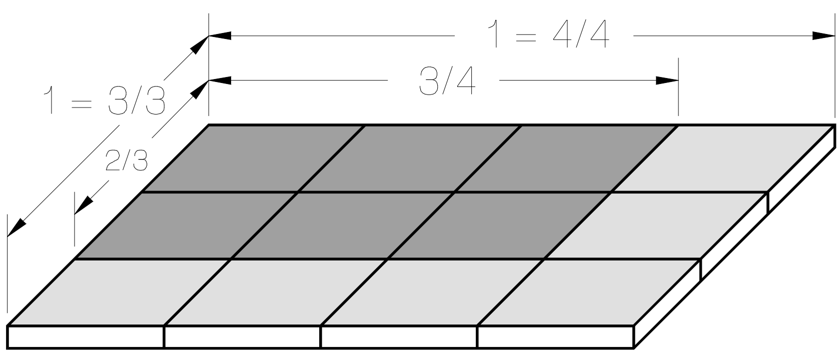
Chapter 4

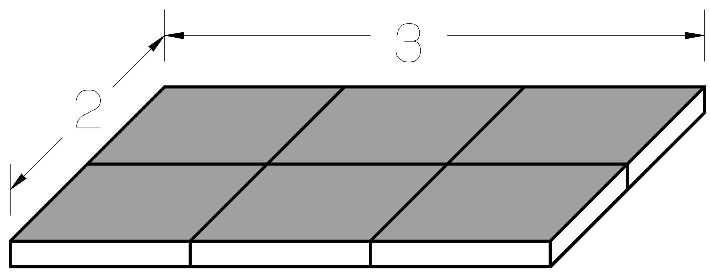
Multiplication and Division of Fractions



Section 1
Multiplication of Fractions

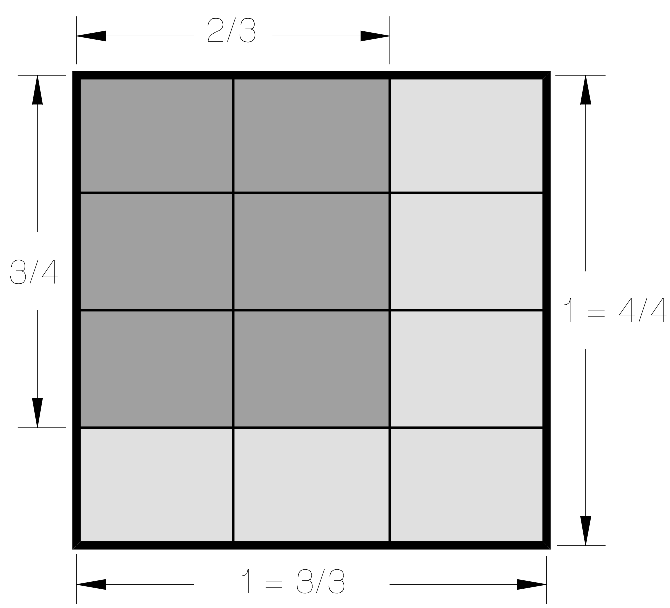
The Meaning of Multiplication

Multiplication has the same meaning with fractions as it does with positive and negative numbers. When we multiply 2 times 3, we make a rectangle that is two units wide and three units long:

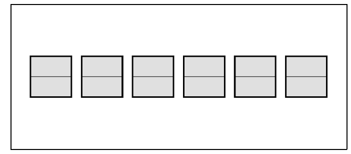


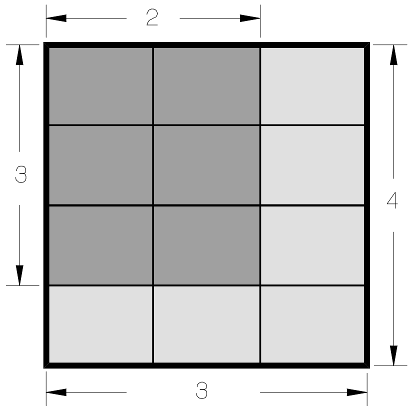
Multiplying Fractions

To multiply two fractions, we also make a rectangle. We start with a unit chip (a small square) and we cut it into smaller pieces. For example, $\frac{2}{3} \cdot \frac{3}{4}$ means:

