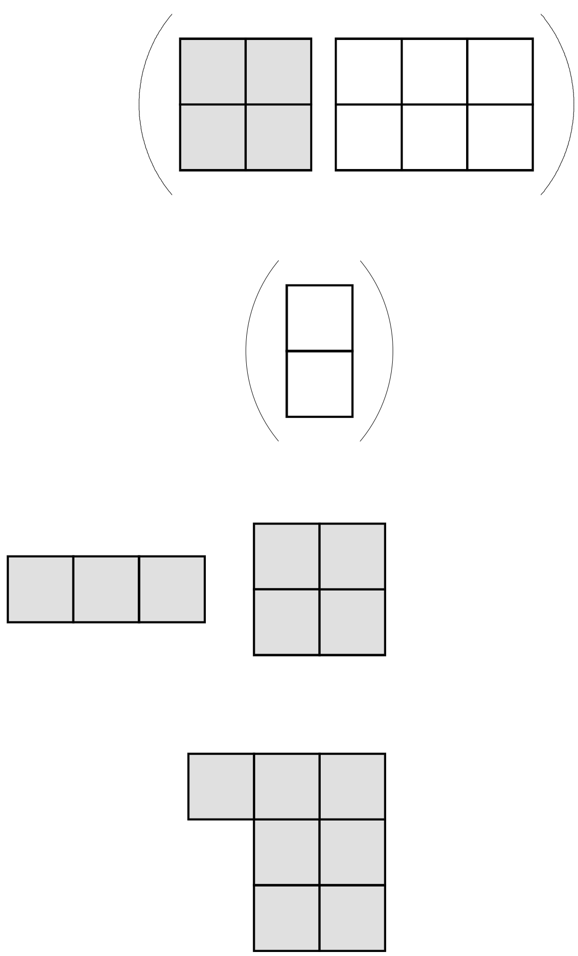
Chapter 3

Symbols and the Order of Operations



Section 1
Rules of Language

Symbols and Grammar

Algebra is a written language, and just like English or French, it has an alphabet of symbols and a set of rules. (Unlike other languages, algebra is usually written and seldom spoken). As we all know, written languages have very specific rules for things like

- **which direction to read**
- **where to start reading**
- **where to pause**
- **how to end one thought and begin a new thought**

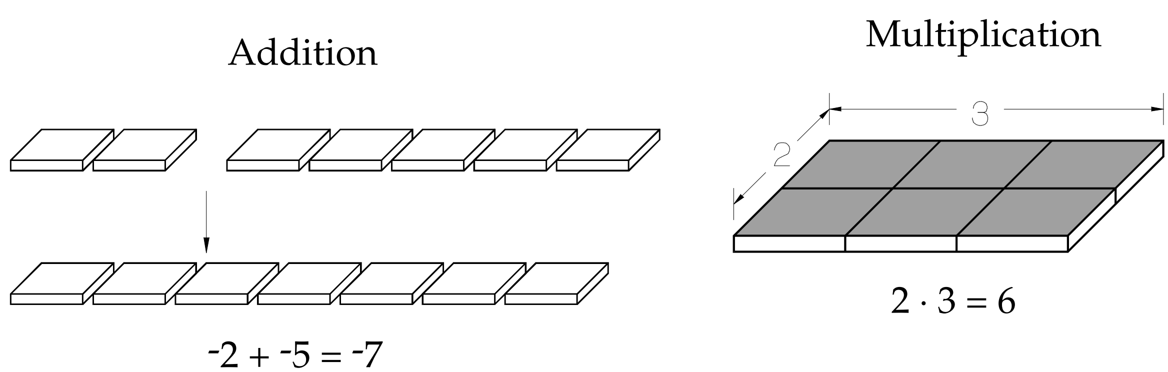
We call these rules **grammar**. In order to write and read effectively we must all agree upon the rules of writing and reading, and upon the meaning of the symbols that we use. With this agreement, the author knows what the readers expect and the readers know what the author means to say.

The differences between symbols can be very subtle. For example, think of the differences in meaning among these symbols in English:

, ; : .

