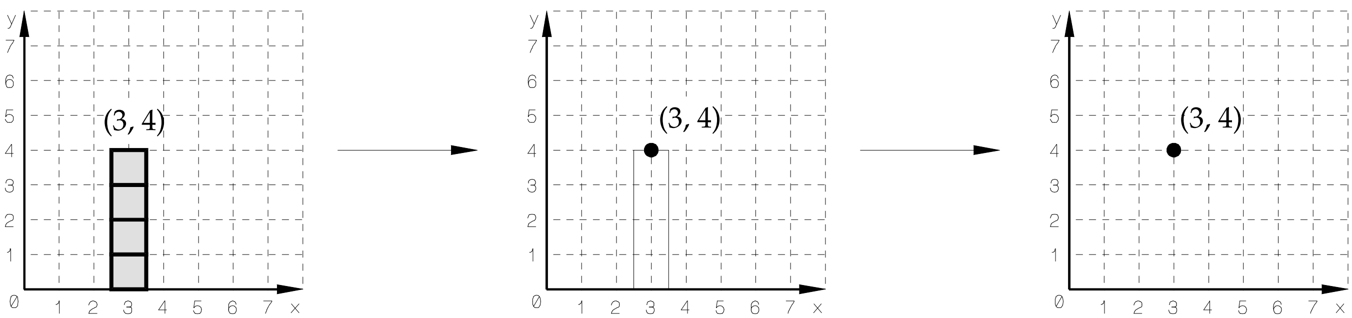


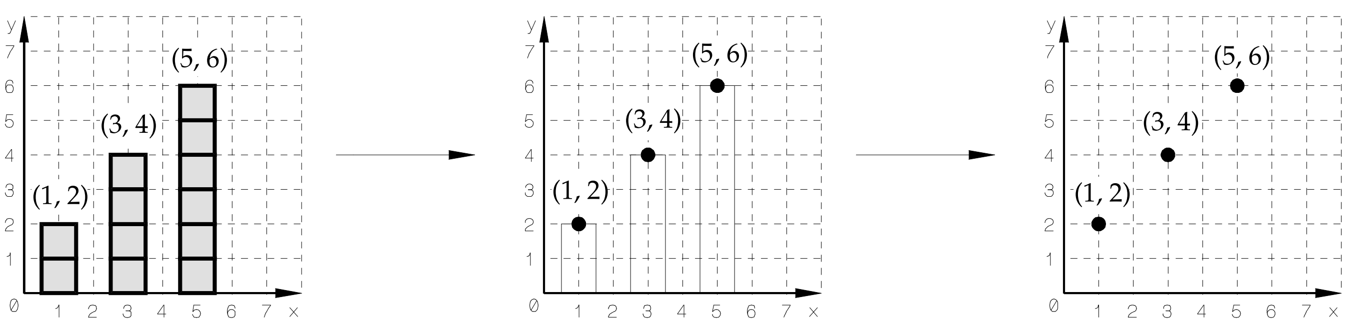

y
7
6
5
4
3
2
1
0
1 2 3 4 5 6 7 x
(2, 5)
(3, 7)
(1, 3)

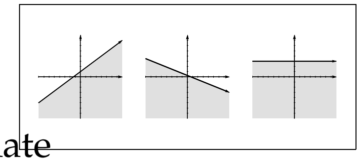


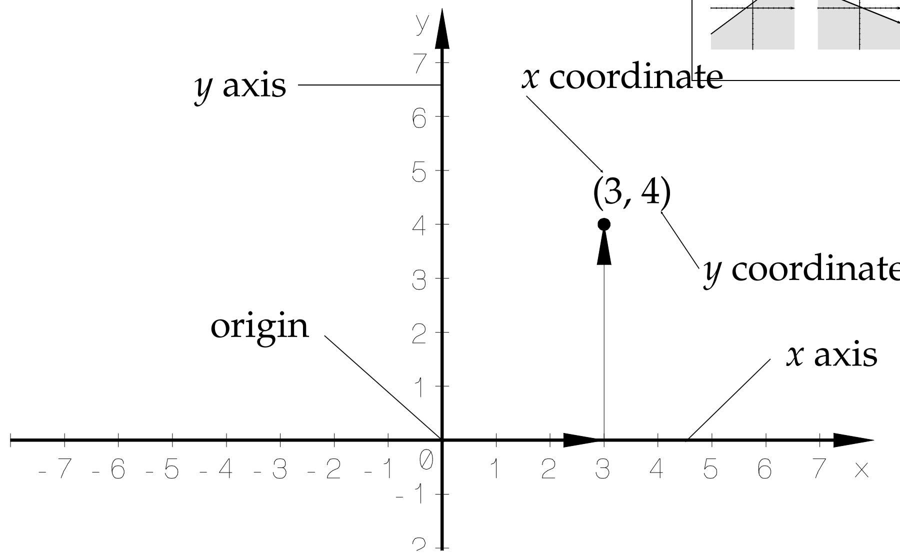

y
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 x
-3 -2
(2, 5)
(3, 7)
(1, 3)
(⁻3, ⁻5)
(⁻2, ⁻3)

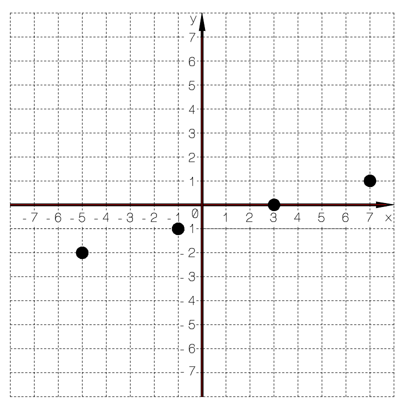

y
-7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 x
y positive
x negative
x positive
y negative

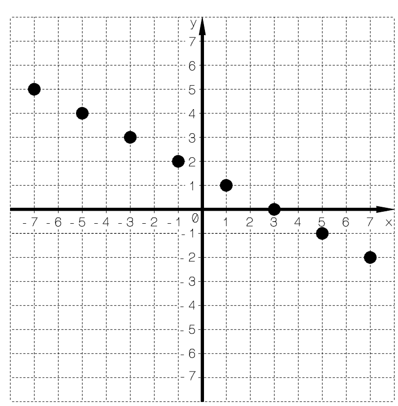

(3, 4)
(3, 4)
(3, 4)

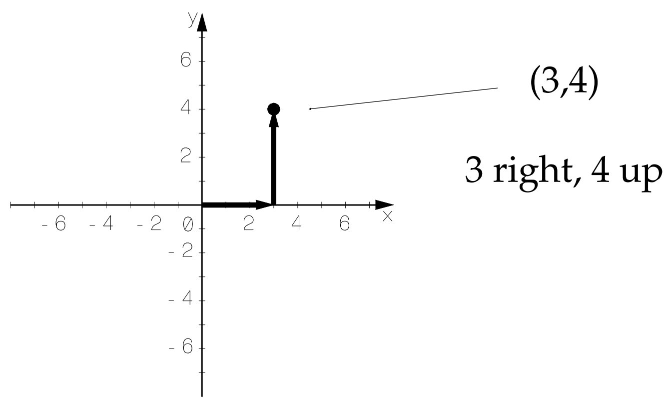

(1, 2)
(3, 4)
(5, 6)
(1, 2)
(3, 4)
(5, 6)
(1, 2)
(3, 4)
(5, 6)