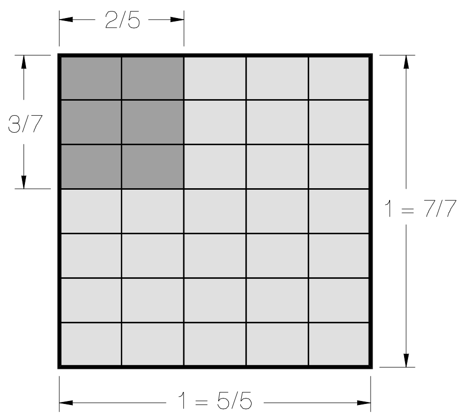


The result is a rectangle smaller than a single unit which is $\frac{2}{3}$ units on one side and $\frac{3}{4}$ on the other side. To make this smaller rectangle, we must cut a unit square (1 by 1) into 4 parts (fourths) along one side and 3 parts (thirds) on the other side. Because there are now 12 equal pieces, and our rectangle has six $(3 \cdot 2)$ of them, we say that the result is 6 out of 12 or $\frac{6}{12}$ of a unit:

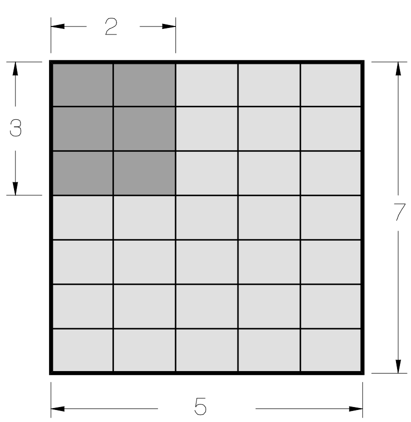




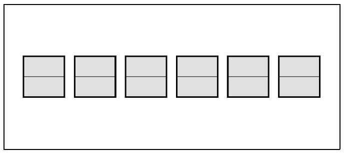
To find $\frac{3}{7} \cdot \frac{2}{5}$ we make another rectangle:



This results in a rectangle that is 6 out of 35 or $\frac{6}{35}$. We have 35 total sections because we ruled the two sides into 7 and 5 pieces and $7 \cdot 5$ is 35. The pattern should now be clear.

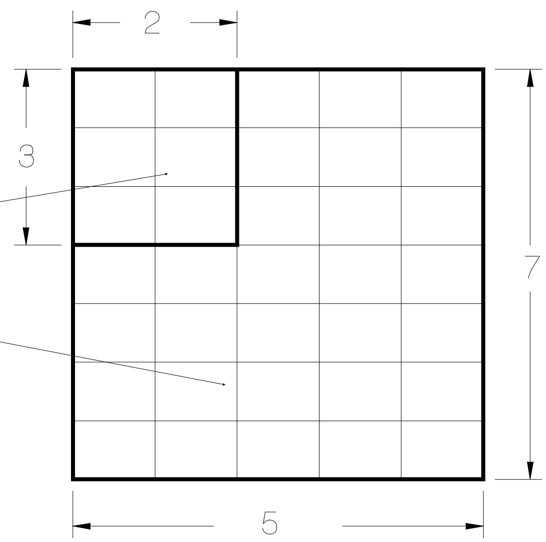


Since we chose a rectangle **three**-sevenths by **two**-fifths, we obtained a 3 by 2 rectangle with $3 \cdot 2$ or 6 pieces out of 35 total.



Multiplying the denominators (bottom numbers) gives us the total number of pieces. Multiplying the numerators (top numbers) gives us the number of pieces in the result. The result is the fraction of the unit square that the multiplication represents.

$$\frac{3}{7} \cdot \frac{2}{5} = \frac{3 \cdot 2}{7 \cdot 5} = \frac{6}{35}$$

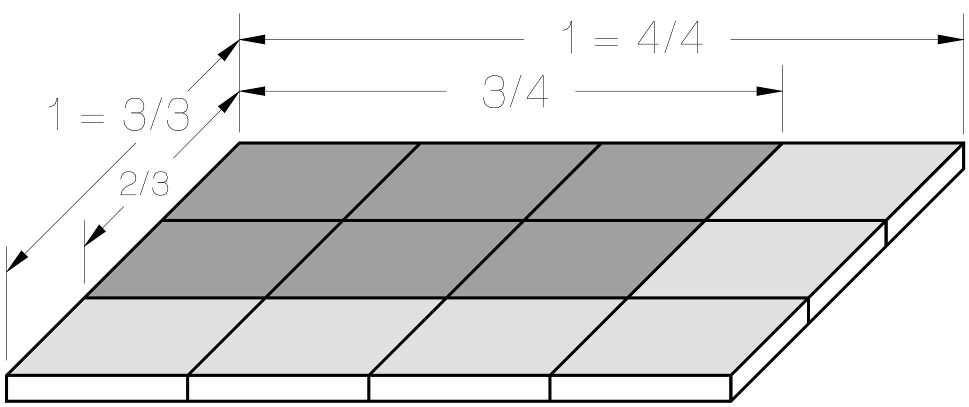


The ratio or fraction of 6 out of 35 ($\frac{6}{35}$) is the final result.