Distinguishing Multiplication from Addition

We know the difference between *adding* two numbers and *multiplying* two numbers:

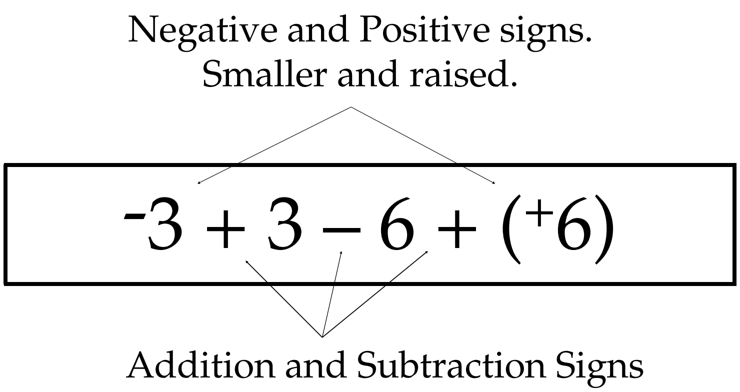


In the written language of algebra, the symbols that indicate adding or multiplying can be quite confusing. There may be several different ways to write the same statement. This is especially true when using signed numbers, since they all have positive or negative signs attached to them, even when they are to be multiplied rather than added together:

$$(^{+}3)\cdot(^{-}5) + (^{-}23)$$

Positive and Negative Signs

As you may have noticed, positive and negative signs are sometimes written differently than addition and subtraction signs. While they have similar meanings, the plus and minus signs of numbers will be shown raised up and slightly smaller than addition and subtraction symbols:



Simple Addition

To indicate addition, we simply write signed numbers in a row with their signs between them, as we have already shown.

$$^{-}5 + 2 - 1$$

The signs tell which color chips to add (colored for +, white for –) and the numbers tell how many chips we have. We slide the chips together and let the different colors cancel each other out, one for one; our answer (the sum) is the number and color of chips remaining after the canceling is done.
The signs between the numbers tell us we are adding.

Notice that with signed numbers we often do not think of subtraction for negative signs. We still add, but we add white chips instead of colored chips.

{ [( )] }

Signs and Parentheses

When individual numbers are used, the positive and negative signs inside of parentheses represent the type of number—positive or negative. Exposed signs between numbers and outside of parentheses represent addition and subtraction:

Enclosed sign means positive or negative

$$^{+}5 - 3 \;=\; ^{+}5 - (^{+}3) \;=\; (^{+}5) + (^{-}3)$$

Exposed sign means add or subtract

If there are double signs on some numbers, it is the exposed signs, those not inside the parentheses, which tell us to add. You must of course carefully use the properties of double signs to know if you should add white or colored chips for each number.

Multiplication and the Dot

If a symbol is used to show multiplication, the symbol is a dot. For example, two ways to show multiplication of $^{+}5$ and $^{-}3$ are

$$(^{+}5)\cdot(^{-}3) \quad \text{or} \quad ^{+}5\cdot(^{-}3)$$

The dot means multiply.

$$(^{+}5)\cdot(-3) \;=\; ^{+}5\cdot(^{-}3)$$

Dot means Multiply

Notice that even when the dot (·) is used to indicate multiplication, the plus or minus sign between the numbers is still enclosed in parentheses.

Missing Symbols: Multiplication

When two quantities are written next to each other *without* a sign between them, the meaning is multiplication:

$$(^{+}5)(^{-}3) \;=\; {}^{+}5(^{-}3) \;=\; -15$$

No sign means multiply

This rule of *no sign between means multiply* works even if only one of the numbers is in parentheses.

Missing Symbols: Addition

With signed numbers we assume that a number is positive (+) unless we see a minus (–) sign. This means that when the plus (+) sign is not needed for the understanding of a number statement, it can usually be left off.

For example, in addition there must be signs (+ or –) *between* the numbers being added, but if the first number in a row is positive the plus sign can be left off that number without confusion.

