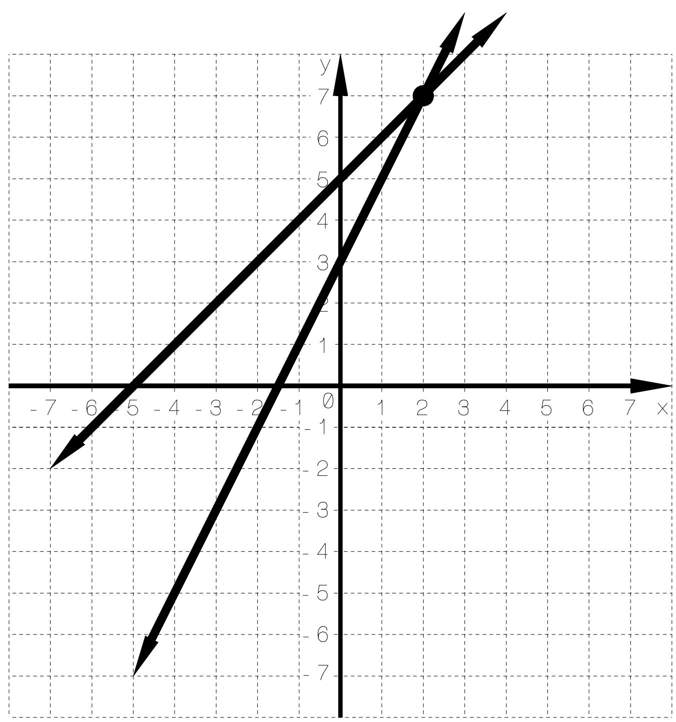
Chapter 13
Systems of Equations



Section 1
Equations and Solutions

Functions and Equations

In the last chapter, we looked at functions as charts, rules, lists of ordered pairs, maps, graphs, and machines. In this chapter, we will continue to develop ideas about ordered pairs, equations, and graphs.

An **equation** is a number statement having one or more unknowns and showing that two expressions are equal in value. For example:

$$y = 2x + 3$$

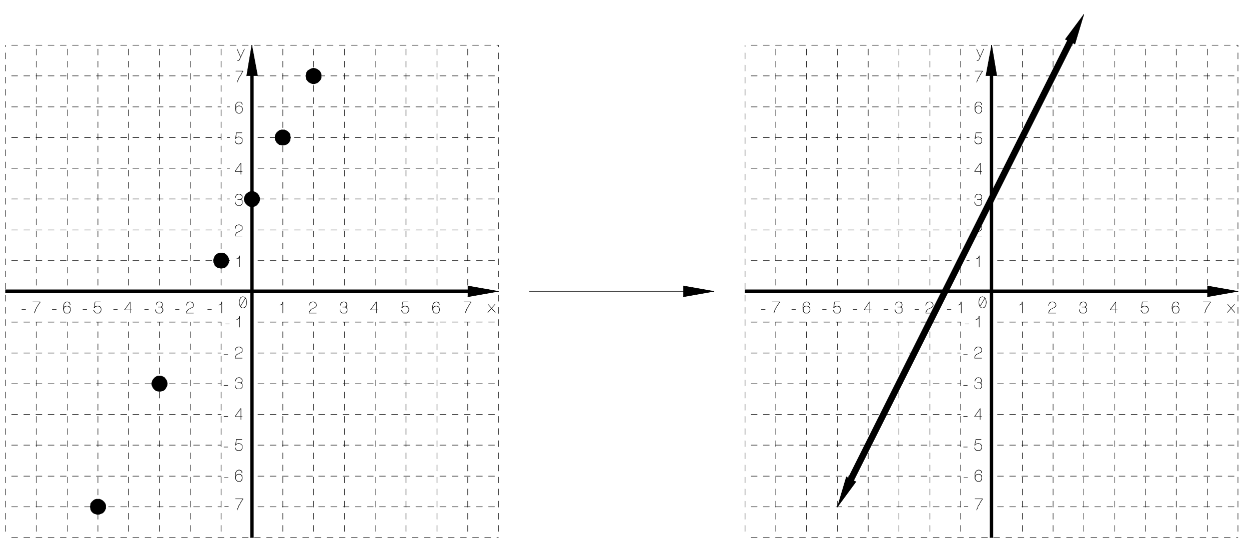
This is a **linear equation in two unknowns.** The **solution** to this equation is not one number—it is an infinite list of ordered pairs:

$$(0, 3), (1, 5), ({}^-1, 1), (1.5, 6), \ldots$$

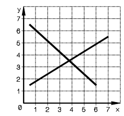
On a graph, we can represent all of the answers by a line. Each point on the line is a solution to the equation.

Individual solutions

All solutions



A linear equation has x terms, y terms, or both. It may also have number terms. Equations having terms with x^2, y^2, $1/x$, $1/y$, or xy are not linear and will not have straight line graphs.



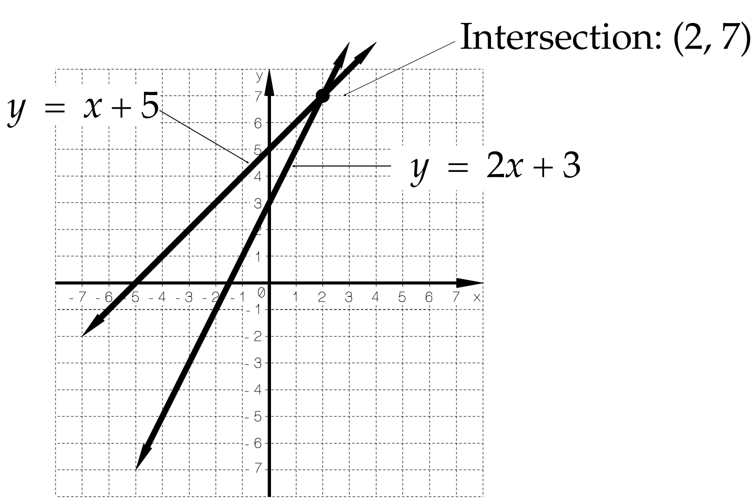
Linear	Not Linear
$y = 3x + 4$	$3 = 3$
$3x + y = 5 + x$	$xy + 3x = 4$
$y = 2$	$x^2 + 3x + 4 = 0$
$x - 2 = 0$	$y + \frac{1}{x} = 3$

Pairs of Equations

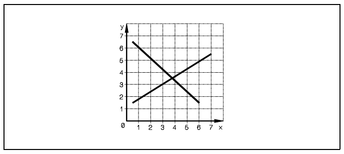
In many real-world situations, we encounter linear equations in related-pairs. A **system of equations** is a group of equations where we are looking for a common solution. The **solution** or **solution set** is one or more ordered pairs that satisfy *all* equations. Here is a system of two equations:

$$y = 2x + 3$$
$$y = x + 5$$

Since a single equation is an instruction to find all possible ordered pairs that make the equation true, *a group of two equations gives us the task of finding the ordered pair(s) that make* ***both*** *equations true at the same time*. Because the equations represent lines, we are looking for the point that is on both lines; this is the place where the graphs cross:



From the graph, we can see that *both* of these equations are satisfied at the point (2, 7) where the two lines cross. The values $x = 2$ and $y = 7$ will make both equations true, so this ordered pair is the solution of the system of equations.



We can confirm that (2, 7) is the solution by testing these x and y values in *both* equations:

$y = 2x + 3$	$y = x + 5$
$(7) = 2(2) + 3$	$(7) = (2) + 5$
$7 = 4 + 3$	$7 = 2 + 5$
$7 = 7$	$7 = 7$

Here is another example of a system of equations:

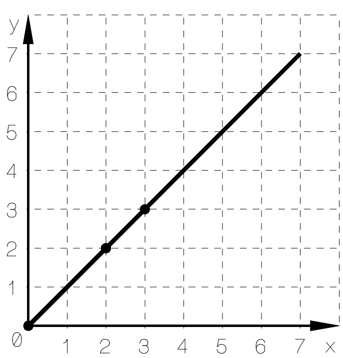
$$y = x$$
$$x + y = 6$$

By trial and error, we could list some solutions (ordered pairs) for each equation and hope to find a pair that works in both equations:

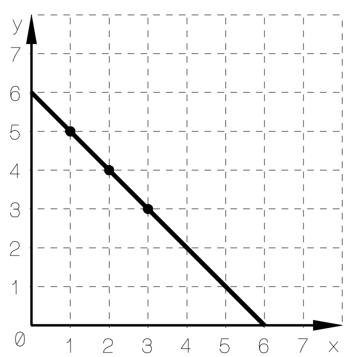
For $y = x$, solutions are (0, 0), (2, 2), **(3, 3)**
For $x + y = 6$, solutions are (1, 5), (2, 4), **(3, 3)**

The common solution is (3, 3). On the graph, we would see the answer as the intersection of the two lines:

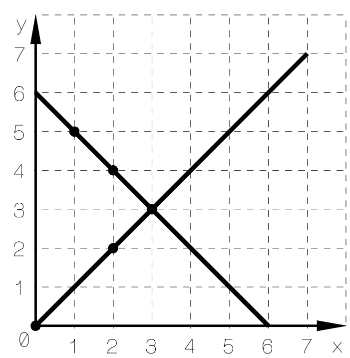
$y = x$



$x + y = 6$

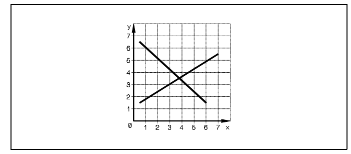


Both equations

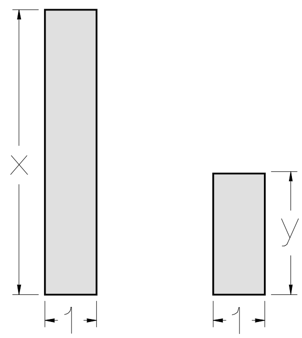


Although trial and error may work to find solutions for some easy situations, it will obviously be a poor way to find solutions for many system of equations; the rest of this chapter will cover several different ways to find the solution in a more efficient manner.