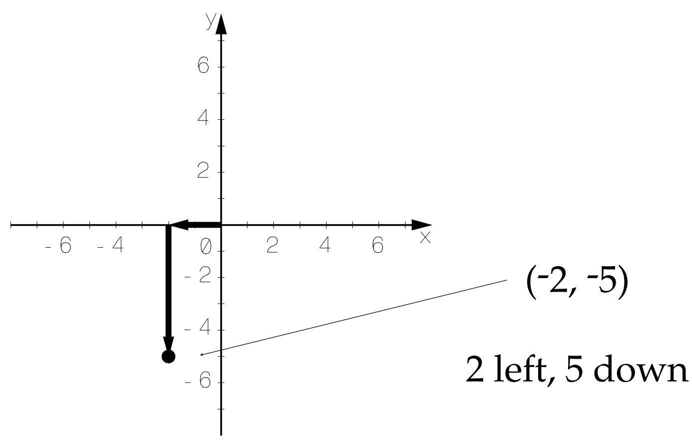


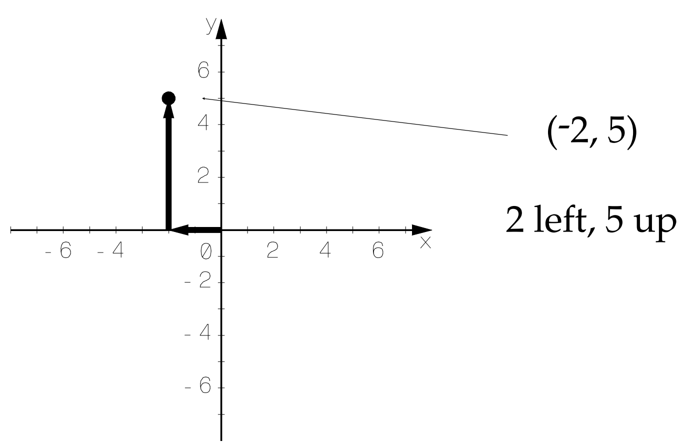

y axis
x coordinate
(3, 4)
y coordinate
origin
x axis

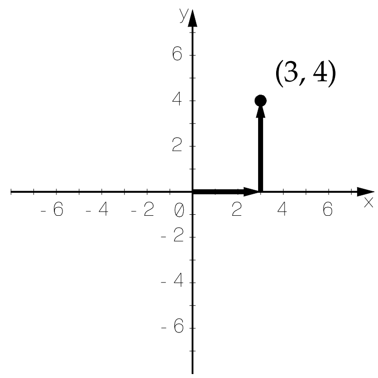


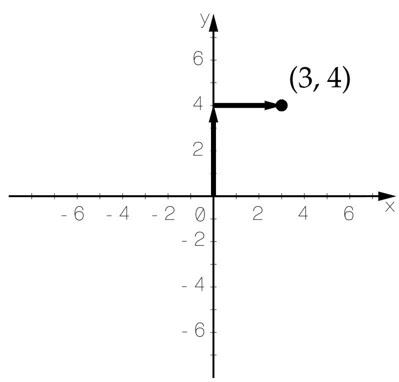


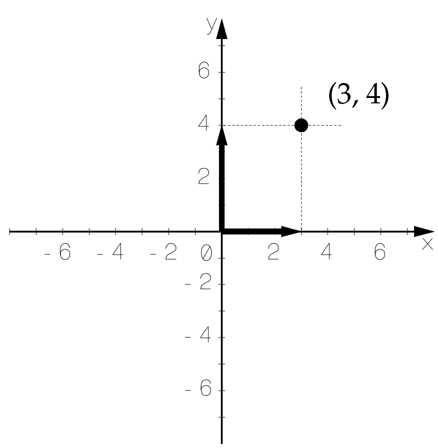

(3,4)
3 right, 4 up

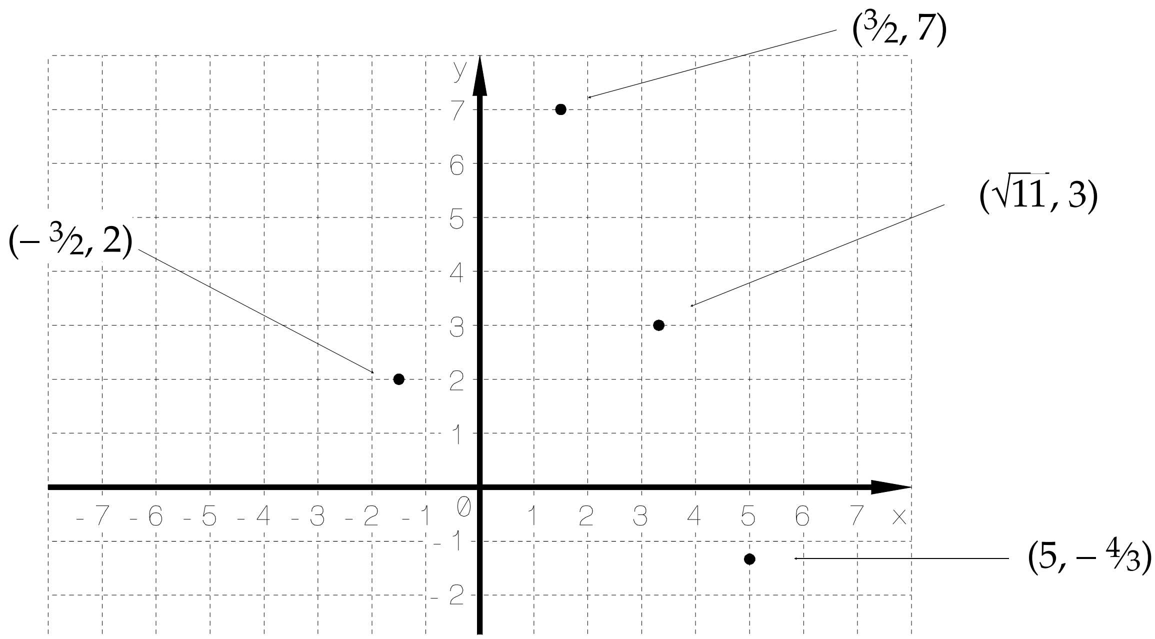

(-2, -5)
2 left, 5 down

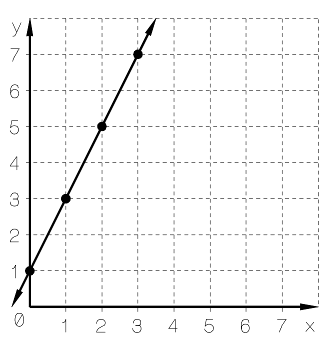

(-2, 5)
2 left, 5 up


(3, 4)


(3, 4)

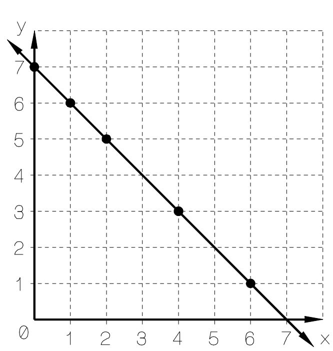

(3, 4)


(3⁄2, 7)
(√11, 3)
(– 3⁄2, 2)
(5, – 4⁄3)
y
x
0
- 7 - 6 - 5 - 4 - 3 - 2 - 1
1 2 3 4 5 6 7
1 2 3 4 5 6 7
- 1 - 2



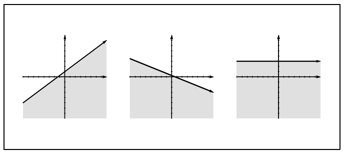
$y = 2x + 1$

x	y
1	3
2	5
3	7
0	1



$y = -x + 7$

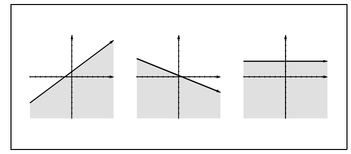
x	y
1	6
2	5
4	3
6	1
0	7

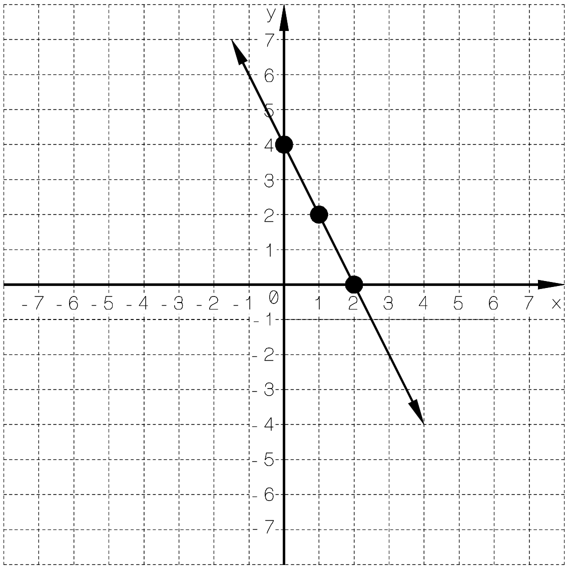


Any rule of form $y = mx + b$ is a line.

Any graph that is a line can be written in the form $y = mx + b$.

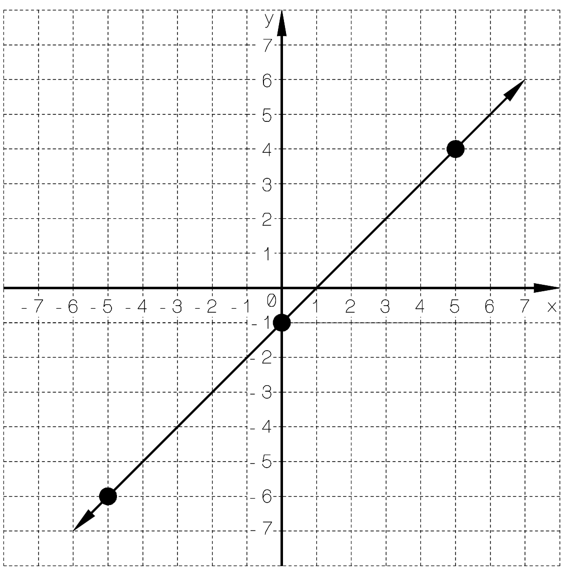
These equations do *not* graph as straight lines.