If the positive number is not the first number in a row, then the sign is still necessary to show addition:

Can't leave off negative sign (–)

$$5 - 3 \;=\; {}^{-}3 + 5$$

Positive first number.
Leave off + sign.

Positive number, not first.
Must have + sign

{ [( )] }

When multiplying, since no sign means *multiply*, a positive number can often be written without a sign:

$$5\cdot(^-3) = (^-3)(5) = {}^-15$$

The important thing to remember is that if you want numbers to be added together, then there must be an exposed sign (+ or –) *between* the numbers. If there is no exposed sign between two numbers, the expression means *multiply*. A number with no sign in front (to the immediate left) of it is understood to be positive.

Summary

- **Positive and negative signs may be shown raised and smaller than addition and subtraction signs.**
- **The dot means multiply.**
- **Enclosed signs refer to the type of number (positive or negative).**
- **Exposed plus and minus signs and signs outside of parentheses stand for addition and subtraction.**
- **No sign between numbers means multiplication.**
- **When the first number in a statement is positive, the positive sign may be omitted. Negative signs are always required.**

Exercises

Read the symbols carefully as you do these exercises:

1. $5 - 3 + 6 = 8$

2. ${}^-3(^-4) = 12$

3. ${}^-2(^+5) = {}^-10$

4. $2 + 5 =$

5. ${}^-3 - 4 =$

6. ${}^-5\cdot(3) =$

7. $2 - 5 =$

8. $2(^-5) =$

9. $7 - 5 =$

10. ${}^-7 - 5 =$

11. ${}^-7(5) =$

12. $7\cdot(5) =$

Section 2
Order of Operations

Number Statements

Number statements in algebra can be quite complex. Statements can have many numbers, some of which are multiplied while others are added together first and then multiplied. To deal with this variety, the language of algebra has a set of rules which tells us which steps to do first and which to do next.

These rules tell us in what order we do the different operations, so that we all agree on the meaning and result of the statement. In English or Spanish, an equivalent rule is that we always agree to read words from left to right, starting at the top of the page and moving down line by line. If we try to change the order of our reading, the statements don't make sense.

But in some other languages, words are read from top to bottom starting at the right edge of the page and moving to the left, column by column. In these languages, you must also follow those rules of order for the statements to make sense.

The first rule of order for algebra is that we always multiply (or divide) before we add (or subtract). When you have a choice between multiplying or adding, always multiply first and then add.

CORRECT:

$3 + 5 \cdot 6$ — First, multiply 5 times 6

$3 + 30$ — Then add 3

33

If you do the operations in the wrong order, you will get a different (and incorrect) result:

INCORRECT:

$3 + 5 \cdot 6$ — Adding first

$8 \cdot 6$

48 — A different result

{ [( )] }

You will notice that in algebra we do not always work from left to right. Wherever we find multiplications in a number statement, we do these first, and then we do the additions.

2(5) – 3(6) + 4	Multiply first
10 – 18 + 4	Then add
$^{-}4$	The result

Writing each step below the one before can make the steps easier to follow. It is also helpful to line up the related numbers below each other, as illustrated above.

In summary, here is the order of operations that we have developed:

Order of Operations

1. **Multiply or divide.**
2. **Add or subtract.**
3. **Finish the operations from left to right**

Exercises

Complete the following exercises by simplifying to one number. Use the chips so you don't miss any steps. Remember, no sign between numbers means multiply.

Example:

$$^{-}3 + 4(2) - 6$$
$$^{-}3 + 8 - 6$$
$$^{-}1$$

$\{\ [\ (\ )\ ]\ \}$

More examples:

$$^{-}2(^{-}3) + 7(2)$$
$$^{+}6 + 14$$
$$+20$$

$$3 - 2(^{-}1) + 5(^{-}5)$$
$$3 \quad + \quad 2 \quad - 25$$
$$^{-}20$$

1. $3(^{-}4) + 5(3)$

2. $^{-}5 + 3(3) - 4(^{-}2)$

3. $5(2) - 3(^{-}2) + 7$

4. $^{-}4(^{-}2) + 6 + 3(^{-}1)$

5. $^{-}5 + 3 - 7(^{-}2) + 4$

6. $(^{-}4)(^{-}3)(^{-}2) - (2)(^{-}3)(^{-}1)$

7. $1 + 23 \cdot 2$

8. $4 - 3 \cdot 2$

9. $^{-}3 \cdot 2 + 4$

10. $1 + 2 \cdot 3 - 4 \cdot 5$

11. $12 - 4 - 3$

12. $12 \div 4 - 3$

13. $3 - 12 \div 3$

14. $12 - 4 + 3$

15. $2 \cdot 3 \cdot 4 - 5 \cdot 6$

16. $^{-}2(^{-}3) + 5(2)$

17. $^{-}2 - 3(5) + 2$

18. $^{-}3(^{-}4) - 5$

19. $^{-}3 - 4(^{-}5)$

20. $^{-}3 - 4 - 5$

21. $2(^{-}5) - 3(^{-}4)$

22. $2(^{-}5)(^{-}3) - 4$

23. $2 - 5(^{-}3)(^{-}4)$

24. $2 - 5(^{-}3) - 4$

25. $^{-}2 - 5(^{-}3) - 4$

26. $^{-}2 - 2(^{-}3)(^{-}4)$

Section 3
Parentheses ( )

Using Parentheses

Even though the first rule in our "order of operations" says that we all agree to multiply before we add, there are times when you might want to write a number statement in which the first step has to be addition, with multiplication coming later.

In the cases where a statement needs to say "add this first," the language of algebra uses special symbols called parentheses ( ). If a number statement has parentheses with some operation inside them like (3 + 5), then the parentheses ( ) are a signal which says "do this step first." For example, watch how the following number statement is simplified:

$\mathbf{{}^{-}3(2) + 5(7 - 3) - 4}$	Do ( ) first
$\mathbf{{}^{-}3(2) + 5(4) - 4}$	Now multiply
$\mathbf{{}^{-}6 + 20 - 4}$	Now add
$\mathbf{10}$	The result

First we do what is inside the ( ), then we multiply, and finally we add. So now we have three rules of order:

- **First do any operations which are inside parentheses ( ).**
- **Second, when you have a choice, multiply before you add.**
- **Finally, work the remaining operations from left to right.**

If there are both additions and multiplications inside of the parentheses, then the "multiply first" rule still holds.

{ [( )] }

For example:

3 – 2(5 + 2 · 4) — Work inside parentheses ( )
Multiply first

3 – 2(5 + 8) — Now add

3 – 2(13) — Outside ( ), multiply first

3 – 26 — Now add

-23 — The result

If the parentheses have only one number inside, there is no operation to do. It is sometimes useful to set off one number with parentheses to show multiplication or to show the effect of a positive or negative sign. Here are some examples (each is separate from the others):

$$^{-}(^{-}3)$$

$$(2)(3)$$

$$(2)(^{-}3)$$

$$5 - (6)$$

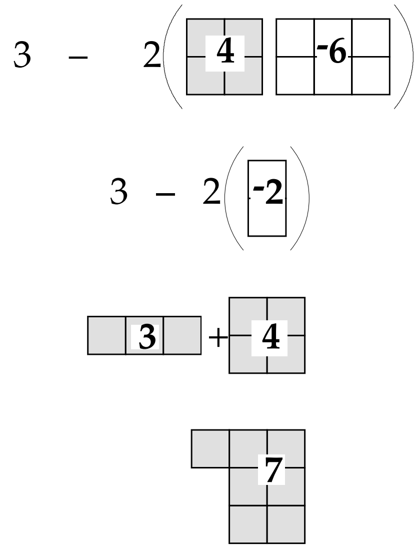
$$5 - (6)(3)$$

$$(^{-}4)(^{+}3)$$

{ [( )] }

Here is how a statement is simplified if we use the chips to do one step at a time:

$$3 - 2(4 - 3 \cdot 2)$$



There is one final order of operations rule. Sometimes, for more complex statements, it is necessary to have one set of parentheses inside of another set of parentheses. Whenever this is required, a different type of symbol is used for each pair of parentheses to avoid confusion:

First level: round **Parentheses** ( )

Second level: square **Brackets** [ ]

Third level: curly **Braces** { }

So a very complex number statement could have parentheses arranged like this:

$$\{43 + 16[9 - 6(2 + 5)] - 13\}$$

Following the rule of "do what's inside the parentheses first," it makes sense that inside the braces { } we do the bracket [ ] part first, and inside the brackets [ ] we do the parentheses ( ) first.

{ [( )] }

So our first rule becomes:

- **Do the operations in the *innermost* parentheses first, and work your way, step by step, to the outside.**

The rules for order of operations can still be written as only three rules, with both parentheses rules combined into one:

- **Do the operations in the *innermost* parentheses first. Work your way, step by step, to the outside.**
- **When you have a choice, multiply before you add.**
- **Work the remaining operations from left to right.**

Watch how these rules work together:

$3 - 2[5 + (4 - 3 \cdot 2)]$ — Inner parentheses first. Multiply.

$3 - 2[5 + (4 - 6)]$ — Now add

$3 - 2[5 + (^{-}2)]$ — Now move out and add

$3 - 2[3]$ — Move out and multiply

$3 - 6$ — Add

$^{-}3$ — The result!

Summary

Order of Operations

1. **Do operations in the innermost parentheses first.**
   Work your way to the outside.

2. **Inside parentheses or when there are no parentheses:**
   Multiply or divide first.
   Add or subtract next.
   Finish the operations from left to right.

{ [( )] }

Exercises

Perform the following operations. Work carefully, and *remember to do only one step at a time*, while the rest of the steps wait. Examples:

$$^{-}5 + 2(3\cdot4 - 6)$$
$$^{-}5 + 2(12 - 6)$$
$$^{-}5 + 2(6)$$
$$^{-}5 + 12$$
$$7$$

$$3(2 - 5\cdot2) - 2(5 + 1)$$
$$3(2 - 10) - 2(5 + 1)$$
$$3(^{-}8) - 2(6)$$
$$^{-}24 - 12$$
$$^{-}36$$

$$5 - 2[3 + 4(2 - 3\cdot3)]$$
$$5 - 2[3 + 4(2 - 9)]$$
$$5 - 2[3 + 4(^{-}7)]$$
$$5 - 2[3 - 28]$$
$$5 - 2[\ ^{-}25\ ]$$
$$5 + 50$$
$$55$$

Simplify to give one number:

1. $^{-}2(4 - 2\cdot3) + 6 - 1$

2. $(5 + 6\cdot2) - 3\cdot4$

3. $^{-}4 + 3(2 - 5\cdot2)$

4. $2(3 - 2\cdot4) + 3(2 + 6)$

5. $4 + 2[5 - 3(2\cdot4 - 3)]$

6. $5 - 3 + [2 + (3 - 5\cdot2)]$

7. $^{-}4\{^{-}3 - 2[1 + (6 - 2\cdot3)] - 1\}$

8. $^{-}1 - 3\{^{-}1 - [2(^{-}4 - 3) + 2] - 1\}$

9. $(4 - 3)(2 - 4)(^{-}7 - 1)$

10. $(^{-}4 + 2)\{2 - 3[^{-}6 - (^{-}1)]\}$

Section 4
Division and Fractions

The Meaning of Fractions and Division

Most number statements in algebra do not use the sign for division (÷). Instead, division steps are usually written as fractions or ratios. For example:

$$6 \div 2 = \frac{6}{2}$$

See Section 3 of the FRACTIONS chapter for a more detailed discussion of why this is true.

As with multiplications, steps involving division or the reducing of fractions are done before steps involving addition. For example, follow through the simplification of the following number statement:

$$3 + 4 \cdot 2 - 12 \div 3 - 1$$ — First multiply and divide

$$3 + 8 - 4 - 1$$ — Then add

$$6$$ — The result!