Bars for x and y



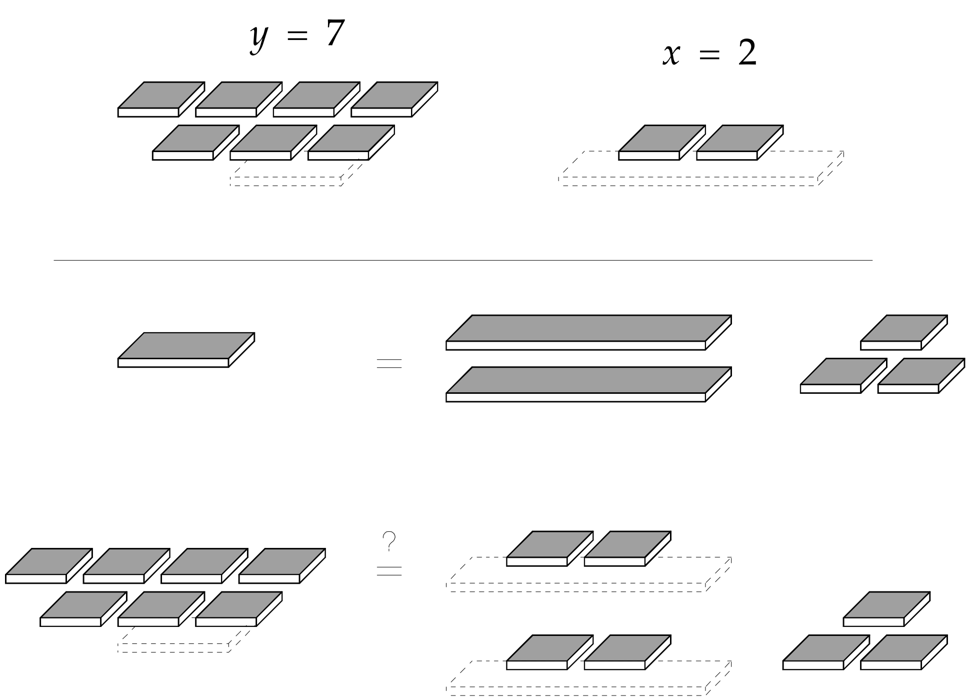
In the previous chapter, we created a new bar for y. We will continue to use the different bars for x and y to help us solve systems of equations. To review, the two bars look like this:



We will now use these bars to help us check the solutions we have already obtained for the two systems of equations given above. For the first system:

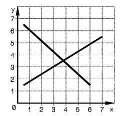
$$y = 2x + 3$$
$$y = x + 5$$
Solution: (2, 7) or $x = 2$, $y = 7$

To verify the solution, take the number for x and put that many unit chips on each x bar. Do the same for y. If both sides balance in *each* equation, the solution is correct:

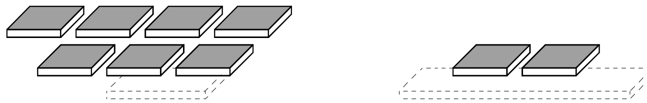


$$(y) = 2(x) + 3$$
$$(7) = 2(2) + 3$$

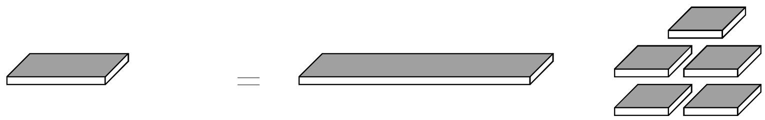
$$7 = 7$$



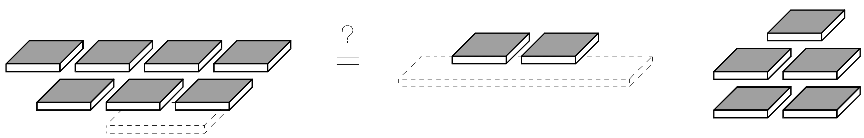
$y = 7$ $x = 2$



$(y) = (x) + 5$
$(7) = (2) + 5$

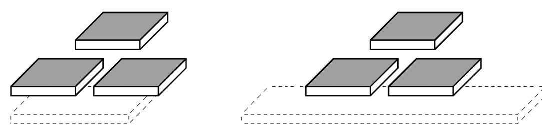


$7 = 7$



For the second system we have discussed ($y = x$ and $x + y = 6$):

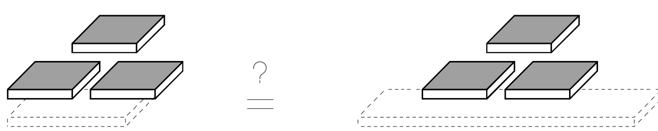
$y = 3$



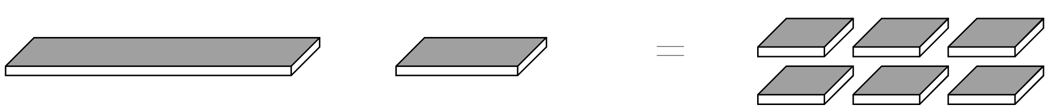
$x = 3$

$(y) = (x)$

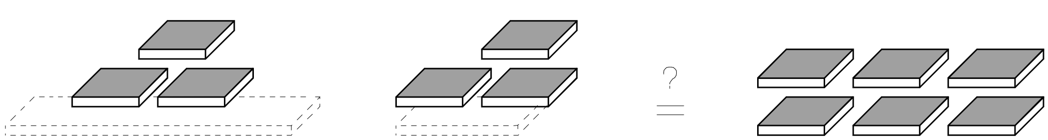
$3 = 3$



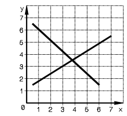
$(x) + (y) = 6$



$3 + 3 = 6$



Exercises



Find three solutions (ordered pairs) to each equation:

1. $y = x + 1$

2. $y = 2x - 3$

Use the chips to verify that the given ordered pair is a solution to the given system of equations:

3. $y = x + 1$ $\quad$ (4, 5)
$y = 2x - 3$

4. $2x + y = 12$ $\quad$ (4, 4)
$y = x$

5. $y = x + 2$ $\quad$ (2, 4)
$y = 6 - x$

6. $2x + y = 8$ $\quad$ (3,2)
$y = x - 1$

7. $y = 2x + 1$ $\quad$ (1, 3)
$y = -x + 4$

8. $x + 2y = 3$ $\quad$ (3, 0)
$y = x - 3$

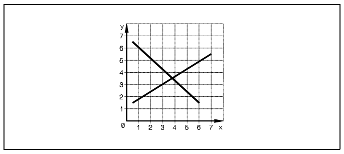
9. $y = 3x - 5$ $\quad$ (3, 4)
$y = x + 1$

10. $3x - y = 1$ $\quad$ (4, 11)
$y = 2x + 3$

Verify the solution without using the chips:

11. $2x - y = 0$ $\quad$ (0, 0)
$y = 3x$

12. $3x - y = 11$ $\quad$ $(3, {}^{-}2)$
$y = x - 5$



13. $2x - y = 1$ $(2, 3)$
$x = y - 1$

14. $y = x + 1$ $(1, 2)$
$x + y = 3$

15. $2x + 2y = 24$ $(4, 8)$
$3x - y = 4$

16. $x = 8 - 4y$ $(^{-}4, 3)$
$3x + 5y = 3$

Section 2
Solving by Graphing

Graphing the System

One method of solving a system of equations is to put the two equations on the same coordinate graph. Since we are dealing with lines, the lines will usually meet in one point. That point represents the ordered pair (x, y) that works for both equations at the same time. For example, consider the following system:

$$y = x + 1$$
$$y = 7 - x$$

First we make a table of 3 or more ordered pairs that satisfy each equation:

$y = x + 1$

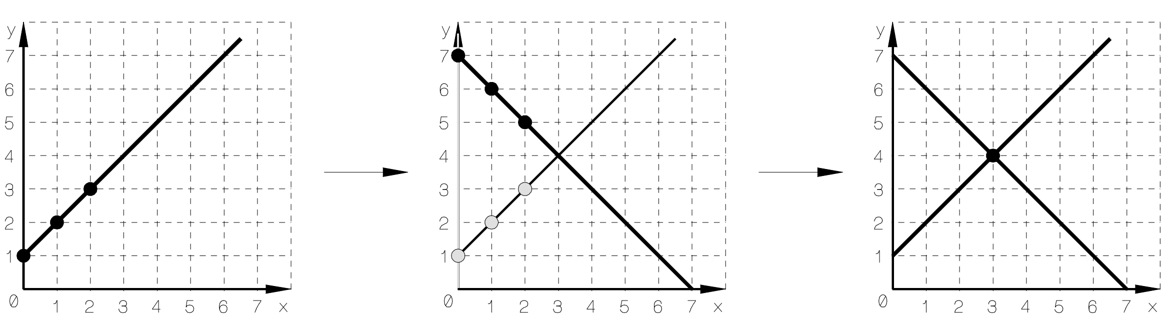
x	**y**
0	1
1	2
2	3

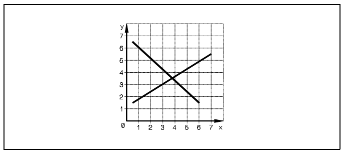
$y = 7 - x$

x	y
0	7
1	6
2	5

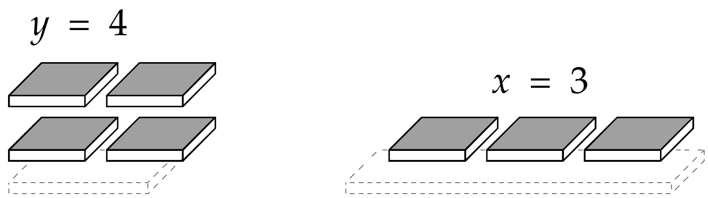
We then graph each line and find the point of intersection. This is our solution:

$y = x + 1$ $\qquad$ $y = 7 - x$ $\qquad$ Solution is (3, 4)

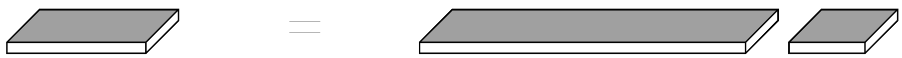




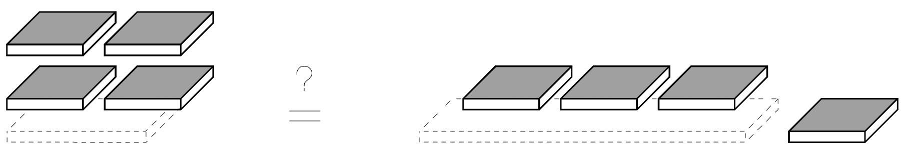
Finally, we must verify that the solution is correct. We test the solution (3, 4) by substituting $x = 3$ and $y = 4$ into each equation separately. If both equations make true statements for these values, our solution is correct:



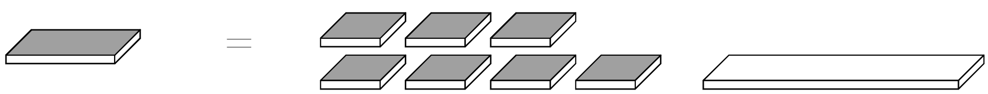
$(y) = (x) + 1$



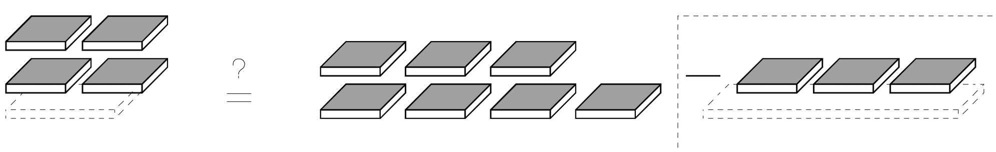
$(4) = (3) + 1$

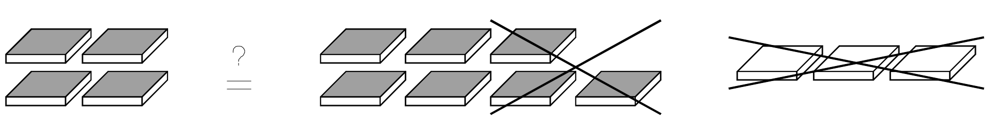


$(y) = 7 - (x)$

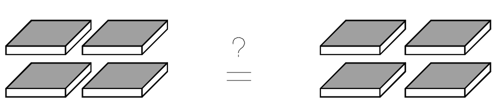


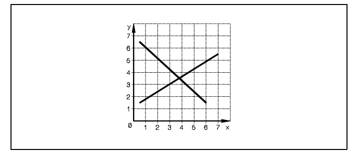
$(4) = 7 - (3)$





$4 = 4$





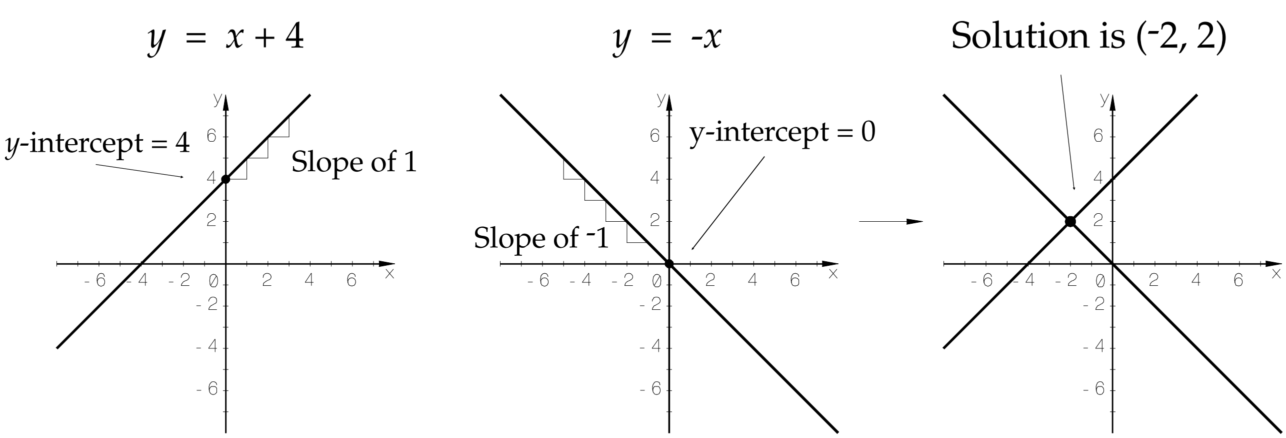
We will do another example to illustrate the graphing method. Consider the system show below:

$$y - x = 4$$
$$x + y = 0$$

Our first step is to solve each equation for y. To do this, we add and multiply on both sides of each equation until we "isolate" y. (See RULES AND GRAPHS, Section 4).

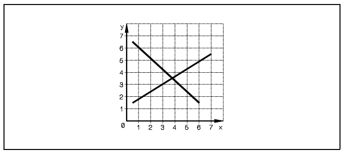
$\boldsymbol{y - x = 4}$	$\boldsymbol{x + y = 0}$
$y - x = 4$	$x + y = 0$
$+x \quad +x$	$-x \quad\quad -x$
$y = x + 4$	$y = -x$
$y = \mathbf{1}x + \mathbf{4}$	$y = {}^{-}\mathbf{1}x + \mathbf{0}$
Slope = 1; y-intercept = 4	Slope = ⁻1; y-intercept = 0

We now have two equations in the slope-intercept form. We plot both equations on the same graph:



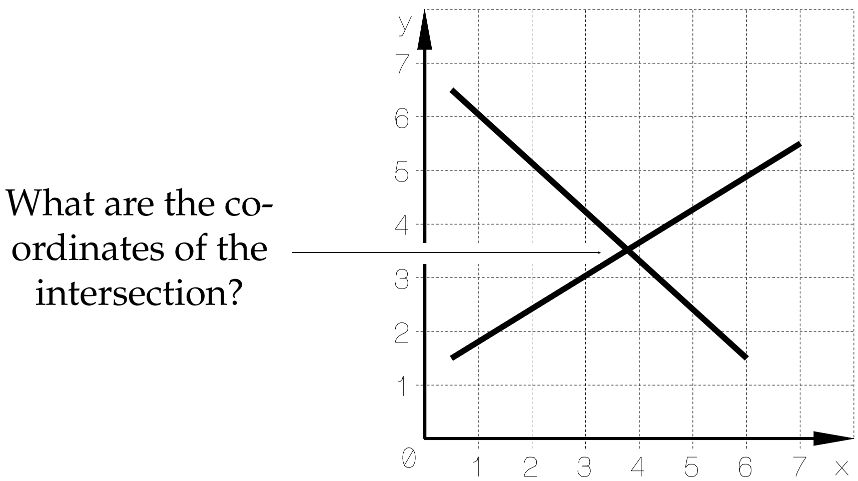
Finally we see the solution is (⁻2, 2) and we verify it in *both* equations:

$\boldsymbol{y - x = 4}$	$\boldsymbol{x + y = 0}$
$(y) - (x) = 4$	$(x) + (y) = 0$
$(2) - ({}^{-}2) = 4$	$({}^{-}2) + (2) = 0$
$4 = 4$	$0 = 0$

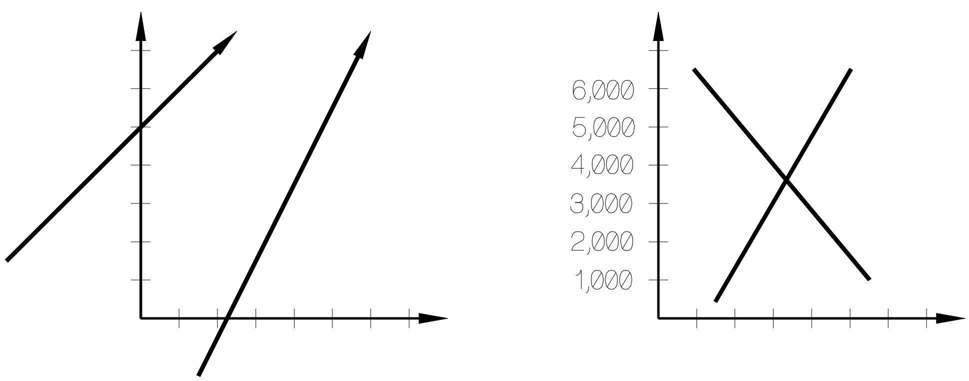


Limitations of the Graphing Method

The graphing method gives us a clear picture of the solution to a system of equations, but this method has some important limitations. First, it may take a great deal of time to graph the lines, especially if the numbers are large or the slopes are fractional amounts. Second, it is not always easy to tell where the lines actually meet; if the intersection is not on a place where the grid lines meet, then we have to guess at a fractional answer:

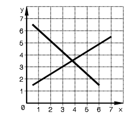


Finally, the lines may meet at a place that has such large coordinates that it is impractical to graph them at all.



In the sections that follow, we will learn two alternative techniques to supplement the graphing method for finding solutions to a system of equations. These new methods will rely on algebraic symbols instead of graphing lines.

Summary



The steps for solving a system of equations using a graph are as follows:

- **If necessary, solve each equation for *y*.**
- **Graph the equations by plotting 3 points for each line or by using the slope-intercept method.**
- **Read the solution—the point at the intersection of the lines.**
- **Check the solution by substituting the x and y values for the point of intersection into *both* of the original equations to be sure that both equations give true statements.**

Exercises

Graph each system and find the ordered pair that is the solution. Check x and y in both equations.

1. $x + y = 3$
$y = x - 5$

2. $y = x + 1$
$y = 2x - 3$

3. $x + 2 = y$
$x + y = 2$

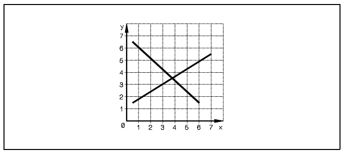
4. $y = x - 3$
$y = {}^-2x + 6$

5. $x + 2y = 5$
$x - y = 2$

6. $x + y = {}^-1$
$y = {}^-5 + x$

7. $y = x$
$2x + y = 6$

8. $y = x + 1$
$y = 2x - 1$



9. $2x - y = 4$

$2x + y = {}^{-}4$

10. $x - y = {}^{-}1$

$x + y = 3$

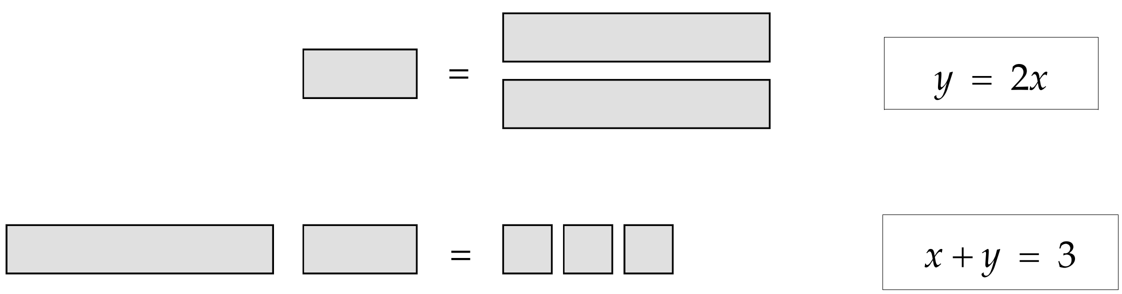
Section 3
The Substitution Method

Using the Equations Together

Another approach to solving systems of equations is called **substitution.** The substitution method gives us the exact solution, even if the values of x and y are fractional or very large. Substitution doesn't give us a picture of the equations; it just gives us the value of the solution. We will use information from both equations to determine the solution. For example, consider this system of equations:

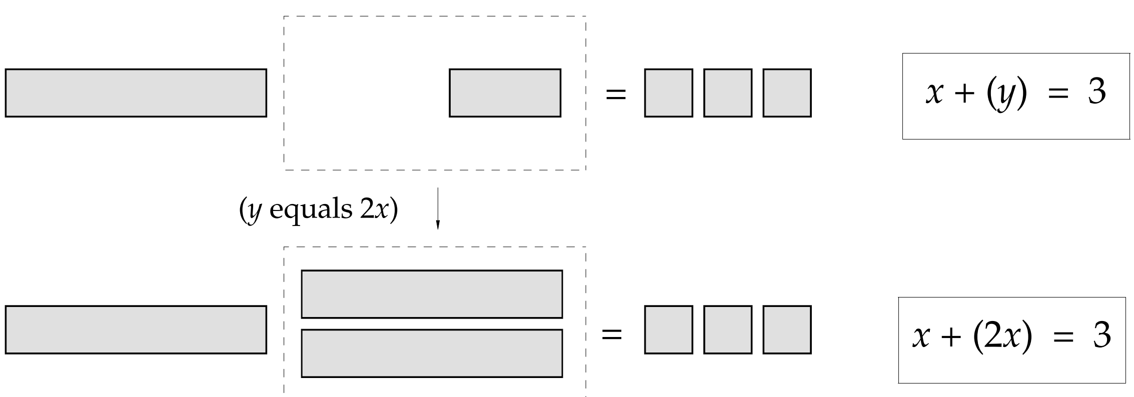
$$\begin{aligned} y &= 2x \\ x + y &= 3 \end{aligned}$$

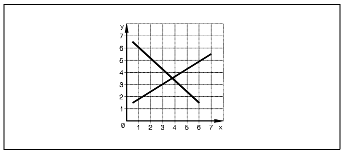
Using our new y bar, here is a picture of this system:



If the system has a solution, then the x and y values of that solution will work (make true statments) in both equations.