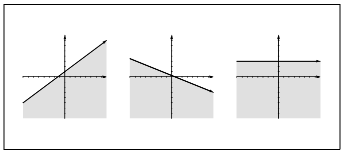
$y = {}^{-}2\,x + 4$

x	y
0	4
1	2
2	0



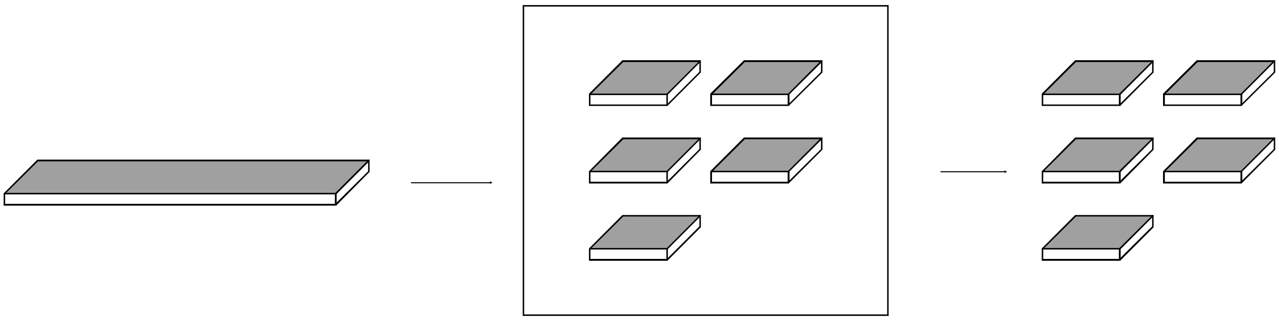
$y = x + {}^{-}1$

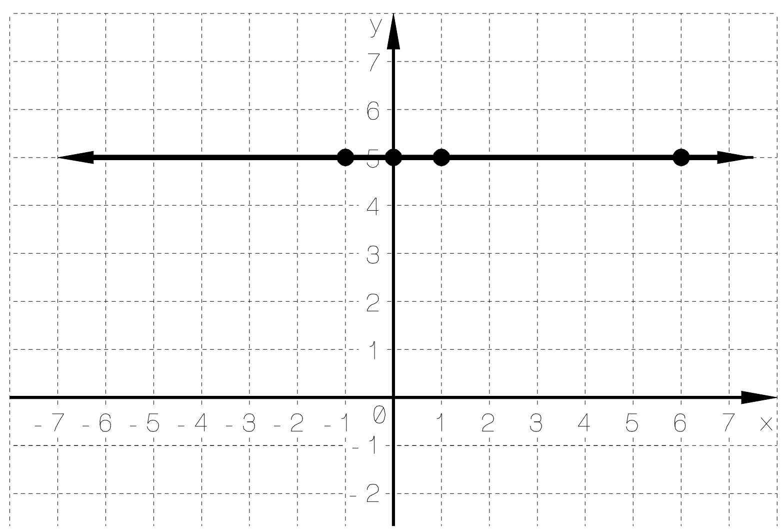
x	y
5	4
0	1
$^{-}5$	$^{-}6$

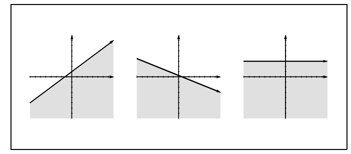


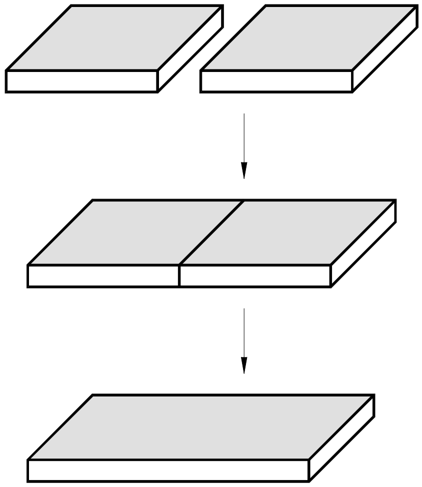
$y = 5$

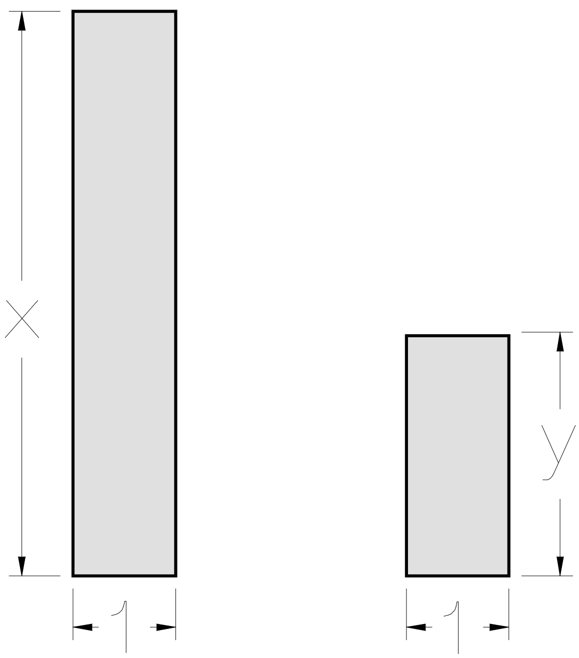
x	y
0	5
1	5
-1	5
6	5

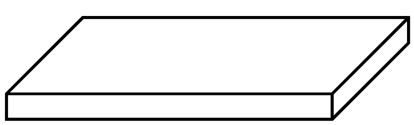


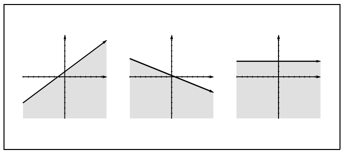




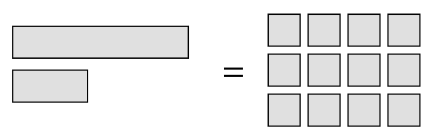



x
y
1
1

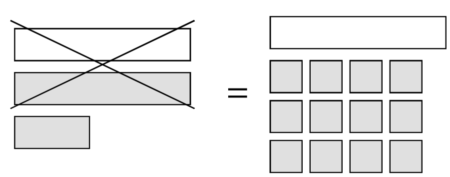




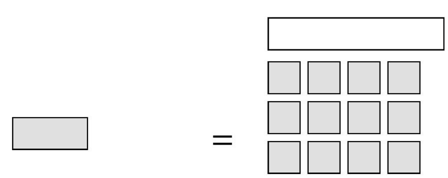
$$x + y = 12$$

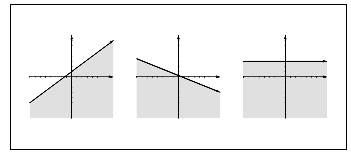


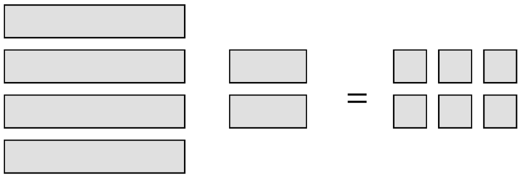
$$x - x + y = -x + 12$$



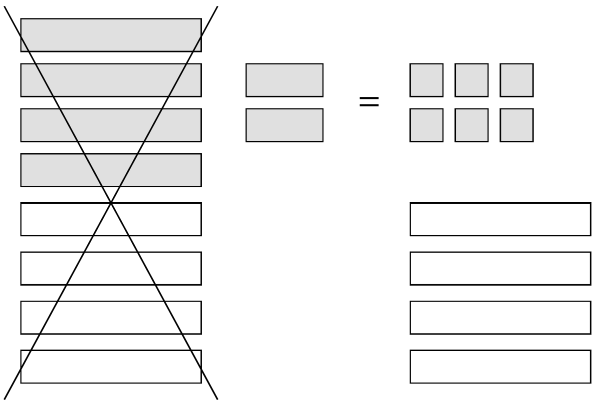
$$y = -x + 12$$





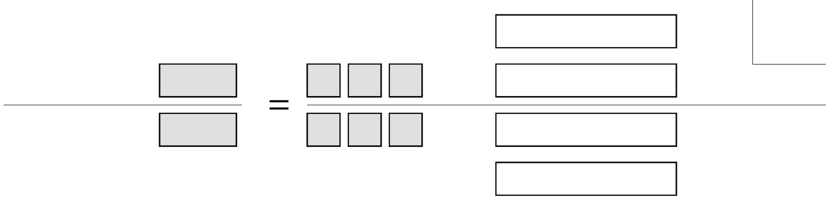


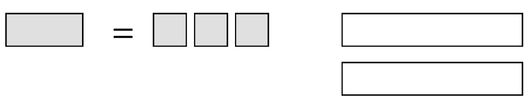
$$4x + 2y = 6$$



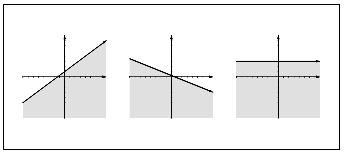
$$4x - 4x + 2y = {}^{-}4x + 6$$