As we stated above, this number statement would usually be written with a fraction rather than with a division sign. First, simplify the fraction and do the multiplication. Next add the results and the remaining numbers:

$$3 + 4 \cdot 2 - \frac{12}{3} - 1$$ — Multiplication and fractions first

$$3 + 8 - 4 - 1$$ — Add next

$$6$$ — The result!

{ [( )] }

If the top (numerator) and the bottom (denominator) of a fraction need simplification before the fraction can be reduced, then this simplification must be done first. For example:

$$\frac{-2 \cdot 9}{4 \cdot 3 - 6}$$ Multiply within fraction

$$\frac{-18}{12 - 6}$$ Now add

$$\frac{-18}{6}$$ Divide

$$-3$$

Since fractions must be simplified before they can be reduced, we can think of fractions as understood to be within their own parentheses saying "do me first." Simplifying a statement having fractions would look like this:

$$6 - \frac{2+7}{3} - 2(3-7)$$ Simplify fractions and parentheses

$$6 - \frac{9}{3} - 2(-4)$$ Divide and multiply

$$6 - 3 + 8$$ Add

$$11$$

Here is our final set of rules:

- **Do the operations in the innermost parentheses first. (Fractions have implied parentheses; they represent a division problem.) Work your way step by step to the outside.**
- **When you have a choice, multiply or divide before you add or subtract.**
- **Work the remaining operations from left to right.**
- **Do only one step at a time, leaving everything else unchanged.**

These rules are convenient agreements that we make to avoid confusion and to simplify writing number statements. This is one of the few parts of algebra that you must memorize; the rest of the properties and techniques will be easy to understand without memorization.

{ [( )] }

Review the rules as you follow these examples:

$$8\cdot(2) + \left(7 - \frac{5+4}{3}\right)$$

$$8 \cdot 2 \quad + \quad (7-3)$$

$$8 \cdot 2 \quad + \quad (4)$$

$$16 \quad + \quad 4$$

$$20$$

$$\frac{6 - 2 \cdot 5}{2} + 3$$

$$\frac{6-10}{2} + 3$$

$$\frac{-4}{2} + 3$$

$$^{-}2 + 3$$

$$+1$$

{ [( )] }

Here is a more complicated example. Remember to work only one small step at a time, while everything else waits.

$$5 - \frac{2+4}{3 \cdot 2} - 2(2 \cdot 5 - 8)$$

$$5 - \frac{2+4}{6} - 2(10 - 8)$$

$$5 - \frac{6}{6} - 2(2)$$

$$5 - 1 - 4$$

A final example:

$$5 - 3\left[6 - 2\left(\frac{6+4}{5} - 2 \cdot 3\right)\right]$$

$$5 - 3\left[6 - 2\left(\frac{10}{5} - 2 \cdot 3\right)\right]$$

$$5 - 3[6 - 2(2 - 6)]$$

$$5 - 3[6 - 2(^{-}4)]$$

$$5 - 3[6 + 8]$$

$$5 - 3[14]$$

$$5 - 42$$

$$^{-}37$$

Exercises

{ [( )] }

Simplify:

1. $\dfrac{6+4}{2} - 3$

2. $\dfrac{5 \cdot 4}{2} + 6$

3. $6 + \dfrac{4 - 2 \cdot 7}{5}$

4. $5 - \dfrac{4+10}{3 \cdot 2 + 1}$

5. $3\left(2 - \dfrac{7+5}{4}\right)$

6. $4 - 3\left(\dfrac{9 \cdot 2}{3} - 8\right)$

7. $\dfrac{7 \cdot 2 + 1}{3 + 2} - (2 \cdot 3 + 1)$

8. $18 - 2\left(\dfrac{3 + 2 \cdot 6}{5} - 4\right)$

9. $3 - \left[18 - 2\left(\dfrac{3 + 2 \cdot 6}{5} - 4\right)\right]$

10. $2\left\{3 - \left[18 - 2\left(\dfrac{3 + 2 \cdot 6}{5} - 4\right)\right]\right\}$