The first equation states that y is equal to $2x$, so y and $2x$ stand for the same amount and we can *substitute* $2x$ instead of y in the second equation:





This substitution gives a new equation:

$$x + (2x) = 3$$

We can solve this equation because it only contains the one variable x. Here is how we complete the solution with our chips:

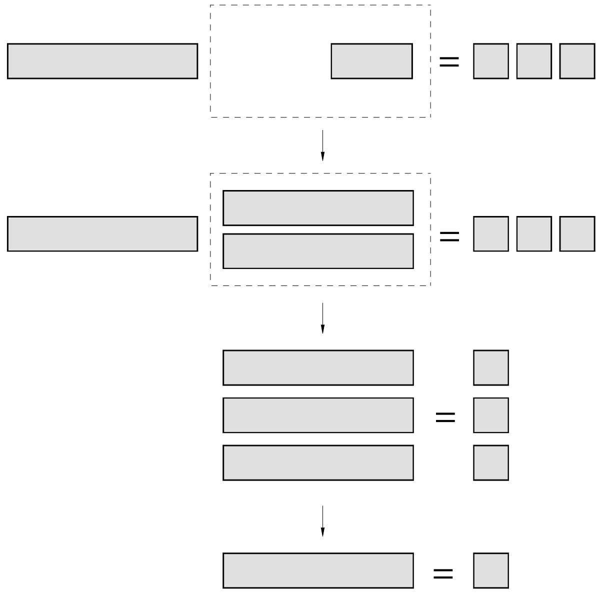
$x + (y) = 3$

$(y = 2x)$

$x + (2x) = 3$

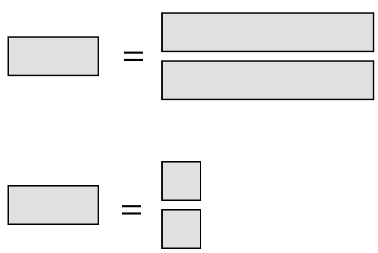
$3x = 3$

$x = 1$



Since we know that x must be 1, our next step is to use this information to help us find y:

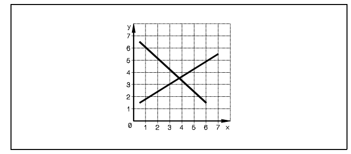
$y = 2(x)$
$y = 2(1)$
$y = 2$



Here is the process with symbols alone:

$$y = 2x$$
$$x + (y) = 3$$

$$x + (2x) = 3$$
$$3x = 3$$
$$\frac{1}{3}(3x) = \frac{1}{3}(3)$$
$$x = 1$$

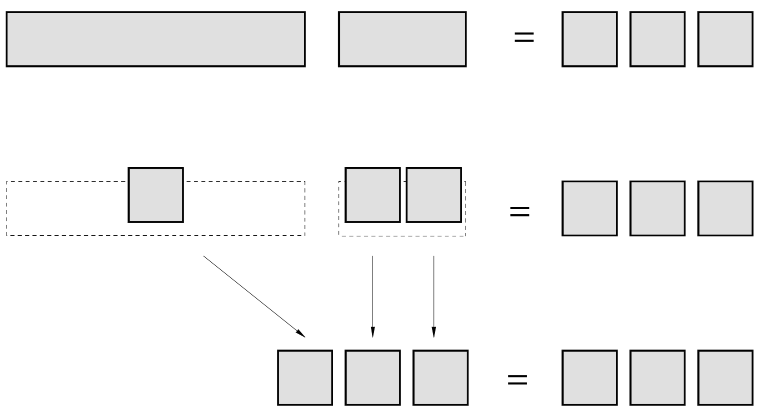


Completing the solution for y:

$$y = 2x$$
$$y = 2(1) \quad \text{(because } x = 1\text{)}$$
$$y = 2$$

Our answer is $x = 1$, $y = 2$. We can also write this as (1, 2).

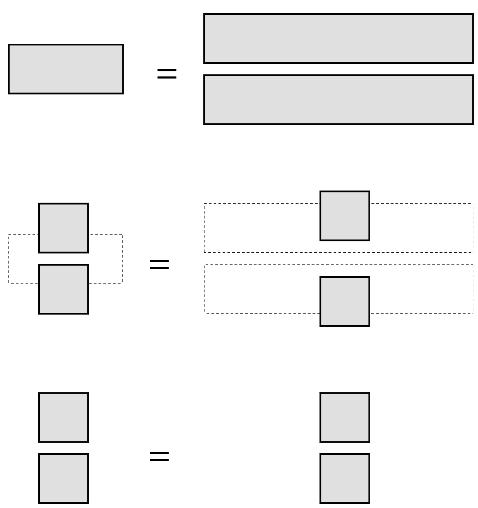
Finally, we check our solution by replacing the x and y in *both* equations with 1 and 2:



$$(x) + (y) = 3$$

$$(1) + (2) = 3$$

$$3 = 3$$



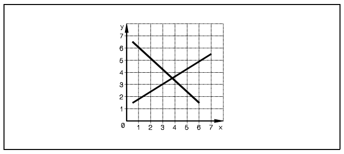
$$(y) = 2(x)$$

$$(2) = 2(1)$$

$$2 = 2$$

More Examples

We will now look at two more examples. We solve each system in the same way, by substituting the value of one variable (from one equation) into the second equation.



Here is the first system:

$$\begin{aligned} y &= 2x+1 \\ 2x+y &= 9 \end{aligned}$$

$y = 2x + 1$

$2x + y = 9$

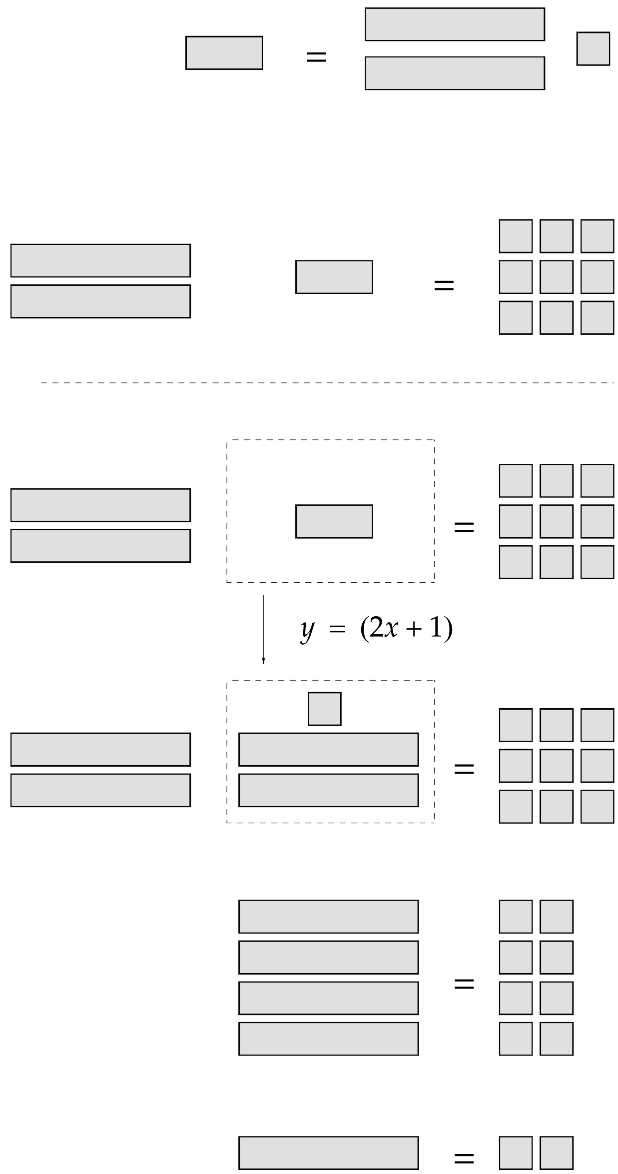
$2x + (y) = 9$

(substitute for y)

$2x + (2x + 1) = 9$
$4x + 1 = 9$

$4x = 8$

$x = 2$

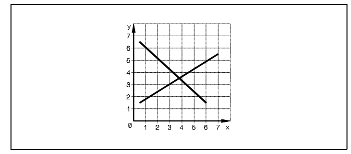


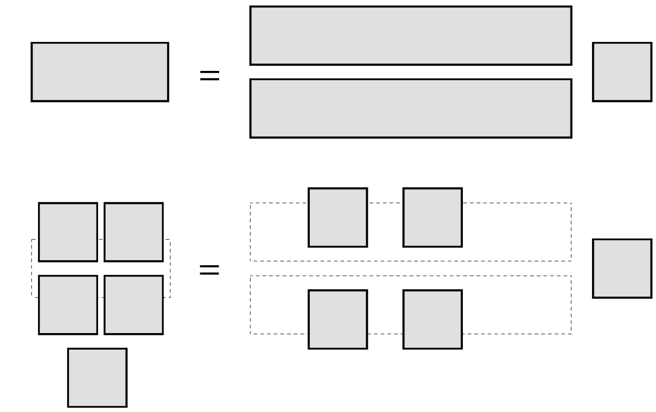
Now that we know x, we can continue on to solve for y:

$$\begin{aligned} y &= 2(x)+1 \\ y &= 2(2)+1 \quad \text{(because } x=2\text{)} \\ y &= 5 \end{aligned}$$

The solution is $x = 2$ and $y = 5$, or (2, 5)

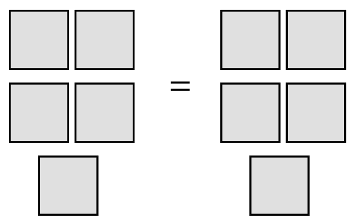
Here is the check:





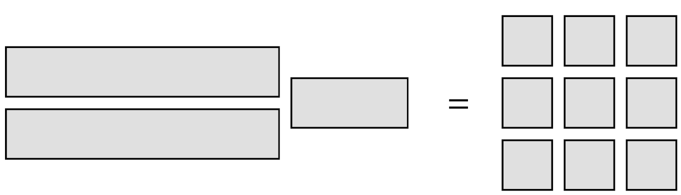
$(y) \;=\; 2(x)+1$

$(5) \;=\; 2(2)+1$

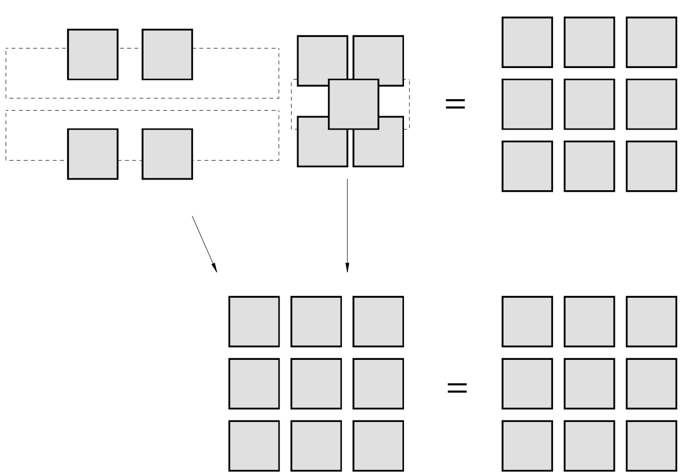


$5 \;=\; 5$

The check continues:

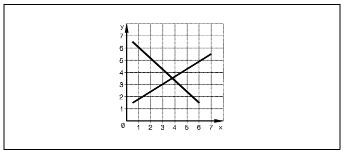


$2(x)+(y) \;=\; 9$



$2(2)+(5) \;=\; 9$

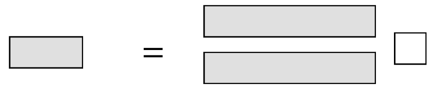
$9 \;=\; 9$



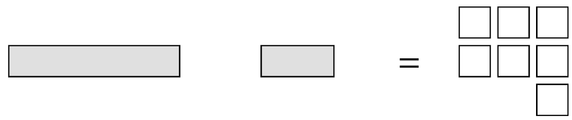
Here is a second example of a system of equations to solve:

$$y = 2x - 1$$
$$x + y = {}^{-}7$$

$y = 2x - 1$

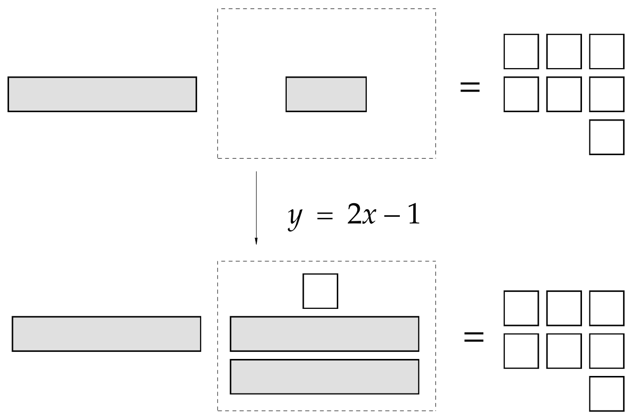


$x + y = {}^{-}7$



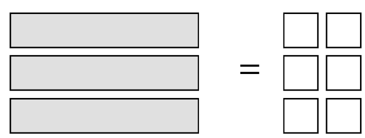
$x + (y) = {}^{-}7$

(substitute for y)