$$\frac{1}{2}(2y) = \frac{1}{2}({}^{-}4x + 6)$$

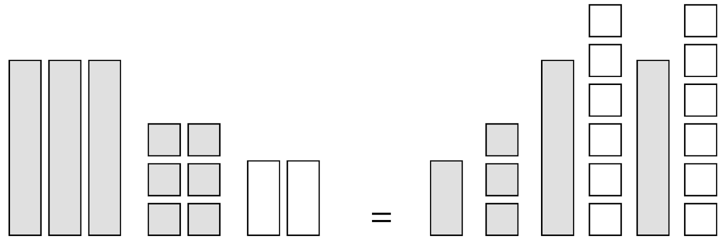




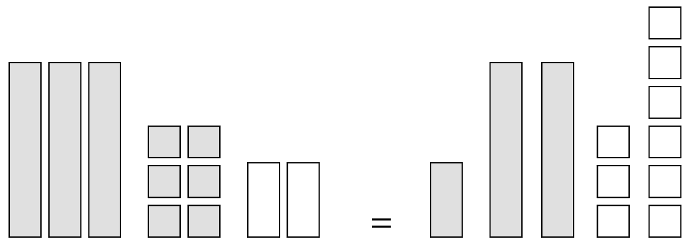
$$y = {}^{-}2x + 3$$



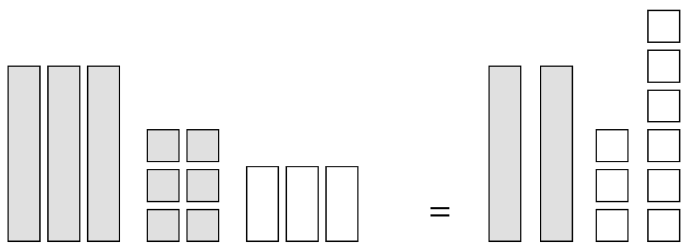
$$3x + 6 - 2y \;=\; y + 3 + 2(x - 6)$$



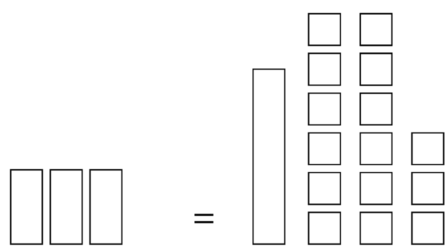
$$3x + 6 - 2y \;=\; y + 2x + {}^{-}9$$



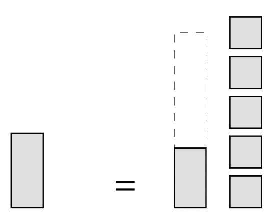
$$3x + 6 - 3y \;=\; 2x + {}^{-}9$$

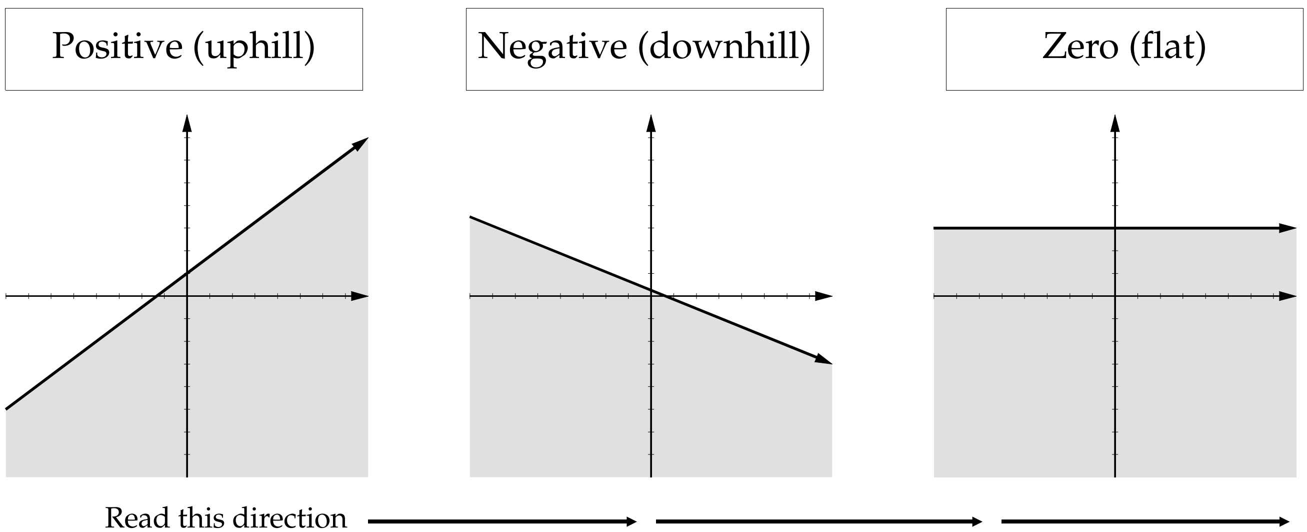


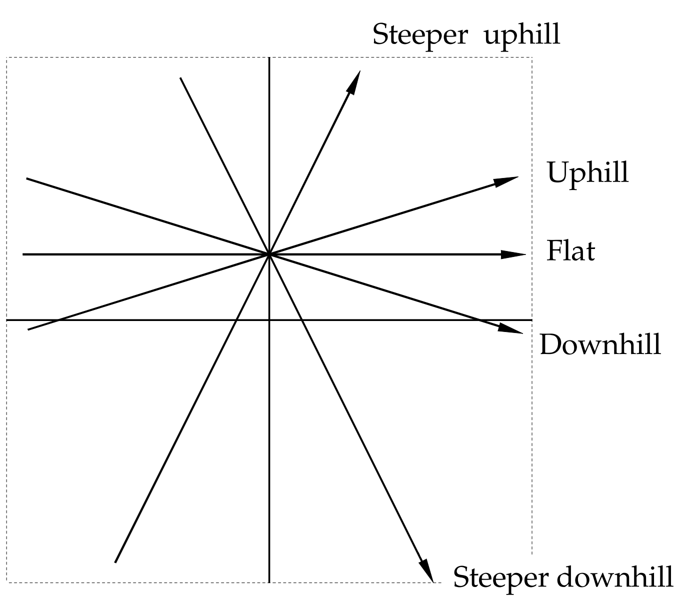
$${}^{-}3y \;=\; -x + {}^{-}15$$

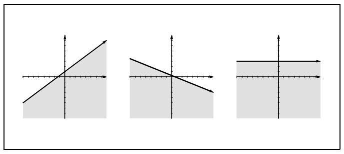


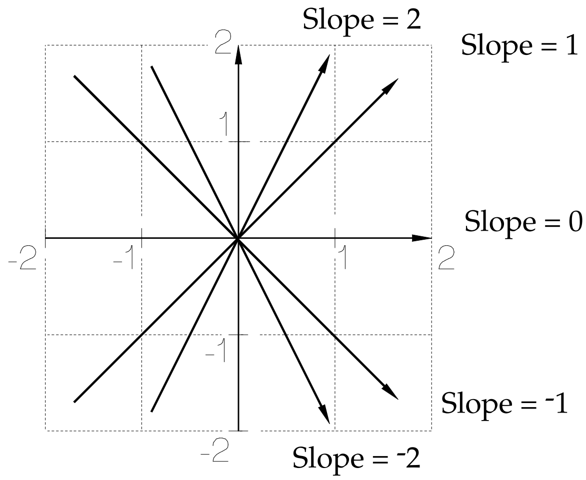
$$y \;=\; \frac{1}{3}x + 5$$




Positive (uphill)
Negative (downhill)
Zero (flat)
Read this direction


Steeper uphill
Uphill
Flat
Downhill
Steeper downhill

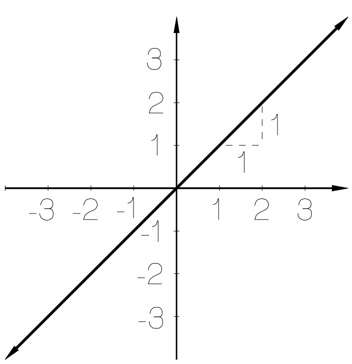




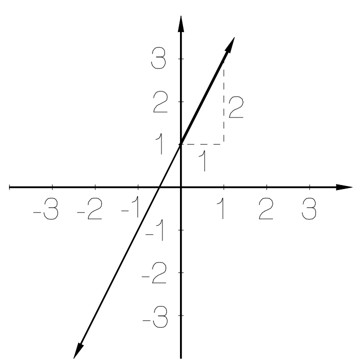
The Slope of a Line

Looking left to right, the slope is the distance travelled up or down for every 1 unit travelled to the right.