Reviewexercises:

1. $^{-}2(3 + 5) + 1$

2. $2(3) + 5(3 - 8)$

3. $^{-}3 - 2(5 - 1)$

4. $5 - 3(2 - 9)$

5. $7 + 2[3 - (4 - 6)]$

6. $^{-}3 + 5(^{-}4 - 2(8 - 3))$

7. $2 + [7 - 3(2 - 6) + 2]$

{ [( )] }

8. $(2 + 3)[4 - 5(3 - 1)]$

9. $\mathbf{(3 - 5)[{}^{-}3(2 - 7) + 1]}$

10. $\mathbf{(7 + 2)[(3 - 4) - 6]}$

11. $\dfrac{5 \cdot 2 - 1}{3 + 6} - (5 \cdot 3 - 1)$

12. $2\left(\dfrac{3 + 2 \cdot 6}{5} - 4 \cdot 2\right) - (0 \cdot 17)$

13. $2 + \left[{}^{-}3 - 2\left(\dfrac{2 + 2 \cdot 6}{7} + 3\right)\right]$

14. $\dfrac{3 + 2 - 4}{2 \cdot 2 + {}^{-}3} + \dfrac{3 + 2}{1 - {}^{-}4}$

15. $1\left\{1 - \left[1 - 1\left(\dfrac{1 + 1 \cdot 1}{1} - 1\right)\right]\right\}$

Section 5
Absolute Value

The Size Of A Number

The *size* of a number (independent of its sign) has a special name in the language of algebra. The size of a number is called its **absolute value.**

We indicate the absolute value of a number by putting a straight vertical line on each side of the number. Thinking of the chips, the absolute value of a group of chips is just the number of chips, independent of their color.

$$|5| = 5 \qquad |{}^{-}5| = 5$$

$$|{}^{-}3| = 3 \qquad |3| = 3$$

The absolute value of a number is *always* positive. Notice that the bars | | around a number indicate an operation to be done to that number. The bars mean that we should *make the number positive and rewrite it without the bars*.

Operations Inside The Absolute Value Sign

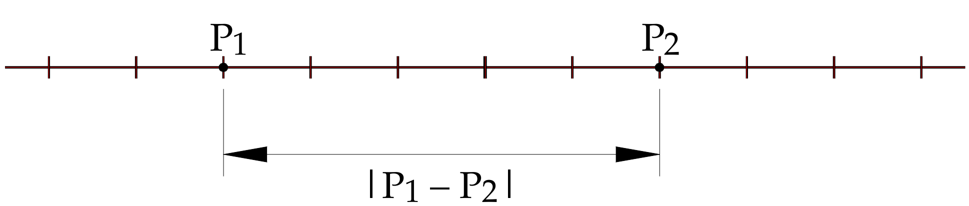
If the absolute value bars are around an expression, and the expression has several numbers and operations, perform the operations first, *before* taking the absolute value.

$$|3-7| = |{}^{-}4| = 4$$

$$|2{\cdot}3-7| = |6-7| = |{}^{-}1| = 1$$

When we say that the absolute value is always positive, this *doesn't* mean that we should turn all the signs inside that absolute value into plus signs. Do the operations as they are indicated, and then take the absolute value of the simplified number at the very end.

Absolute value is often used to describe the separation between two points on the number line. This separation is obviously the difference between the values of the points, but we generally want the separation to be expressed as a *positive* number. This can be indicated using the absolute value of the difference in the values of the points.