$x + (2x - 1) = {}^{-}7$
$3x - 1 = {}^{-}7$

$3x = {}^{-}6$

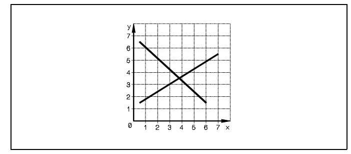


$x = {}^{-}2$

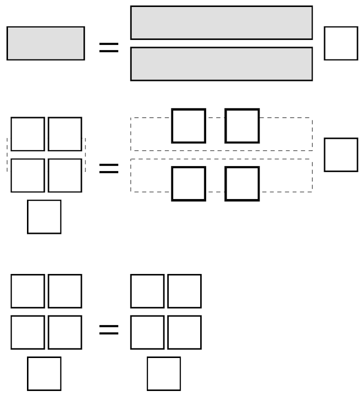
Now that we know x is ${}^{-}2$, we can continue on to solve for y:

$$y = 2(x) - 1$$
$$y = 2({}^{-}2) - 1 \quad \text{(because } x = {}^{-}2\text{)}$$
$$y = {}^{-}5$$

The solution is $x = {}^{-}2$ and $y = {}^{-}5$, or $({}^{-}2, {}^{-}5)$



Here is the check. For the first equation:

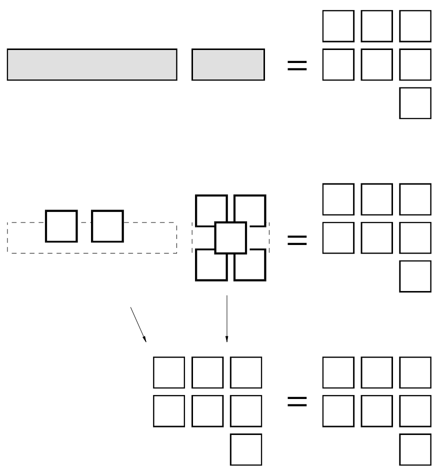


$$(y) = 2(x) - 1$$

$$(^-5) = 2(^-2) - 1$$

$$^-5 = {}^-5$$

The second equation:



$$(x) + (y) = {}^-7$$

$$(^-2) + (^-5) = {}^-7$$

$$^-7 = {}^-7$$

Working with Fractions

Systems of equations can contain fractions. When we solve for one unknown and get a fraction, we substitute the answer in the second problem in the same way as we did before:

$$y = \frac{1}{2}x \quad \rightarrow \quad y = \left(\frac{1}{2}x\right)$$

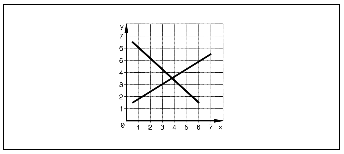
$$2x + 4y = 8 \quad \rightarrow \quad 2x + 4(y) = 8$$

$$2x + 4\left(\frac{1}{2}x\right) = 8$$

$$2x + 2x = 8$$

$$4x = 8$$

$$x = 2$$



We now return the x value (2) into the original equation and solve for y.

$$y = \frac{1}{2}x$$

$$y = \frac{1}{2}(2) \quad \text{(because } x = 2\text{)}$$

$$y = 1$$

The answer is (2, 1)

Working with Negatives

When one of the equations has a negative number of y's, we must be careful to remember that -y means the opposite of y. For example:

$$y = 3x$$

$$x - y = 2$$

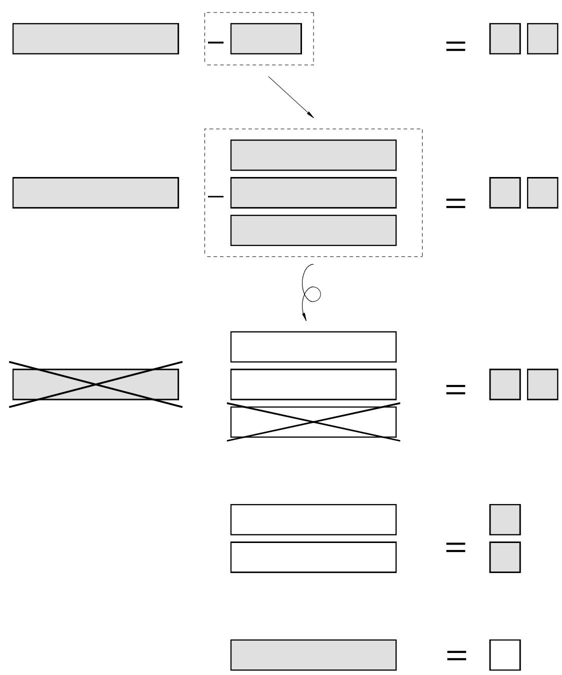
$$x - (y) = 2$$

$(y = 3x)$

$$x - (3x) = 2$$

$$^-2x = 2$$

$$x = {}^-1$$



We now have the value of x, so we can continue on in the usual way:

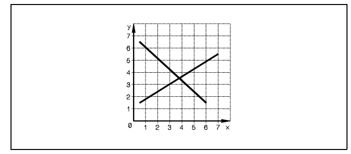
$$y = 3(x)$$

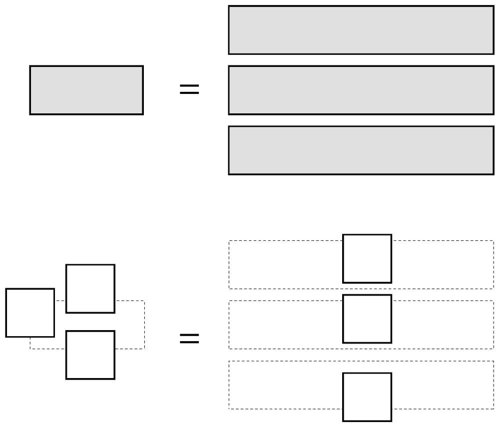
$$y = 3(^-1) \quad \text{(because } x = {}^-1\text{)}$$

$$y = {}^-3$$

The solution is ($^-1$, $^-3$)

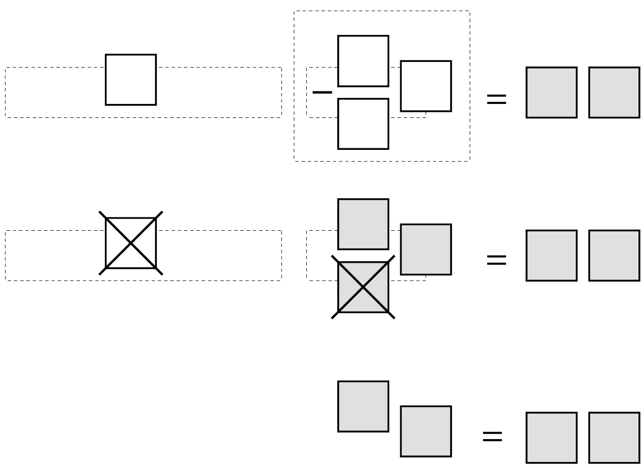
Notice that we substituted $3x$ for y, but that y was being subtracted, so the $3x$ was flipped to its opposite. When we check, we must also be careful; since y is $^-3$, we substitute $^-3$ in for y but do not ignore the negative sign:





$$(y) = 3(x)$$

$$(^-3) = 3(^-1)$$
$$^-3 = {}^-3$$



$$(x) - (y) = 2$$

$$(^-1) - (^-3) = 2$$

$$^-1 + 3 = 2$$

$$2 = 2$$

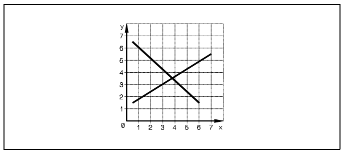
Different Forms

For simplicity, we have been using examples where one equation is already solved for y in terms of x. Systems may be in other forms where one equation is solved for x or where neither equation is in the desired form.

For example, here is a system where the second equation is solved for x:

$$2x - 3y = 7$$
$$x = 2y + 3$$

We can substitute from either equation at the beginning, so we choose the



second equation; since it is solved for x, we substitute $2y + 3$ in for x in the first equation:

$$2(x) - 3y = 7$$

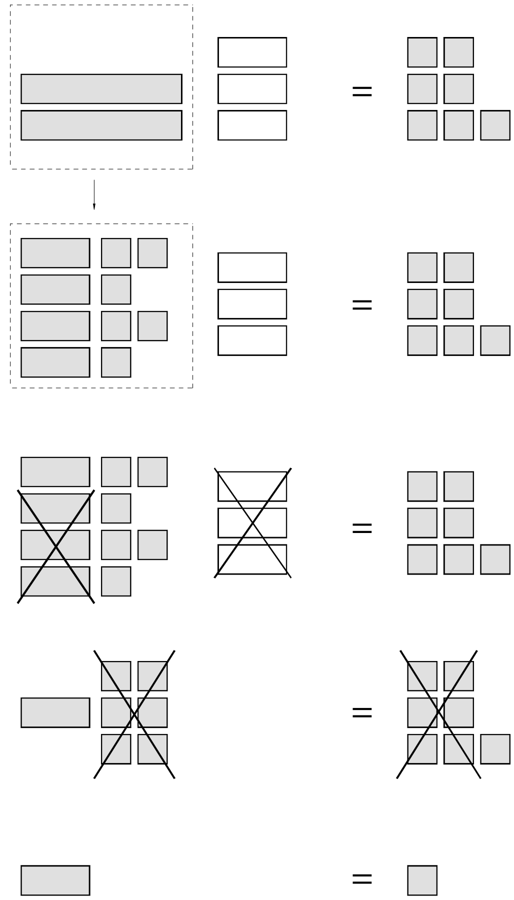
$$\downarrow \quad (x = 2y + 3)$$

$$2(2y + 3) - 3y = 7$$

$$4y + 6 - 3y = 7$$

$$y + 6 = 7$$

$$y = 1$$



Continuing on to solve for x:

$$x = 2(y) + 3$$

$$x = 2(1) + 3 \quad \text{(because } y = 1\text{)}$$

$$x = 5$$

The solution is (5, 1)

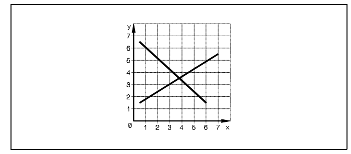
Changing the Form

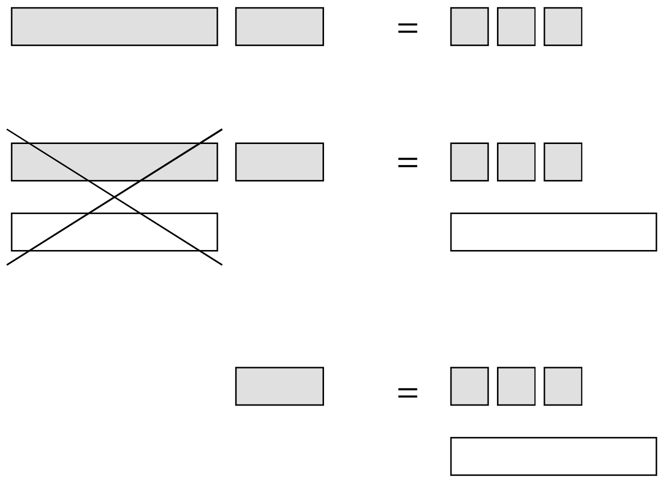
If neither equation is solved for x or y, we use the technique from RULES AND GRAPHS, Section 4. We pick one equation and rearrange the terms by adding and multiplying both sides until we have the equation solved for either x or y. For example, given the equations:

$$x + y = 3$$

$$x - y = 1$$

Our first step is to solve one equation for y; choose $x + y = 3$:





$$x + y = 3$$
$$-x \qquad -x$$
$$y = -x + 3$$

We continue on by substituting for y in the same manner as before:

$$y = (-x + 3)$$
$$x - (y) = 1$$

$$x - (-x + 3) = 1$$
$$x + x - 3 = 1$$
$$2x - 3 = 1$$
$$2x = 4$$
$$x = 2$$

$$y = -(x) + 3$$
$$y = {}^{-}(2) + 3$$
$$y = 1$$

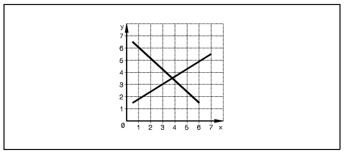
The solution is (2, 1).