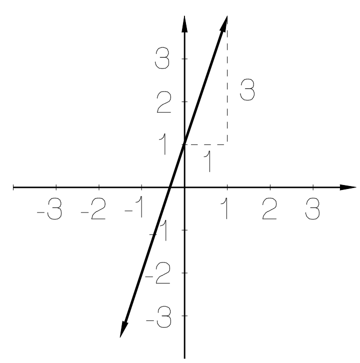
Slope = 1

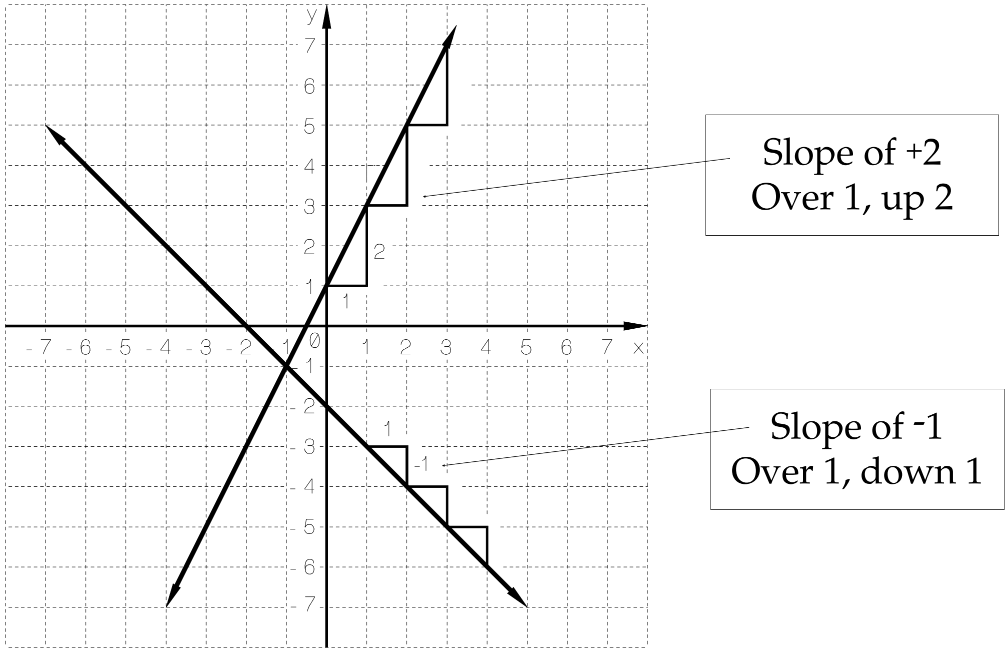


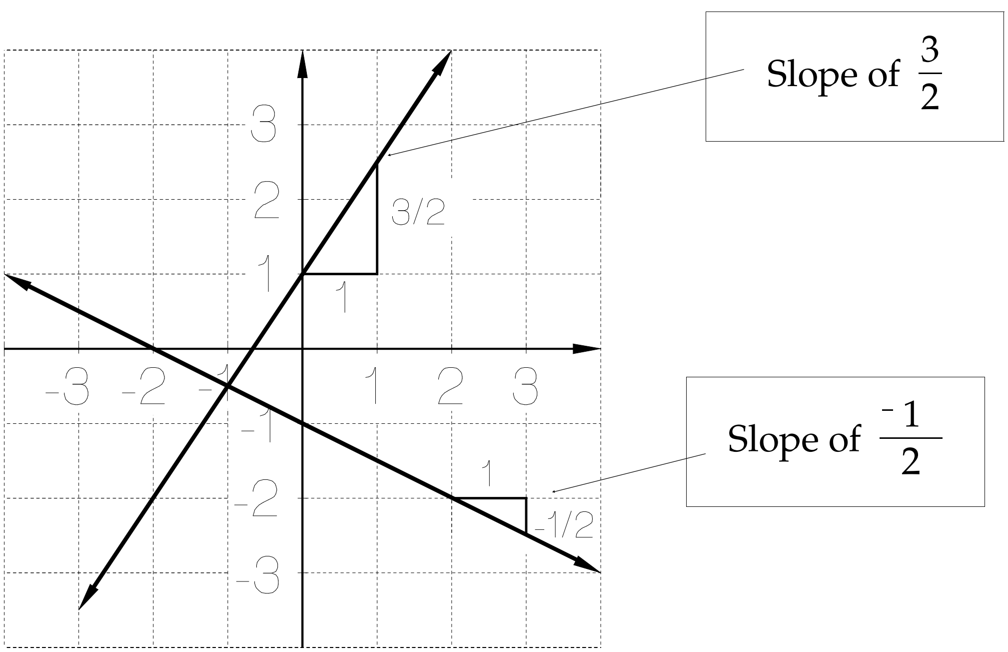
Slope = 2

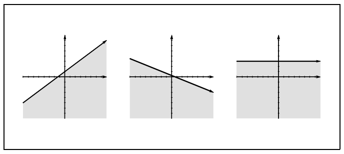


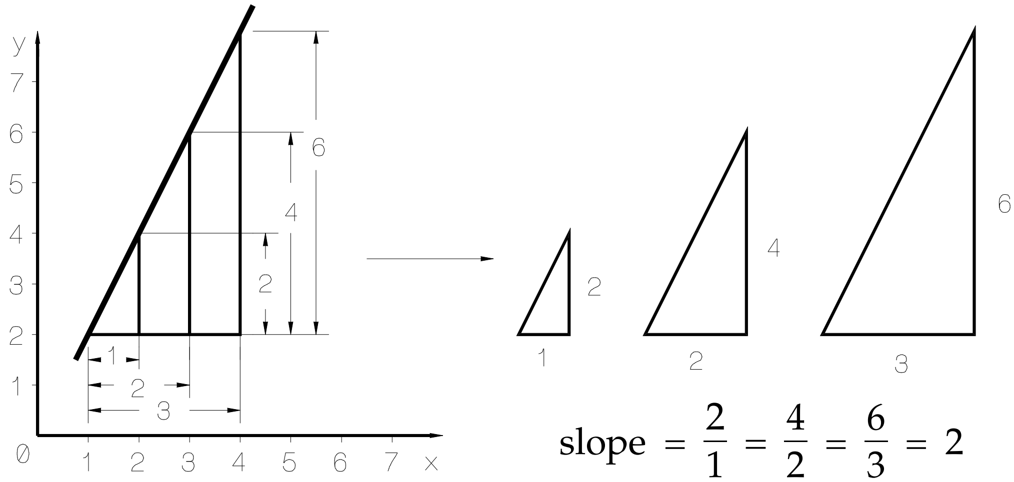
Slope = 3




Slope of +2
Over 1, up 2
Slope of -1
Over 1, down 1
y
x
1
2
-1


Slope of 3/2
Slope of -1/2
3/2
1
-1/2



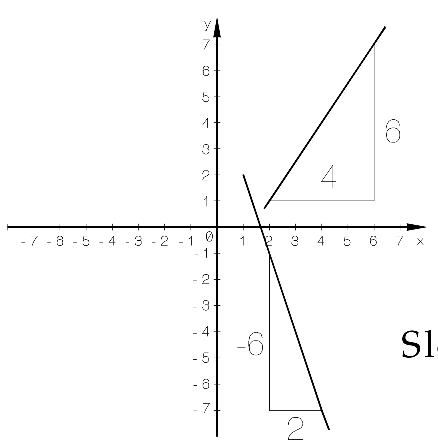


$$\text{slope} = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = 2$$

Slope of a Line (Alternate definition)

The ratio of the rise (change in *y*) to the run (change in *x*)

$$\textbf{slope} = \frac{\textbf{change in } y}{\textbf{change in } x} = \frac{\textbf{rise}}{\textbf{run}}$$



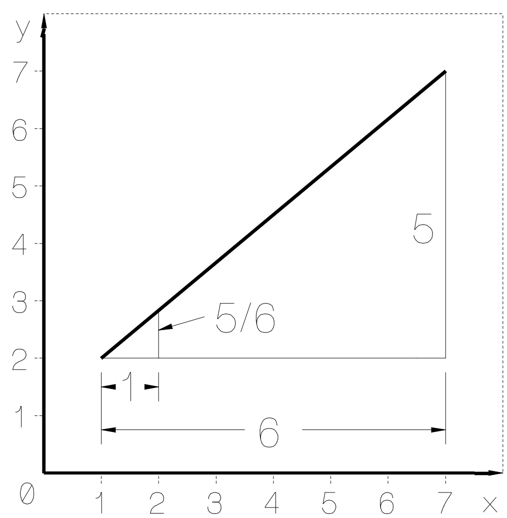
$$\text{Slope} = \frac{6}{4} = \frac{3}{2}$$

$$\text{Slope} = \frac{^{-}6}{2} = {}^{-}3$$

Definitions of Slope

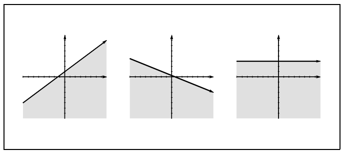
1. The distance traveled up or down for every 1 unit travelled to the the right.

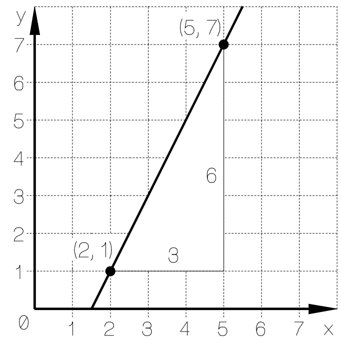
2. $\textbf{Slope} = \dfrac{\textbf{change in } y}{\textbf{change in } x} = \dfrac{\textbf{rise}}{\textbf{run}}$

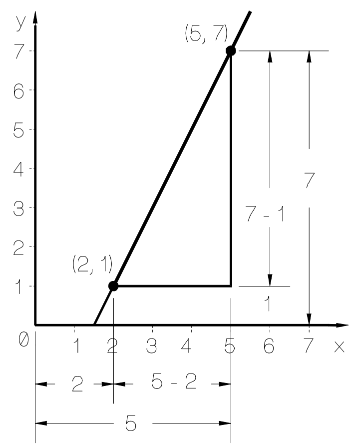


Definition 1: $\frac{5}{6}$ up for every 1 over

Definition 2: Up 5 and over 6, slope = $\frac{5}{6}$







Slope between (2, 1) and (5, 7):

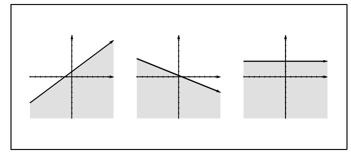
$$\frac{\text{Diff. of } y}{\text{Diff. of } x} = \frac{7-1}{5-2} = \frac{6}{3} = 2$$