Using Chips

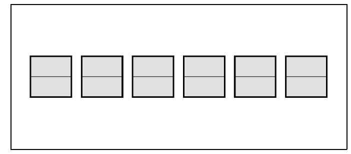
To use the chips for this process, we will have to change our system of representing fractions. Instead of a using a square for one whole unit, we will use a rectangle that is made up of small chips. If we choose the correct number of chips for each side, we will not need to draw lines or cut up the chips.

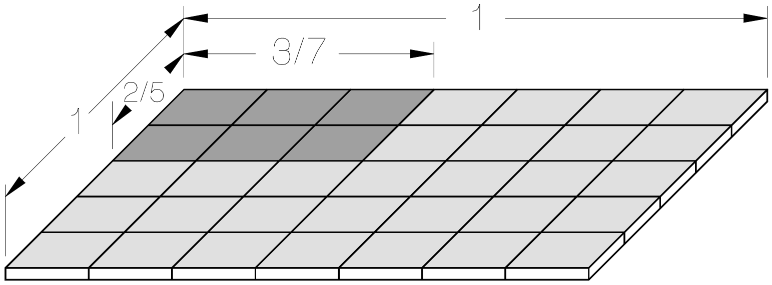
For the first example above of $\frac{2}{3} \cdot \frac{3}{4}$ we will use a 3 by 4 rectangle to represent one whole unit. This will allow us to measure thirds in one direction (where there are 3 chips) and fourths in the other direction:



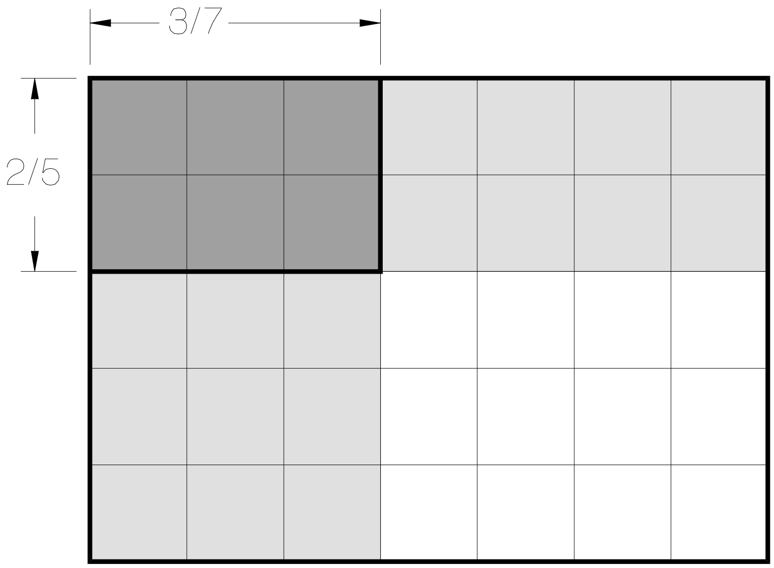
The large rectangle represents one whole. Again, we see that $\frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12}$.

To multiply $\frac{3}{7} \cdot \frac{2}{5}$ with the chips, we lay out a 7 by 5 rectangle and then measure three-sevenths and two-fifths:





You may also have realized that $\frac{3}{7}$ times $\frac{2}{5}$ can mean three-sevenths *of* two-fifths.



To summarize this property with symbols:

Multiplying fractions

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}$$

Exercises

Multiply the two fractions. Use chips or pictures to illustrate each problem.

1. $\frac{1}{2} \cdot \frac{1}{4}$ (Use a 2 by 4 rectangle)

2. $\frac{1}{3} \cdot \frac{2}{3}$ (Use a 3 by 3 rectangle)