Here is a more difficult example:

$$2x + 3y = 8$$
$$3x + 3y = 9$$

We solve the first equation for y:

$$2x + 3y = 8$$
$$2x - 2x + 3y = 8 - 2x$$
$$3y = 8 - 2x$$
$$y = \frac{8}{3} - \frac{2}{3}x$$



We continue by substituting the expression for y in the second equation:

$$y = \left(\frac{8}{3} - \frac{2}{3}x\right)$$

$$3x + 3(y) = 9$$

$$3x + 3\left(\frac{8}{3} - \frac{2}{3}x\right) = 9$$

$$3x + \frac{24}{3} - \frac{6}{3}x = 9$$

$$3x + 8 - 2x = 9$$

$$x + 8 = 9$$

$$x = 1$$

$$y = \frac{8}{3} - \frac{2}{3}x$$

$$y = \frac{8}{3} - \frac{2}{3}(1) \quad \text{(because } x = 1\text{)}$$

$$y = \frac{8}{3} - \frac{2}{3}$$

$$y = \frac{6}{3}$$

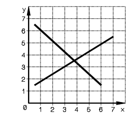
$$y = 2$$

To begin, we can choose *either* equation and solve for *either* unknown. While any choice will work eventually, it is best to "look ahead" to find the equation that looks easiest to solve. To avoid fractions, you can try to pick an equation where division isn't required to solve for either x or y.

Summary: Substitution Method

- **If necessary, solve one equation for either unknown. If an equation is already solved, use that one. We will call this the first equation.**
- **Substitute the expression for y or x in the second equation, leaving this equation with only one unknown remaining.**
- **Solve this second equation for the one unknown.**
- **Put this value back into the first equation and solve for the other unknown.**
- **Check the solution (x, y) by putting the values for x and y into both equations and confirming that these values make true statements in both equations.**

Exercises



Solve the following systems by substituting the expression for x from the first equation into the second equation:

1.
$$\begin{aligned} x &= 2y + 5 \\ 2y + x &= 9 \end{aligned}$$

2.
$$\begin{aligned} x &= 3 - y \\ y - 3x &= 7 \end{aligned}$$

3.
$$\begin{aligned} x &= y + 5 \\ x + 2y &= 8 \end{aligned}$$

4.
$$\begin{aligned} x &= 6 - y \\ y &= {}^{-}2x + 6 \end{aligned}$$

Solve the following systems by solving the first equation for y (if necessary), and substituting the expression for y into the second equation:

5.
$$\begin{aligned} y &= x + 5 \\ x + y &= 9 \end{aligned}$$

6.
$$\begin{aligned} x + y &= {}^{-}5 \\ x + 2y &= {}^{-}4 \end{aligned}$$

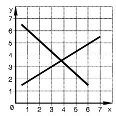
7.
$$\begin{aligned} y &= 3 - 2x \\ 3y - 2x &= 1 \end{aligned}$$

8.
$$\begin{aligned} 2y &= x + 6 \\ 4x + y &= 12 \end{aligned}$$

Solve the following systems by substitution:

9.
$$\begin{aligned} x + y &= 5 \\ 3x + 2y &= 13 \end{aligned}$$

10.
$$\begin{aligned} x + 2y &= {}^{-}5 \\ x - 3y &= 10 \end{aligned}$$



11.
$$\begin{aligned} y &= 2x + 3 \\ 2x + 5y &= 3 \end{aligned}$$

12.
$$\begin{aligned} 2x + y &= 7 \\ 5x + 2y &= 6 \end{aligned}$$

13.
$$\begin{aligned} x &= 4 \\ 3x - 4y &= 0 \end{aligned}$$

14.
$$\begin{aligned} 2x + 3y &= 2 \\ x + y &= \frac{5}{6} \end{aligned}$$

15.
$$\begin{aligned} x + y &= 0 \\ x - y &= 0 \end{aligned}$$

Section 4
The Addition Method

Adding Equations

Another method of solving systems of equations is to add the equations together. If the equations are arranged properly, this can result in a quick solution.

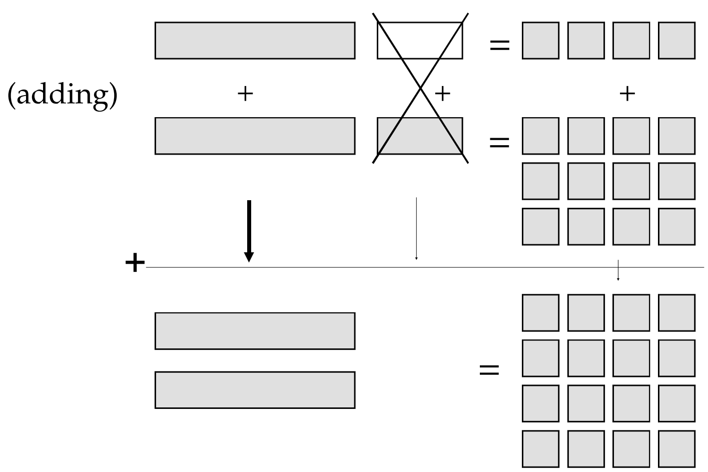
When we solve equations, we are able to add the same amount to both sides. This works because we start with two equal expressions; if we make the same changes to both sides of an equation, the expressions are still equal.

This idea will give us another method of solving a system of equations:

$$\begin{aligned} x - y &= 4 \\ x + y &= 12 \end{aligned}$$

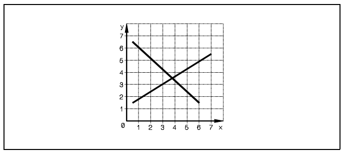
The second equation tells us that $x + y$ is equal to 12. We start with the first equation; instead of adding 12 to both sides, we add $x + y$ to the left side and 12 to the right side. We call this process **the addition method**:

We added the two equations together and the y's canceled out, giving a new equation with x's and units only. We solve this in the usual way for x:



$$\begin{array}{ccccc} x & - & y & = & 4 \\ x & + & y & = & 12 \\ \hline 2x & & & = & 16 \end{array}$$

$$\begin{aligned} 2x &= 16 \\ \frac{1}{2}(2x) &= \frac{1}{2}(16) \\ x &= 8 \end{aligned}$$



To find the value of y, we put the value of x back into either of the original equations; we then solve for y:

$$\begin{aligned}(x) + y &= 12\\ 8 + y &= 12\\ y &= 4\end{aligned}$$

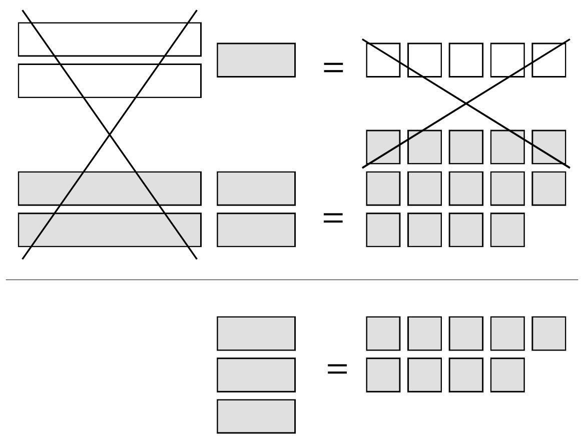
The answer is (8, 4).

Here is another example of a solving a system with the addition method:

$$\begin{aligned}{}^{-}2x + y &= {}^{-}5\\ 2x + 2y &= 14\end{aligned}$$

We "add" the two equations together. This time, the x's cancel out:

$$\begin{array}{rcccccl} -2x & + & y & = & -5\\ 2x & + & 2y & = & 12\\ \hline & & 3y & = & 9\end{array}$$



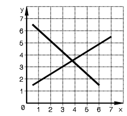
Remember that we are adding *equal amounts* to both sides of the equation, because $2x + 2y$ and 12 are equal. We finish the solution in the usual way:

$$\begin{aligned}3y &= 9\\ \frac{1}{3}(3y) &= \frac{1}{3}(9)\\ y &= 3\end{aligned}$$

$$\begin{aligned}2x + 2y &= 14\\ 2x + 2(3) &= 14\\ 2x + 6 &= 14\\ 2x &= 8\\ x &= 4\end{aligned}$$

The answer is (4, 3)

Rewriting One Equation



The examples in this section have all worked out very neatly. When we added the equations together, one of the unknowns canceled out, allowing us to solve for the other unknown. This will not always happen.

In the system shown below, adding the equations together *does not* cancel either variable. The resulting equation still has both variables and we cannot solve it for either one:

$$\begin{array}{rcrcr} 2x & + & 3y & = & 4 \\ -x & + & 2y & = & 5 \\ \hline x & + & 5y & = & 9 \end{array}$$

This difficulty can be overcome if we rewrite one of our original equations *before* we add the two equations together. If we multiply both sides of the second equation by 2, it will then have a ^-2x term which will cancel the $2x$ term from the first equation:

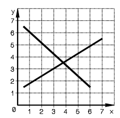
$\mathbf{2x} + 3y = 4$	$\rightarrow \quad \rightarrow$	$2x + 3y = 4$
$\mathbf{-x} + 2y = 5 \rightarrow$	$\mathbf{2}(-x + 2y) = \mathbf{2}(5) \rightarrow$	$^-2x + 4y = 10$
		$7y = 14$
		$y = 2$

We changed our original equations so their x terms were opposites (equal numbers and opposite signs). When we added the equations together, the x terms canceled out, leaving only y terms and numbers. We could then solve for y.

Here is another example:

$2x + \mathbf{3y} = {}^-1$	$\rightarrow \quad \rightarrow$	$2x + 3y = {}^-1$
$5x + \mathbf{y} = 4 \rightarrow$	$\mathbf{^-3}(5x + y) = \mathbf{^-3}(4) \rightarrow$	$^-15x - 3y = {}^-12$
		$^-13x = {}^-13$
		$x = 1$

This time we multiplied to make the y terms cancel (same number of y's and opposite signs). Since both y terms were positive, we multiplied the second equation by $^-3$. This method works whenever the x's in one equation



are a multiple of the x's in the other equation, or the y's in one equation are a multiple of the y's in the other equation.