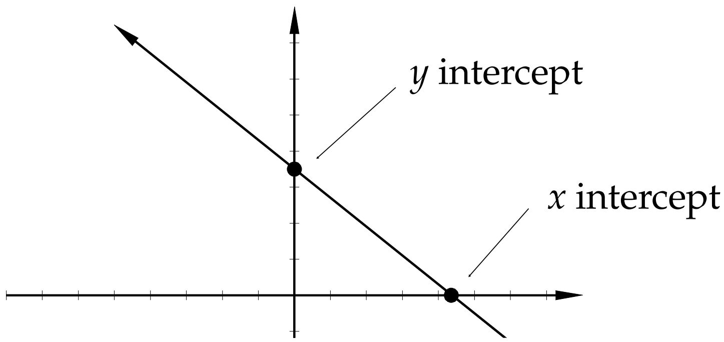
$$\text{or } = \frac{1-7}{2-5} = \frac{^-6}{^-3} = 2$$

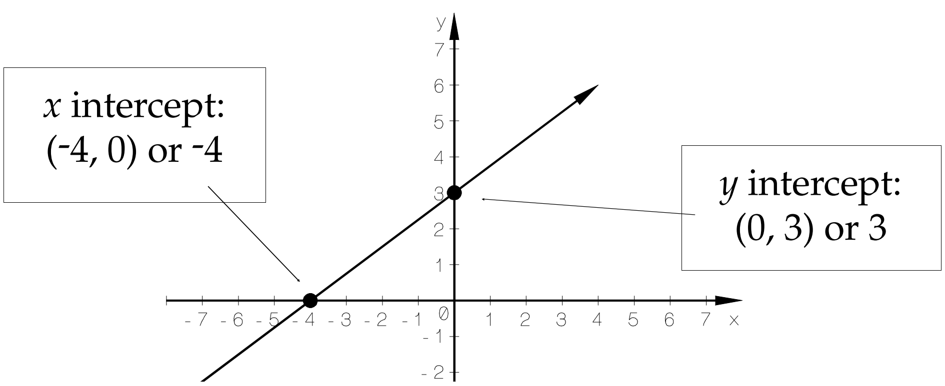
The Slope Between 2 Points (*a*, *b*) and (*c*, *d*)

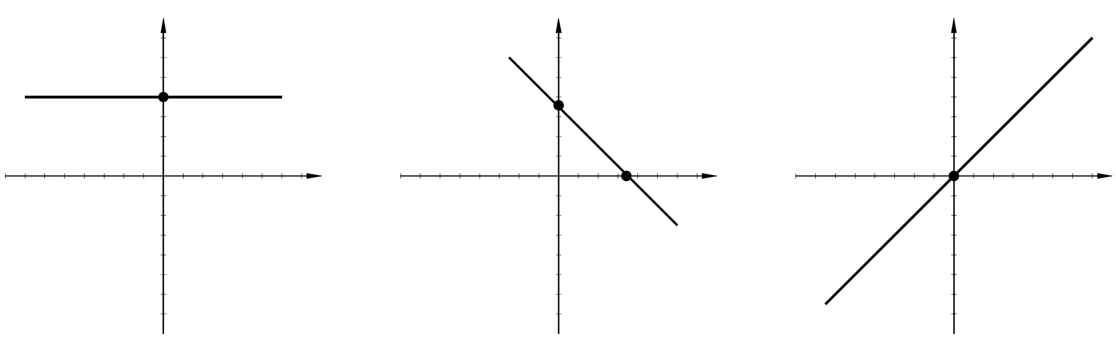
$$\textbf{Slope} = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } y}{\textbf{change in } x}$$

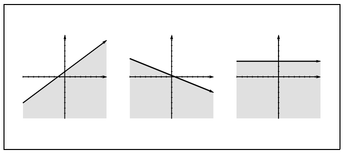
$$= \frac{d-b}{c-a} \textbf{ or } \frac{b-d}{a-c}$$

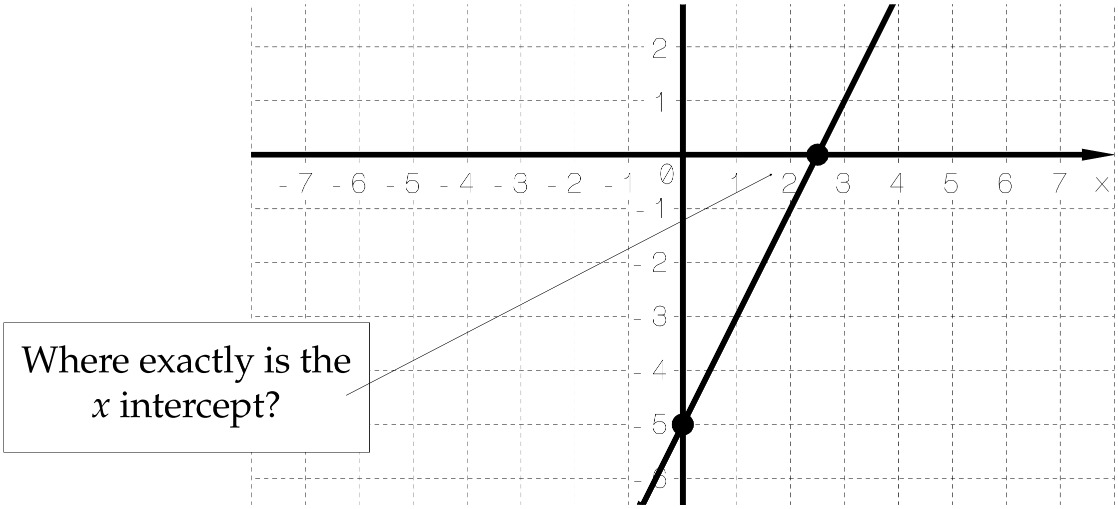



y intercept
x intercept

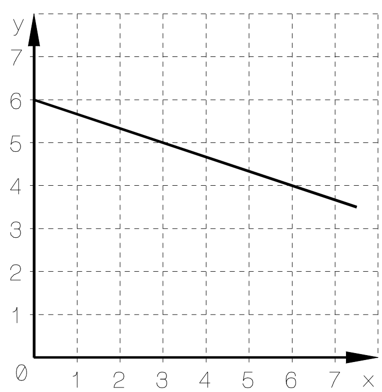

x intercept:
(-4, 0) or -4
y intercept:
(0, 3) or 3
y
x



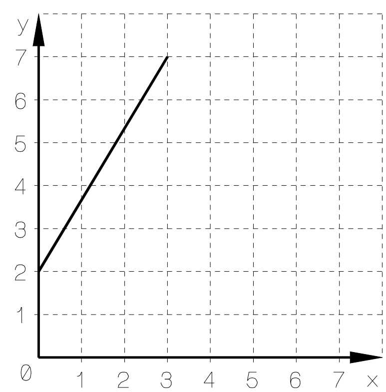



Where exactly is the
x intercept?
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 x
2 1 -1 -2 -3 -4 -5

1.

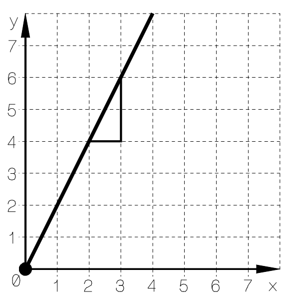


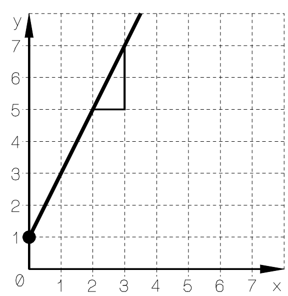
2.

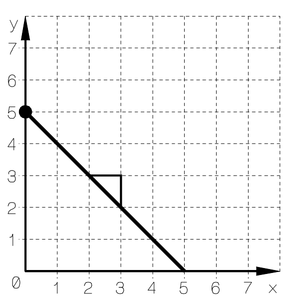


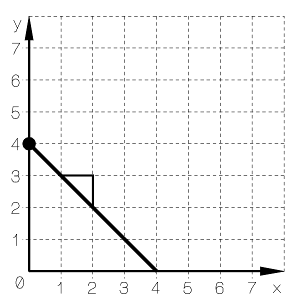
$y = 2x + 0$
Slope = 2
y intercept = 0

$y = 2x + 1$
Slope = 2
y intercept = 1



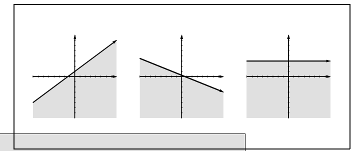






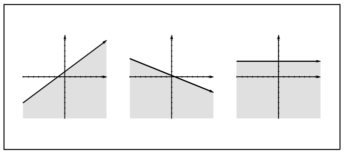
$y = {}^{-}1x + 5$
Slope = $^{-}1$
y intercept = 5

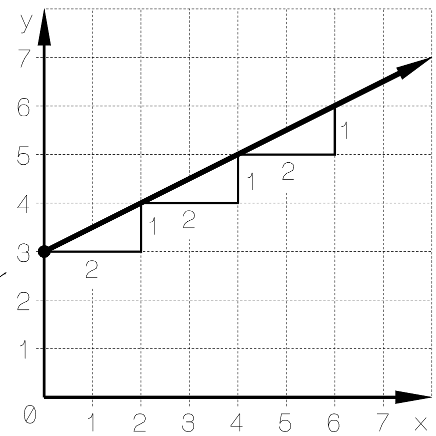
$y = {}^{-}1x + 4$
Slope = $^{-}1$
y intercept = 4