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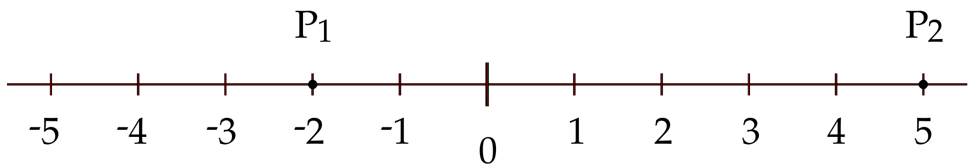
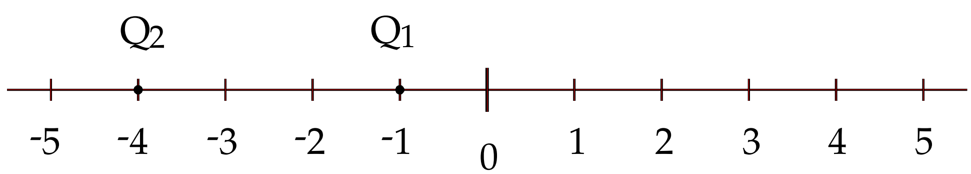
The separation between points P_1 and P_2 is $|P_1 - P_2|$. Written in this way, it does not matter which of the points has the larger value; the separation between them will always be expressed as a positive number.

Here are two other examples:



The separation between P_1 and P_2 is

$$|P_1 - P_2| = |{}^{-}2 - 5| = |{}^{-}7| = 7$$



The separation between Q_1 and Q_2 is

$$|Q_1 - Q_2| = |{}^{-}1 - {}^{-}4| = |3| = 3$$

Altitude and temperature are other quantities where a difference is often discussed as a positive number. For example, consider the statements:

"From the mountain top to the valley floor was 8,000 vertical feet."

"The variation in the temperature during the experiment was 52°."

In mathematical language these statements would be expressed using absolute values.

Operations Outside The Absolute Value

If an absolute value is indicated as part of a larger expression, first simplify inside the absolute value; then take the absolute value and put the result inside parentheses ( ) within the larger expression. Finally, continue to simplify the remaining expression.

{ [( )] }

For example, consider the expressions below:

$$
\begin{aligned}
5 - 2|6 - 10| &= 5 - 2|{}^{-}4| \\
&= 5 - 2(4) \\
&= 5 - 8 \\
&= {}^{-}3
\end{aligned}
$$

$$
\begin{aligned}
3 \cdot 5 - |2 - 8| &= 15 - |{}^{-}6| \\
&= 15 - (6) \\
&= 9
\end{aligned}
$$

Notice that a negative sign inside an absolute value is not directly affected by the negative sign outside the absolute value. In simplifying the absolute value, the negative sign on the outside is not used until the whole expression inside is simplified.

Absolute Value Signs Differ From Parentheses

The concept of the absolute value of a number is not difficult to understand, but the notation for absolute value can sometimes be confusing because it resembles parentheses or brackets.

Absolute value signs are different from parentheses and brackets in that absolute value signs indicate an operation to be done to the number inside. We *never multiply across* an absolute value sign; we must simplify and remove the absolute value, putting its result inside parentheses, before continuing with simplifying the rest of the expression.

For example:

$$\mathbf{3 + {}^{-}2|{}^{-}6 + 4|}$$

CORRECT	NOT CORRECT
$3 + {}^{-}2|{}^{-}6 + 4|$	$3 + {}^{-}2|{}^{-}6 + 4|$
$3 + {}^{-}2|{}^{-}2|$	$3 + {}^{-}2|{}^{-}2|$
$3 + {}^{-}2 \cdot (2)$	$3 + |({}^{-}2) \cdot ({}^{-}2)|$
$3 + ({}^{-}4)$	$3 + (4)$
-1	7

Simplify absolute value first

Wrong: Do not multiply through absolute value signs!

{ [( )] }

Exercises

Simplify to a single number:

1. $|{}^-7|$
2. $|12|$
3. $|8-3|$
4. $|2-11|$
5. $|6-9|$
6. $|10-3|$
7. $|5-3\cdot 8|$
8. $|2\cdot 4-6|$
9. ${}^-|7+2|$
10. ${}^-|{}^-3-6|$
11. $3|15-3|$
12. ${}^-2|3-11|$
13. ${}^-5|{}^-3-{}^-5|$
14. $3|2-14|$
15. $8-|7-5|$
16. $3+|5-9|$
17. ${}^-5+2|7-11|$
18. ${}^-3|5-9|+7$