3. $\frac{2}{3} \cdot \frac{3}{5}$

4. $\frac{3}{4} \cdot \frac{4}{5}$

5. $\frac{4}{5} \cdot \frac{4}{5}$

6. $\frac{5}{8} \cdot \frac{2}{3}$

7. $\frac{5}{9} \cdot \frac{3}{4}$

8. $\frac{1}{3} \cdot \frac{3}{4}$

9. $\frac{5}{8} \cdot \frac{3}{4}$

10. $\frac{5}{7} \cdot \frac{4}{5}$

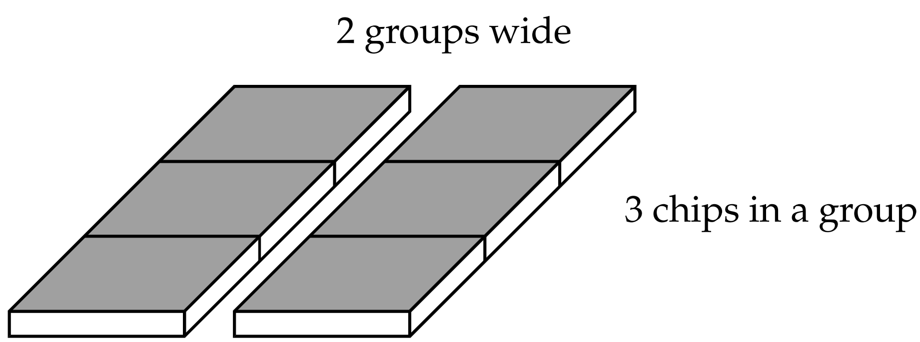
Section 2
Division of Fractions

The Meaning of Division

With whole numbers, 6 ÷ 3 has meant

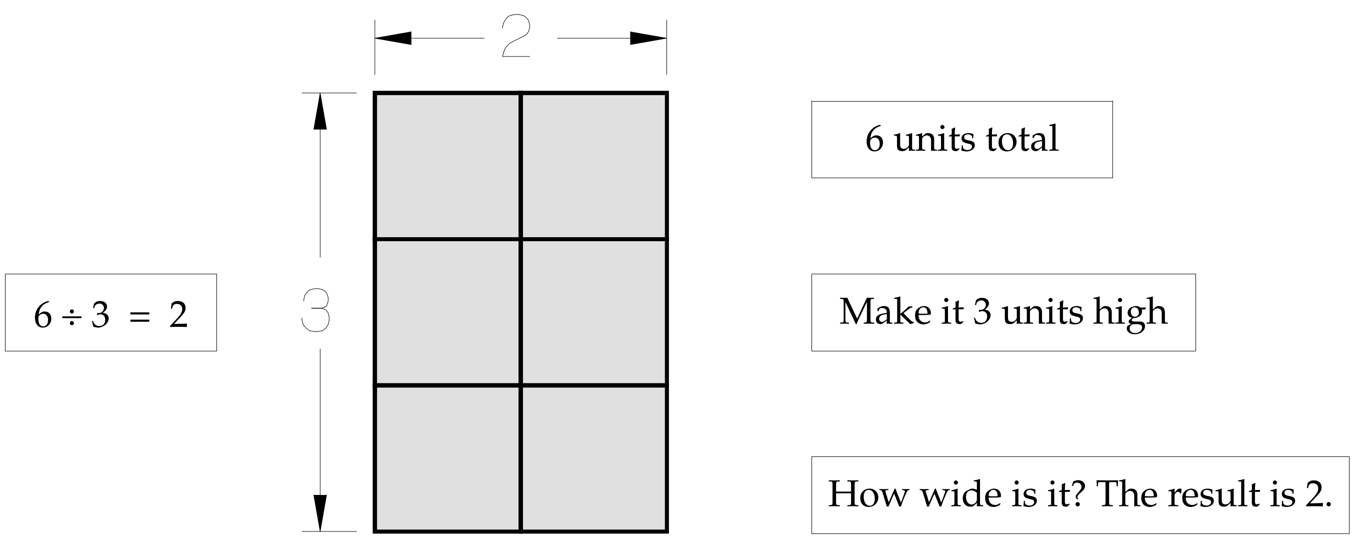
"How many threes in six?"

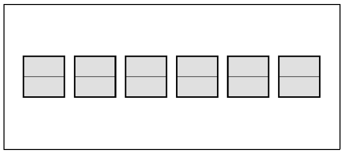
This means that we get out six chips and make groups of three; the result is 2 because there are 2 groups.



Another way to state the problem is to ask:

"If we put 6 into a rectangle that is 3 units high, how wide will it be?"