Graphing: Slope-Intercept Method

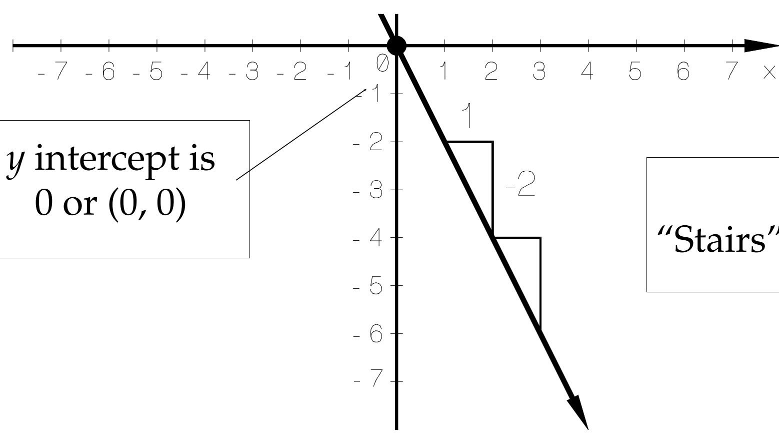
$y = mx + b$ **m is slope, b is y intercept**





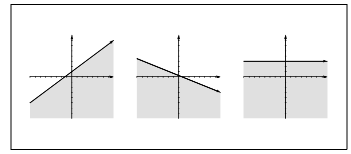
y intercept is 3 or (0, 3)

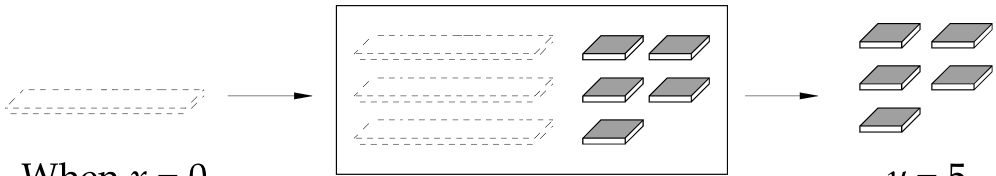
Slope is $\frac{1}{2}$
"Stairs" are 2 over and 1 up



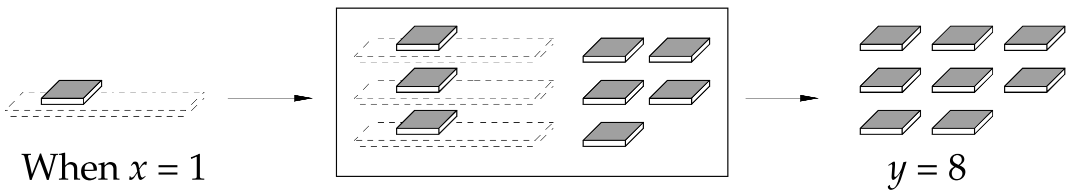
y intercept is 0 or (0, 0)

Slope is -2
"Stairs" are 1 over and 2 down

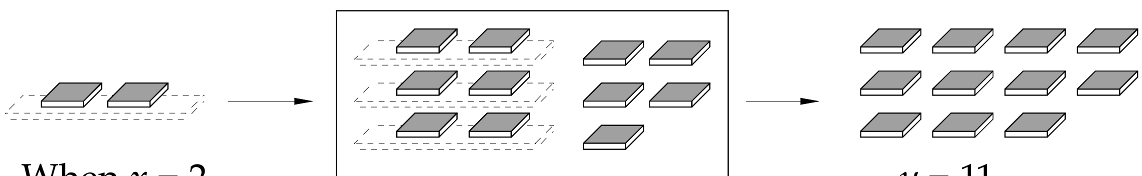




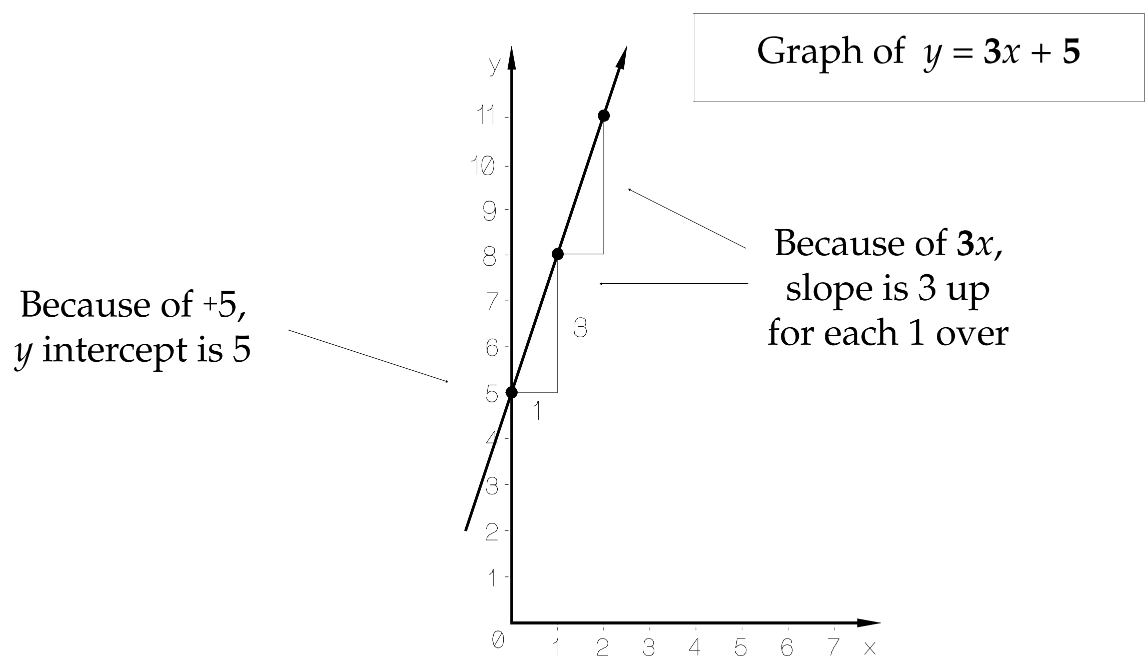
When $x = 0$ $y = 5$

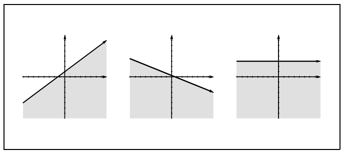


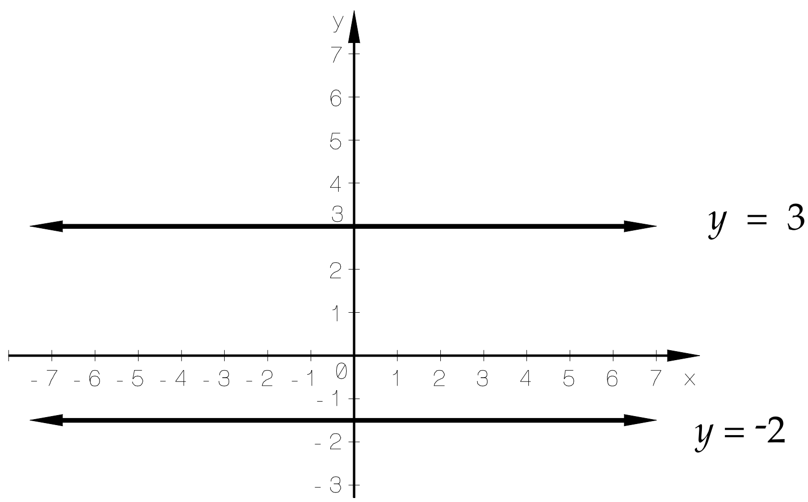
When $x = 1$ $y = 8$

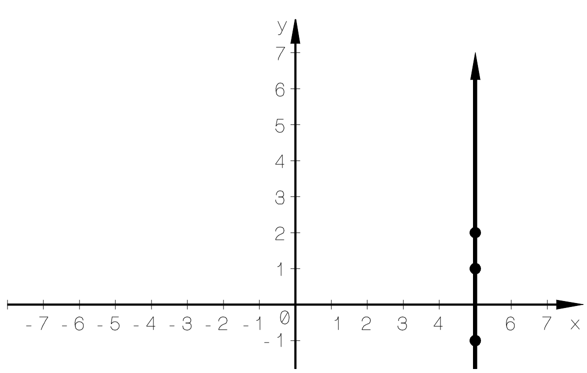


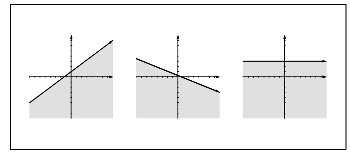
When $x = 2$ $y = 11$






y
7
6
5
4
3
2
1
- 7 - 6 - 5 - 4 - 3 - 2 - 1 0 1 2 3 4 5 6 7 x
- 1
- 2
- 3
y = 3
y = -2

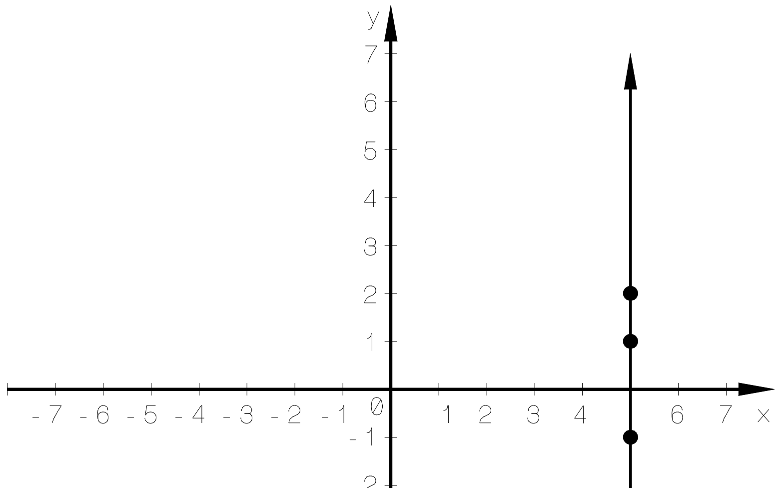

y
7
6
5
4
3
2
1
- 7 - 6 - 5 - 4 - 3 - 2 - 1 0 1 2 3 4 6 7 x
- 1



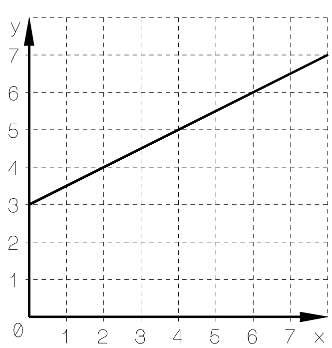
Vertical Line:

Rise is 3.
Run is 0.