To finish the solution:

$$
\begin{aligned}
5(x) + y &= 4 \\
5(1) + y &= 4 \\
y &= {}^{-}1
\end{aligned}
$$

Rewriting Both Equations

It is not always possible to multiply both sides of one equation and then cancel out by adding. Consider the system below:

$$
\begin{aligned}
2x + 3y &= {}^{-}8 \\
3x + 4y &= {}^{-}11
\end{aligned}
$$

We cannot multiply $2x$ by any number to cancel $3x$ and we cannot multiply $3y$ to cancel $4y$. Instead we have to separately multiply *both* equations to make them cancel:

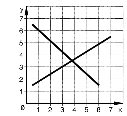
$2x + 3y = {}^{-}8 \rightarrow$	$\mathbf{3}(2x + 3y) = \mathbf{3}({}^{-}8) \rightarrow$	$6x + 9y = {}^{-}24$
$3x + 4y = {}^{-}11 \rightarrow$	$\mathbf{{}^{-}2}(3x + 4y) = \mathbf{{}^{-}2}({}^{-}11) \rightarrow$	${}^{-}6x - 8y = 22$
(Eliminate x. 6 is the common multiple of 2 and 3.)		$y = {}^{-}2$

Did you notice that we multiplied one equation by 3 and the other equation by ${}^{-}2$? *We must multiply both sides of each equation by the same number, but we can multiply the two different equations by two different numbers.* Our object is to get the terms of one variable to be opposite in the two equations so that they will cancel when the equations are added.

To do this, we must find the *least common multiple* of the original numbers of x's or y's. In this example, we chose x, and the least common multiple of 2 and 3 is 6. We then must choose the multipliers so that one equation has a ${}^{+}6x$ and the other equation has a ${}^{-}6x$. To finish the solution:

$$
\begin{aligned}
2x + 3(y) &= {}^{-}8 \\
2x + 3({}^{-}2) &= {}^{-}8 \\
2x + {}^{-}6 &= {}^{-}8 \\
2x &= {}^{-}2 \\
x &= {}^{-}1
\end{aligned}
$$

The solution is $({}^{-}1, {}^{-}2)$



Here is a more complex example:

$$\begin{aligned} 4x + 5y &= 23 \\ 6x + 7y &= 33 \end{aligned}$$

The steps are as follows:

$4x + 5y = 23 \rightarrow$	$\mathbf{-3}(4x + 5y) = \mathbf{-3}(23) \rightarrow$	$-12x - 15y = -69$
$6x + 7y = 33 \rightarrow$	$\mathbf{2}(6x + 7y) = \mathbf{2}(33) \rightarrow$	$12x + 14y = 66$
(Eliminate x. The common multiple of 4 and 6 is 12.)		$-y = -3$ $y = 3$

We finish the solution in the usual way:

$$\begin{aligned} 4x + 5(y) &= 23 \\ 4x + 5(3) &= 23 \\ 4x + 15 &= 23 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

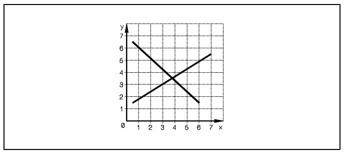
The solution is (2, 3)

The steps for adding equations are as follows:

- **Choose a variable to eliminate.**
- **Find a common multiple of the numbers of x's or y's.**
- **Multiply the original equations to give new equations where the terms for the chosen variable are opposite.**
- **Add the equations together, letting the chosen variable cancel out.**
- **Solve for the remaining variable.**
- **Plug this solution back into the original equation and solve for the remaining variable.**

How Do We Multiply?

From the examples above, we can develop a plan for multiplying the equations. Since our only tool is multiplication, we get multiples of the original numbers of x and y. We are looking for a common multiple,



preferably a **least common multiple—**familiar from the **least common denominator**.

In the last example, we looked at $4x$ and $6x$ and found the least common multiple of 4 and 6. This is 12. We then multiplied the equation containing $4x$ by $^{-}3$, so $4x$ became $^{-}12x$, and we multiplied the equation containing $6x$ by 2, so that $6x$ became $12x$.

How do we know whether we should cancel x or y? We simply pick the one that looks easiest—usually the one where we will have less multiplying to do (where the least common multiple is smaller). Here is a summary of these ideas:

- **If either x's or y's in both equations are ready to cancel, then no multiplying is required.**

- **If the x's (or y's) in one equation are a multiple of the x's (or y's) in the other equation, then multiply both sides of one equation only. Remember to use a negative number when multiplying if necessary.**

- **If the x's (or y's) in one equation are *not* a multiple of the x's (or y's) in the other equation, then separately multiply both sides of *both* equations to get a common multiple of one variable (x or y). Use a negative number if necessary.**

Changing the Form of the Equations

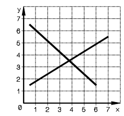
When equations are not given in the form we have been using, it is easy to rearrange the terms. This is a slightly different process than solving an equation for x or y. In this case, we are adding to both sides with a different goal—to have the equation in **standard form**, as shown below:

$$\_\_\,x + \_\_\,y = \_\_$$

For example, in the single equation below, we add to both sides until the x's and y's are on one side (usually the left) and the units are on the other side (usually the right):

$$\begin{aligned} \mathbf{3x + 4} &= \mathbf{y + 7} \\ 3x + 4 - 4 &= y + 7 - 4 \\ 3x &= y + 3 \\ 3x - y &= y - y + 3 \\ \mathbf{3x - y} &= \mathbf{3} \end{aligned}$$

We are not solving for x or y. Our goal is to have the x and y terms on the left side of the equation and the number term on the right side of the equation.



A second example is shown below:

$$\begin{aligned}
\mathbf{9x + 3y + 4} &= \mathbf{7(x - 2) + 1} \\
9x + 3y + 4 &= 7x - 14 + 1 \\
9x + 3y + 4 &= 7x - 13 \\
9x + 3y + 4 - 4 &= 7x - 13 - 4 \\
9x + 3y &= 7x - 17 \\
9x + 3y - 7x &= 7x - 7x - 17 \\
\mathbf{2x + 3y} &= \mathbf{{}^{-}17}
\end{aligned}$$

Notice that we need to multiply out parentheses but that we do not need to divide at the end.

Here are the steps to get an equation into standard form:

- **Multiply out all parentheses.**

- **Combine similar terms on both sides.**

- **Decide how to get the equation into the form: $\_\,x + \_\,y = \_$**
  Choose the left side for the unknown and the right for the units.

- **Add to cancel the x's and y's on the right side.**

- **Add to cancel the units on the left side.**

Summary

We now have a complete addition method for solving systems of equations. Here are the steps we have developed:

- **Rewrite each equation in the standard form by multiplying out parentheses, combining terms, and arranging the unknowns on one side.**

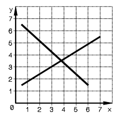
- **If necessary, multiply one or both equations so that either the x's or the y's will cancel.**

- **Add the two equations. The chosen variable will cancel out.**

- **Solve the resulting equation for the remaining variable.**

- **Substitute that result back into either original equation and solve for the other unknown.**

- **Check by substituting x and y into both equations.**



Exercises

Solve the following systems by addition. Change the form if necessary:

1. $$\begin{aligned} 2x + y &= 8 \\ x - y &= 4 \end{aligned}$$

2. $$\begin{aligned} -x + y &= 1 \\ x + y &= 3 \end{aligned}$$

3. $$\begin{aligned} ^{-}2x - 3y &= {^{-}12} \\ 2x + y &= 8 \end{aligned}$$

4. $$\begin{aligned} x - y &= 3 \\ y &= 6 - 2x \end{aligned}$$

5. $$\begin{aligned} 2x + y &= 9 \\ y &= 3x - 3 \end{aligned}$$

6. $$\begin{aligned} 3x + 2y - 7 &= 0 \\ x + 11 &= 2y \end{aligned}$$

Solve the following systems by multiplying one equation. Change the form first if necessary:

7. $$\begin{aligned} 2x - y &= 7 \\ 3x + 2y &= 0 \end{aligned}$$

8. $$\begin{aligned} 3x + 2y &= 8 \\ 3x + y &= 7 \end{aligned}$$

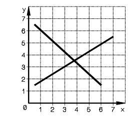
9. $$\begin{aligned} x - 3y &= {^{-}9} \\ 2x + 4y &= 22 \end{aligned}$$

10. $$\begin{aligned} 3x + 2y &= 1 \\ y &= 3 - 2x \end{aligned}$$

11. $$\begin{aligned} x - 2y &= {^{-}5} \\ x &= y - 1 \end{aligned}$$

12. $8 - y = 5x$

$4y + 14 = 3x$



Solve the following equations by multiplying both equations. Change the form if necessary:

13. $3x + 5y = 28$

$5x - 3y = 24$

14. $7x - 5y = 8$

$^{-}5x + 4y = {}^{-}1$

15. $3x + 4y = 13$

$^{-}7x + 3y = {}^{-}18$

16. $2x + 3y + 2 = 0$

$3x = 2y + 10$

17. $3x - 8y = 7$

$^{-}5y = 45 - 10x$

18. $4x - 14 = {}^{-}3y$

$2y = 9x - 14$

Section 5
Choosing a Method

Three Methods

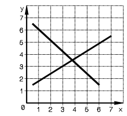
We have learned three different ways to solve a system of linear equations—**graphing**, **substitution**, and **addition**. Each method has advantages and disadvantages, but all three are useful for solving systems and illustrating the solutions.

In this book, the graphing method is included mainly for the purpose of illustration; it is important to remember that a system of linear equations can be represented by the two lines—the solution is the intersection of the lines. Graphing is obviously not very efficient as a practical method to solve linear equations exactly. In statistics and other fields, graphing is very useful when we have the *ordered pairs* instead of the equations.

Here is a comparison of some of the advantages and disadvantages of the three methods:

Method	Advantages	Disadvantages
Graphing	• Shows a picture of the solution • Not much algebraic calculation to do	• Not very accurate • Difficult with very large or very small numbers • Takes time to graph two equations
Substitution	• Only one equation to rewrite • No need to multiply equations	• Fractions can get complicated • Solving one equation for y or x can be difficult
Addition	• Adding and solving can be very fast.	• It may be difficult to find a common multiple

Substitution versus Addition



When you have a choice of methods, it will usually be a choice between the substitution and addition methods. *Either method will always give the correct solution*. It is useful, however to develop a plan to decide which method will be easier for each given system of equations. *For each system, one method is often much quicker than the other.* Here are some important ideas to help you choose:

Substitution

A good choice if:

- One equation is solved for an unknown or is close to being solved. For example, $y = 3x + 2$.
- It will be easy to substitute into the second equation and easy to solve that equation.

A poor choice if:

- It will be difficult to solve either equation for an unknown without creating an expression with fractions. For example: $3y + 4 = 15 - 35x$.
- The other equation (where we substitute for the unknown) has a complicated expression containing the unknown. For example, it will take a great deal of work to substitute for x in the equation $y = 6(3 - 45x) + 234x$.

Addition

A good choice if:

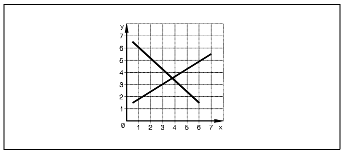
- The number of x's (or y's) in the first equation is the *same* as the number of x's (or y's) in the second equation.
- The number of x's (or y's) in the first equation is *a multiple of* the number of x's (or y's) in the second equation.
- The equations are arranged (or close to being arranged) in the format with unknowns on one side and units on the other. ($\_\, x + \_\, y = \_$)

A poor choice if:

- It will be difficult to find a common multiple.
- Fractions or decimals are present.
- It will take a great deal of work to rewrite the equations in the proper form.