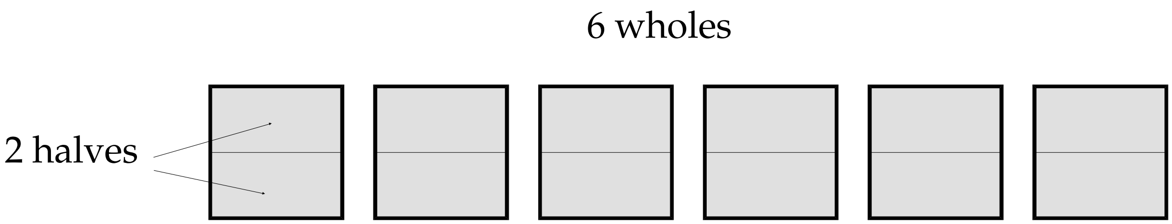


Dividing Fractions

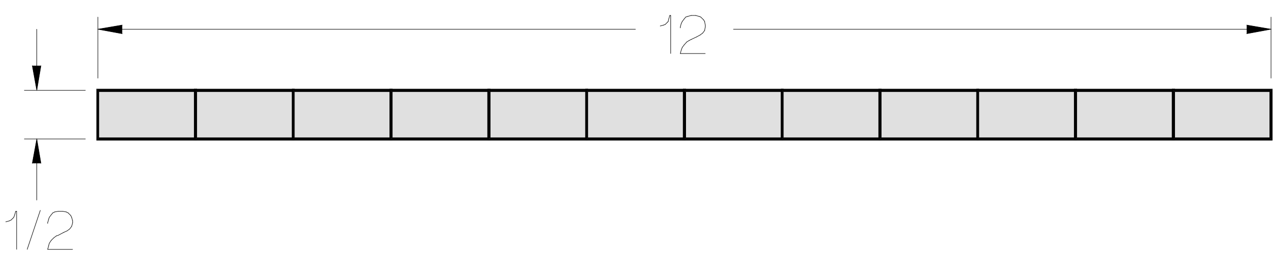
Division problems with fractions have the same meaning as the familiar examples above:

$6 \div \frac{1}{2}$ means "How many halves in six?"

We take six wholes and count the number of halves. There are 12 halves, so the result is 12:



Again, we can look at the problem as a question of using 6 units to construct a rectangle that is $\frac{1}{2}$ unit high. The result is the width, or 12:

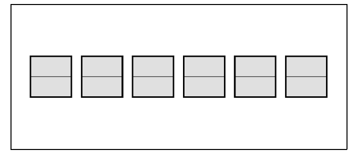


1. Take 6 units.
2. Make a rectangle that is $\frac{1}{2}$ unit high.
3. How wide is it? This is the result.

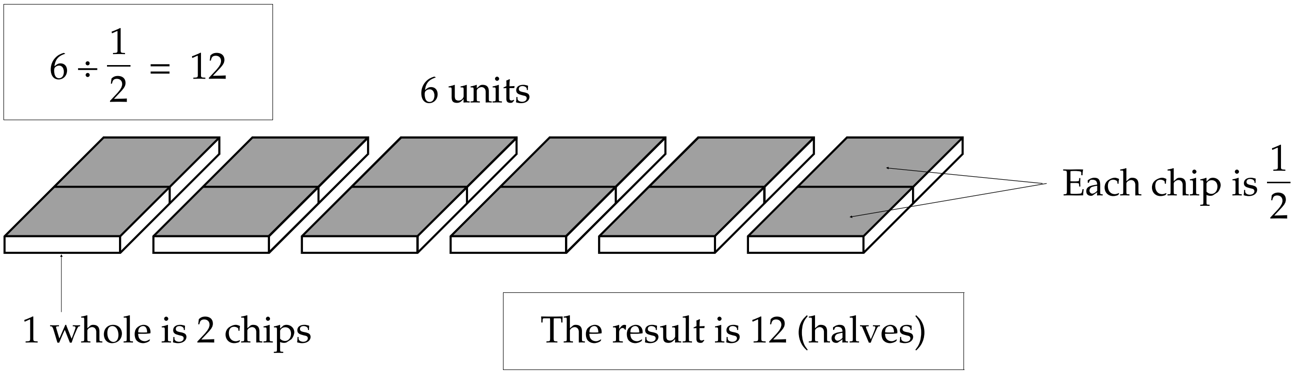
We could count each half, but it would be faster to notice that each whole has 2 halves and that there are 6 groups of 2 or 6·2 = 12 in all. This suggests that