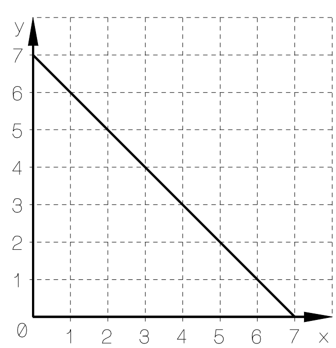
Slope is $\frac{\text{rise}}{\text{run}} = \frac{3}{0} = $ **?**

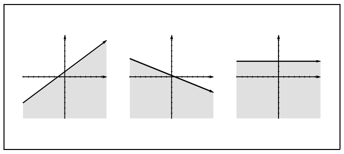


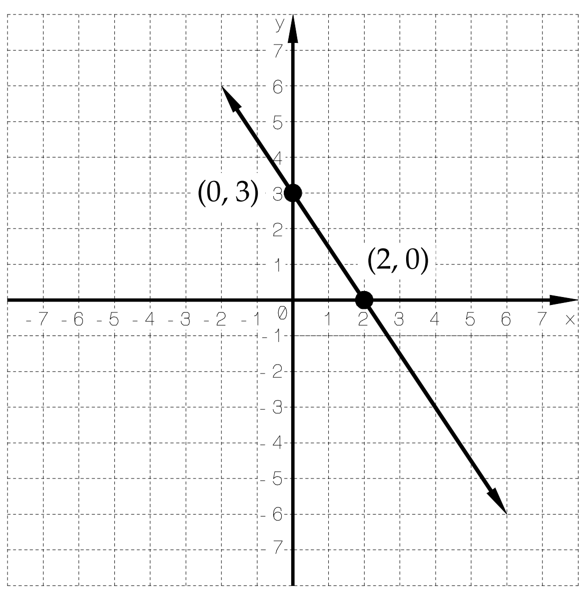
6.



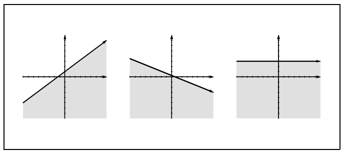
7.



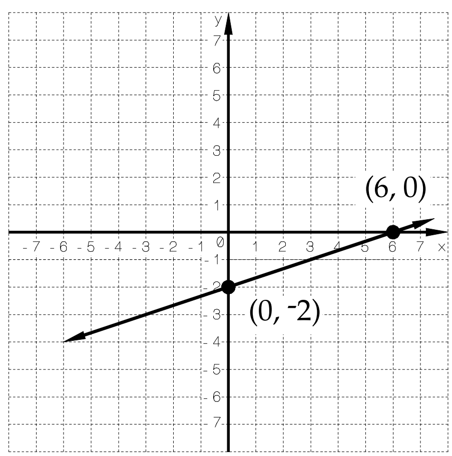


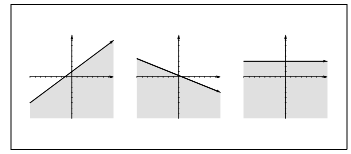


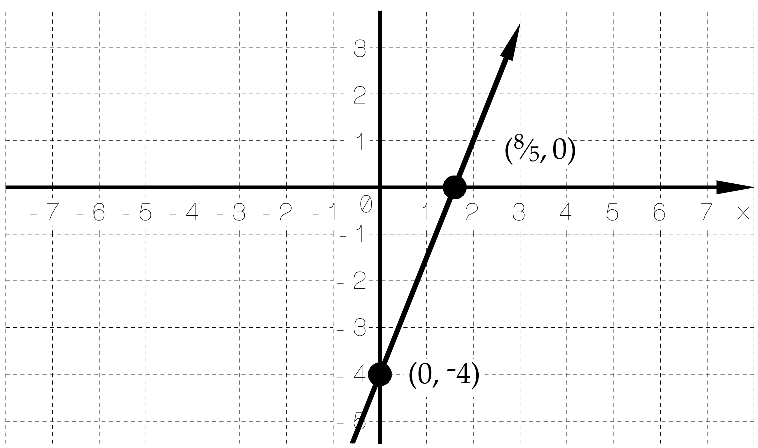
$3x + 2y = 6$



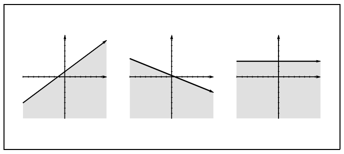
$$2x - 6y = 10$$

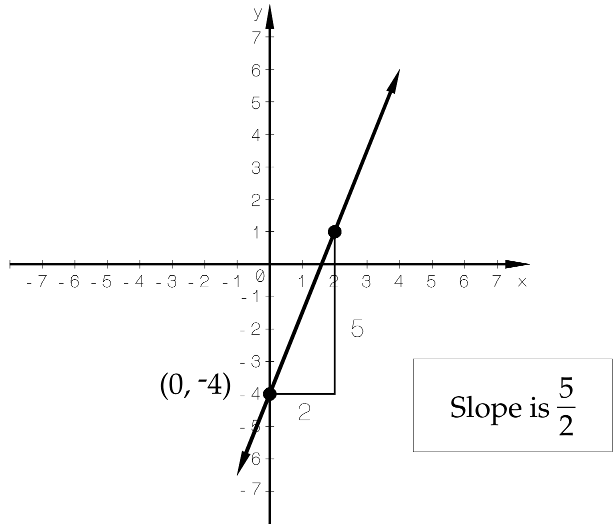






$$5x - 2y = 8$$




y
x
5
2
(0, -4)
Slope is $\frac{5}{2}$

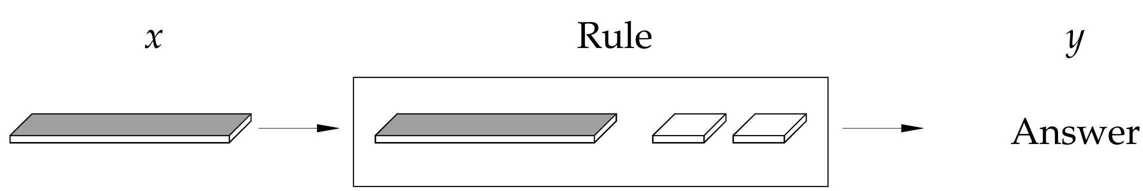
□ **A *table* and *rule*:**

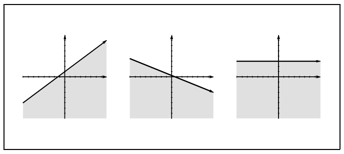
Rule: **y = x + ⁻2**

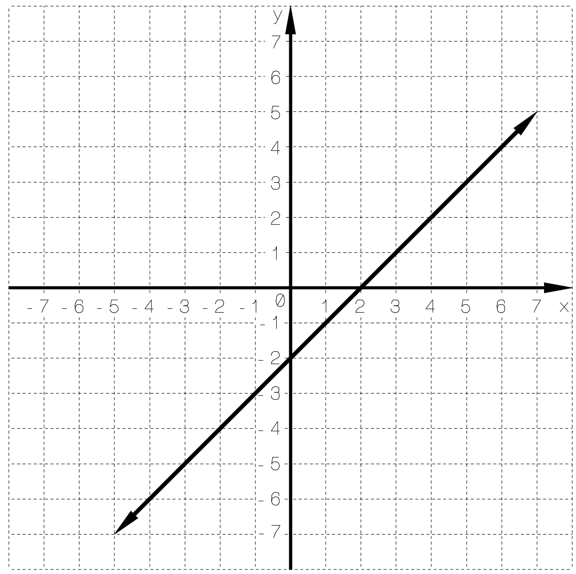
x	y
0	⁻2
1	⁻1
2	0
⁻1	⁻3

- **A list of *ordered pairs*:**

(0, ⁻2), (1, ⁻1), (2, 0), (⁻1, ⁻3)






y
7
6
5
4
3
2
1
0
- 1
- 2
- 3
- 4
- 5
- 6
- 7
- 7 - 6 - 5 - 4 - 3 - 2 - 1
1 2 3 4 5 6 7 x