For example, consider this system:

$$3x - 2y = 7$$
$$5x = 33 - 2y$$



The second equation can easily be rewritten as $5x + 2y = 33$, so the system would look like this:

$$\begin{aligned} 3x - 2y &= 7 \\ 5x + 2y &= 33 \end{aligned}$$

The addition method is the best choice. Substitution would be more difficult because solving either equation for x or y would result in fractions.

If one equation is easy to solve for x or y, then it is usually best to choose substitution. For example:

$$\begin{aligned} 3x + y &= 7 \\ 5x &= \frac{3}{2}y + \frac{17}{2} \end{aligned}$$

The addition method is less attractive because it would be difficult to find a common multiple for the x's or y's, but the first equation can easily be solved for y, allowing for substitution.

In summary:

- **Choose *substitution* if:**
  One equation is easily solved for x or y.
  The other equation has a simple expression for the variable which you are substituting.

- **Choose *addition* if:**
  The equations are easy to arrange in the proper form.
  The common multiple is not too large and is easily found.

- **If you are not sure, *either method will always work.***

Exercises

Solve by graphing:

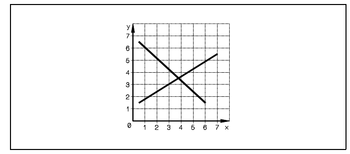
1. $$\begin{aligned} y &= 2x + 1 \\ y &= x - 3 \end{aligned}$$

2. $$\begin{aligned} x &= 2y \\ x + 2y &= 4 \end{aligned}$$

3. $$\begin{aligned} y &= 2x + 1 \\ 3x + y &= 1 \end{aligned}$$

4. $$\begin{aligned} y &= {}^{-}2x - 2 \\ y &= x + 4 \end{aligned}$$

Solve by substitution:



5. $2x + y = {}^{-}1$

$5x + y = {}^{-}13$

6. $3x - y = {}^{-}8$

$x = 6y + 3$

7. $2x + 5y = 12$

$x - y = {}^{-}1$

8. $x - 4y = {}^{-}5$

$3x - 2y = 5$

Solve by addition:

9. $4x + 2y = 6$

$3x + 2y = 1$

10. ${}^{-}4x + 5y = 10$

$4x + 3y = {}^{-}26$

11. ${}^{-}4x + 3y = {}^{-}1$

$2x - 3y = 0$

12. $2x - 3y = 4$

$4x + y = {}^{-}6$

Solve by the best method:

13. $3x - 4y = 11$

$2x + 3y = {}^{-}4$

14. $4x + 3y = 13$

$3x - 7y = {}^{-}18$

15. $3x - y = {}^{-}1$

$x - y = 1$

16. $y = x + 1$

$x + y = 3$

Section 6
Special Cases

Fractions and Decimals

When fractions and decimals appear in systems of equations, we treat them in the same manner as we did when solving one equation. We multiply both sides of the equation by an appropriate number so that the fractions and decimals are gone. For example, consider these equations:

$$\frac{1}{2}x + y = \frac{10}{3}$$

$$\frac{1}{18}y = x$$

We multiply the first equation times 6 (the least common multiple of 2 and 3) and the second equation times 18:

$$\begin{aligned} \frac{1}{2}x + y &= \frac{10}{3} \\ 6\left(\frac{1}{2}x + y\right) &= 6\left(\frac{10}{3}\right) \\ \frac{6}{2}x + 6y &= \frac{60}{3} \\ 3x + 6y &= 20 \quad \text{(no fractions remain)} \end{aligned}$$

$$\begin{aligned} \frac{1}{18}y &= x \\ 18\left(\frac{1}{18}y\right) &= 18(x) \\ y &= 18x \quad \text{(no fractions remain)} \end{aligned}$$

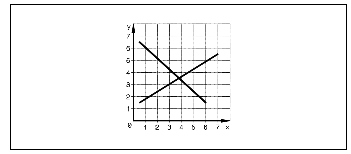
We are now ready to solve the system by substitution or addition.

If decimals are present in an equation, we multiply both sides by 10, 100, 1000, etc. For example:

$$1.2x + .32y = 6$$

$$\begin{aligned} 100(1.2x + .32y) &= 100(6) \\ 120x + 32y &= 600 \end{aligned}$$

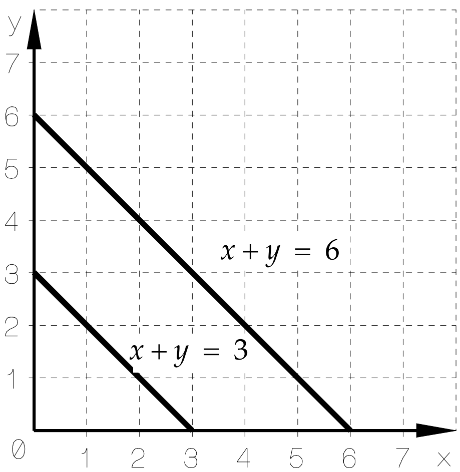
We choose a power of 10 that is large enough to give a new equation without decimals. Because decimal numbers are really fractions with denominators of 10, 100, 1000, and so on, we are really multiplying by the least common denominator.



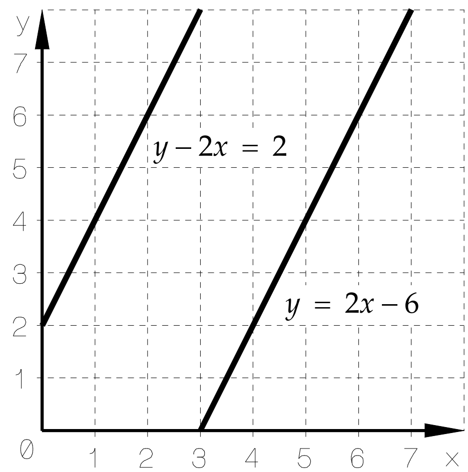
Unusual Solutions

Two lines do not always meet in one single point. The lines may be **parallel** to one another; parallel lines have no common point and no solution:

$$\begin{aligned} x + y &= 3 \\ x + y &= 6 \end{aligned}$$



$$\begin{aligned} y - 2x &= 2 \\ y &= 2x - 6 \end{aligned}$$



We call such a system **inconsistent.** *Inconsistent equations are parallel lines on a graph and have no solution.*

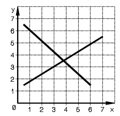
The two equations in a system may also represent the same line. The equations are actually two different forms of the same equation:

$$\begin{aligned} 2x + 3y &= 6 \\ 4x + 6y &= 12 \end{aligned}$$

If we multiply the first equation by 2, it is the same as the second equation:

$$\begin{aligned} 2x + 3y &= 6 \\ 2(2x + 3y) &= 2(6) \\ 4x + 6y &= 12 \end{aligned}$$

It is now clear that the two original equations are really the same equation; they both represent the same line. We call this type of system **dependent** because the two lines are not independent of each other. We also say that the two lines **coincide.** *Dependent equations share the same line on the graph, and all points of that line are solutions.*



Identifying Inconsistent and Dependent Systems

We have just learned how to identify the *graphs* of inconsistent and dependent equations, but how can we identify these situations when we are solving by substitution or addition? Consider the system we just looked at:

$$\begin{aligned} 2x + 3y &= 6 \\ 4x + 6y &= 12 \end{aligned}$$

$2x + 3y = 6 \rightarrow$	$\mathbf{-2}(2x + 3y) = \mathbf{-2}(6) \rightarrow$	$-4x - 6y = -12$
$4x + 6y = 12 \rightarrow$	$\rightarrow \qquad \rightarrow$	$4x + 6y = 12$
		$0 + 0 = 0$
		$\mathbf{0 = 0}$

We obtained a result that is true (zero *is* equal to zero), but useless. This means that the two equations (lines) are the same. They are dependent (the graphs of the lines coincide).

Next, let's look at two parallel lines:

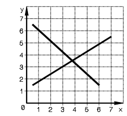
$$\begin{aligned} 2x + 3y &= 6 \\ 4x + 6y &= 7 \end{aligned}$$

$2x + 3y = 6 \rightarrow$	$\mathbf{-2}(2x + 3y) = \mathbf{-2}(6) \rightarrow$	$-4x - 6y = -12$
$4x + 6y = 7 \rightarrow$	$\rightarrow \qquad \rightarrow$	$4x + 6y = 7$
		$0 + 0 = -5$
		$\mathbf{0 = -5}$

This time, we obtained a result that is false. This means that *there is no solution —the two equations are inconsistent and the lines are parallel*. When we combined the two equations, we assumed that there was an x and a y value that were on both lines. Since this was a false assumption, we reached a false conclusion.

Note: These results will be the same whether we are using the addition or the substitution method. Meaningless results (dependent or coinciding systems) may be 3 = 3 or 5 = 5, not just 0 = 0. False results (inconsistent or parallel systems) may come out to be any false statement. In both cases, the variables all cancel out.

Summary



The table below is a summary of how we can identify these two special cases:

If both variables cancel out:		
Results	**Meaning**	**Why**
True, but not helpful: $0 = 0$ $3 = 3$ $^{-}1 = {}^{-}1$	**Dependent (Coinciding) System** Equations are equivalent Equations represent the same line	Since we start with two lines that are the same, we do not have enough information to find an answer.
False: $3 = 5$ $7 = {}^{-}2$ $0 = 3$	**Inconsistent or Parallel system** No solution Lines are parallel	Since we start with a false idea that the lines meet, we come to a false conclusion.

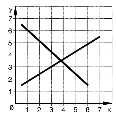
Exercises

Solve the following systems by first multiplying each equation (if necessary) by the least common multiple:

1. $$\frac{x}{5} + \frac{y}{7} = 1$$
$$x + y = 11$$

2. $$\frac{2}{3}x + \frac{2}{5}y = \frac{1}{3}$$
$$-\frac{2}{3}x - \frac{3}{5}y = \frac{1}{6}$$

3. $$\frac{3}{4}x + \frac{1}{3}y = 8$$
$$\frac{1}{2}x - \frac{5}{6}y = {}^{-}1$$



4. $$\frac{2x}{3} + \frac{2y}{6} = \frac{10}{6}$$
$$\frac{5x}{12} + \frac{y}{4} = \frac{11}{12}$$

Solve the following systems by first multiplying each equation (if necessary) by the appropriate power of 10 (10, 100, 1000):

5. $$0.1x + 10y = 30$$
$$^{-}4.1x - 10y = 10$$

6. $$.02 + .01y = .05$$
$$.5x + .3y = 1.1$$

7. $$.5x + 1.0y = 17$$
$$.1x + .1y = 2.2$$

8. $$y = 0.1x + 3.5$$
$$y + .08x = 8$$

Solve the following systems. State the unique solution or state that the system is inconsistent (two parallel lines) or it is dependent (two lines that coincide):

9. $$3x - 6y = 6$$
$$x - 2y = 4$$

10. $$x + 3y = 3$$
$$^{-}x + y = 5$$

11. $$2x - 3y = 6$$
$$4x - 12 = 6y$$

12. $$x + y = {}^{-}1$$
$$y = -x + 1$$

13. $$x - y = 3$$
$$2x - 2y = 6$$

14. $$y = 2x$$
$$4x + y = 6$$