$$6 \div \frac{1}{2} = 6 \cdot 2 = 12$$

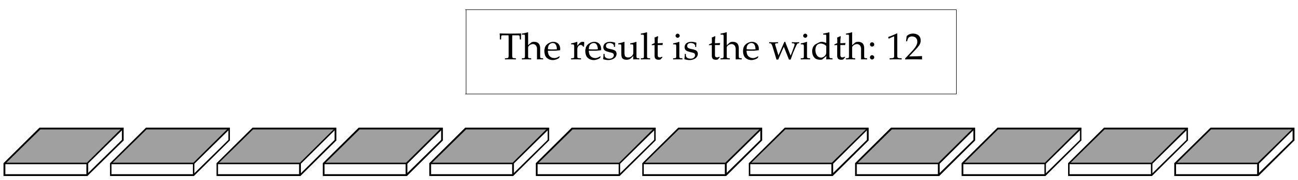
Using Chips



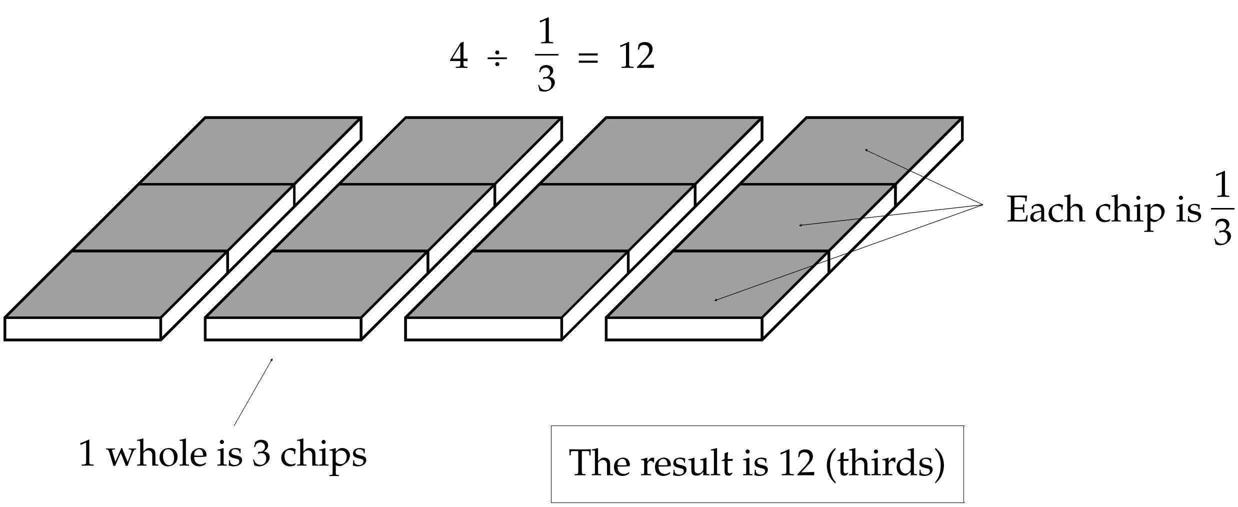
To use the chips, we do the same thing that we did for multiplication—we change the size of a whole so that we do not have to cut up the chips. In this case, because we need 2 halves in each whole, we use 2 *chips* for each whole:

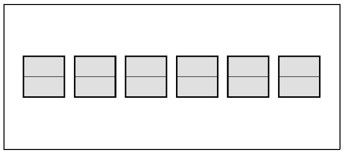


If we arrange the chips in a rectangle that is ½ (1 chip) high, the result is a width of 12:



Here is a second example using the chips: How many thirds in 4?



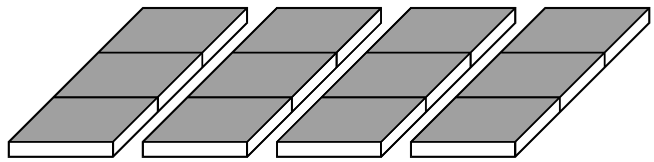


Division: Method I

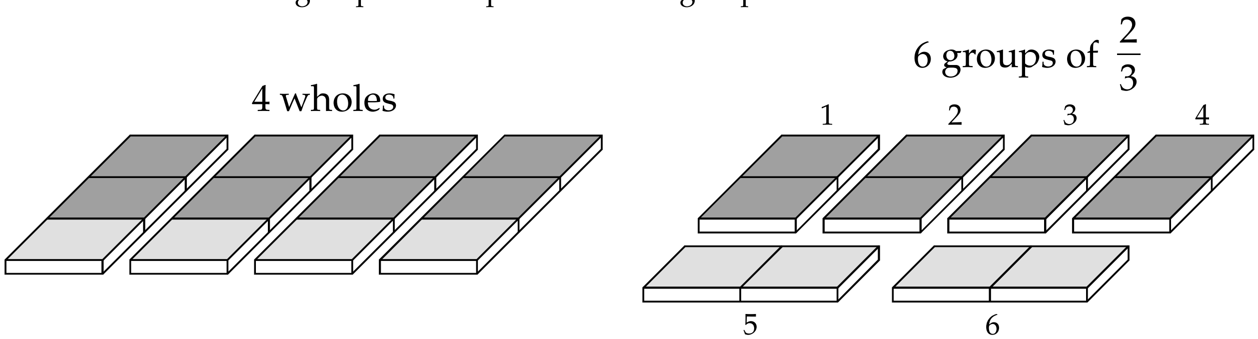
There are two different ways of visualizing the division of more complicated fractions. Consider:

$$4 \div \frac{2}{3}$$

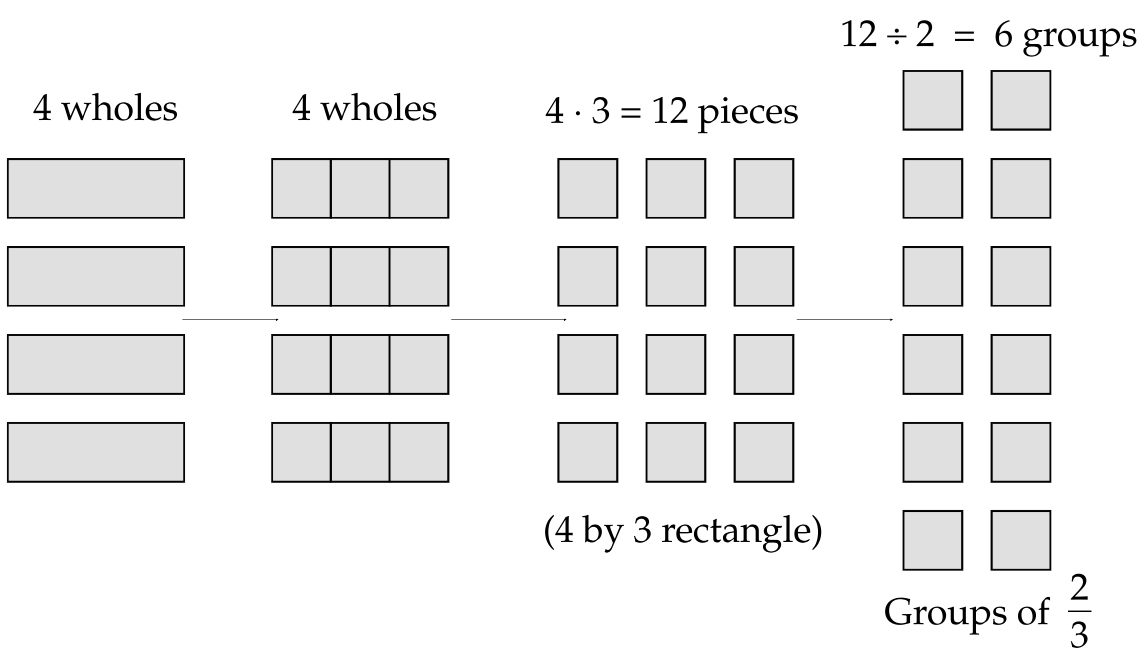
This means we should count how many groups of ⅔ are in 4, so we set up 4 wholes, each made of 3 chips:

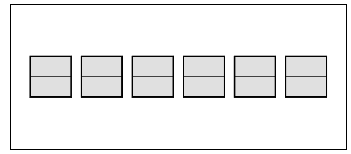


There are really 12 chips, with each chip representing ⅓, so groups of ⅔ are groups of 2 chips. There are 6 groups in all:



When making groups of two chips, be sure to use all the chips—separate pieces from different wholes are joined to be one group. Here is a summary of the process:



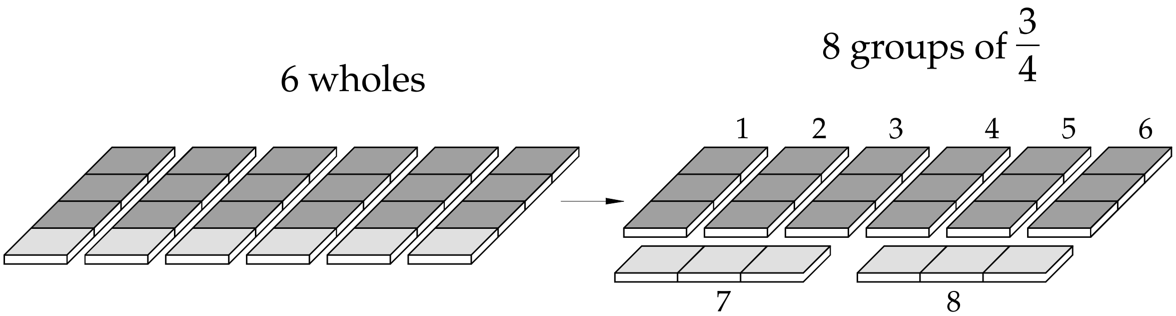


We multiply the first number (4) by the denominator of the fraction (3) to find out how many pieces we have (12). Then we divide by the numerator (2) to find out how many groups we have (6).

Consider another example:

$$6 \div \frac{3}{4}$$

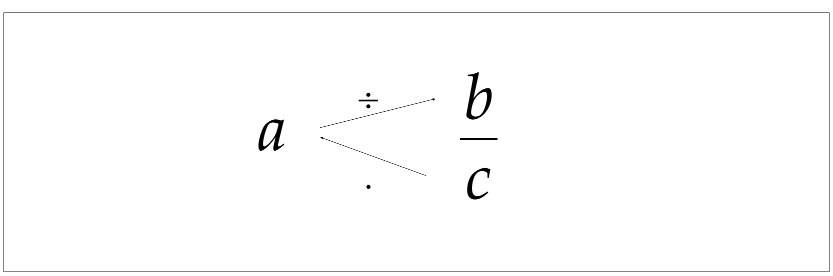
We make each whole from 4 chips, where each chip is ¼. This gives us a total of 6·4 or 24 chips. Since we want groups of ¾ (3 chips), we divide 24 by 3 to get 8:



Since we actually multiply by the *denominator* of the fraction and then divide by the *numerator* we write the technique this way:

Method I: Dividing by a fraction

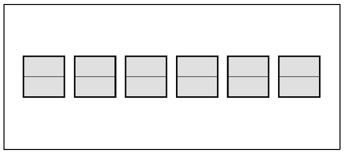
$$a \div \frac{b}{c} = (a \cdot c) \div b = \frac{ac}{b}$$



Division: Method II

There is a second way to visualize the division of fractions. In our first examples, we looked at problems like

$$6 \div \frac{1}{2}$$

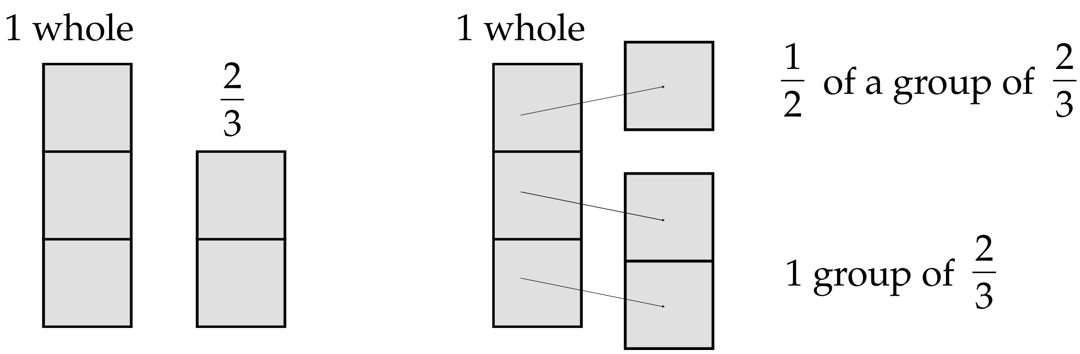


We noticed that there were 2 halves in each whole and then multiplied 6 times 2.

For the problem

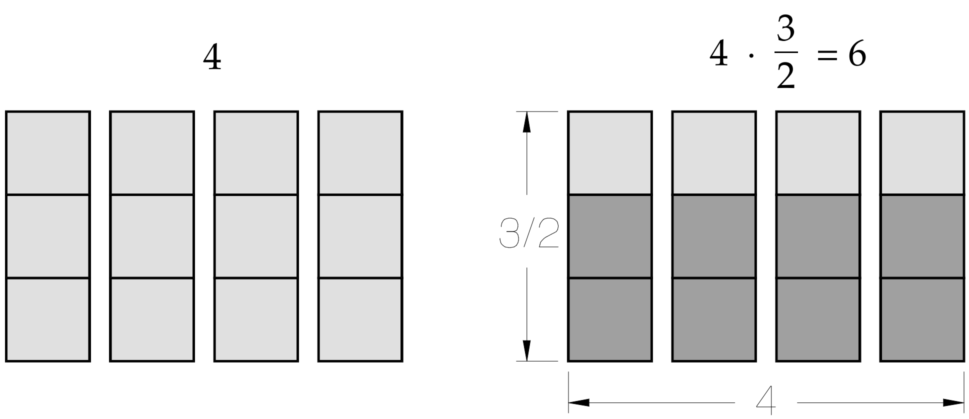
$$4 \div \frac{2}{3}$$

we could find out how many groups of $\frac{2}{3}$ are in each whole, and then multiply that times 4:



We can see that there are 1 ½ or $\frac{3}{2}$ groups in each whole. *Notice that we are looking at 1 ½ groups of $\frac{2}{3}$, not at simply 1 ½ wholes.*

Now we can count 4 groups of 1 ½, or 4 times 1 ½:



It is not an accident that $\frac{3}{2}$ looks like $\frac{2}{3}$ “flipped over.” When we ask

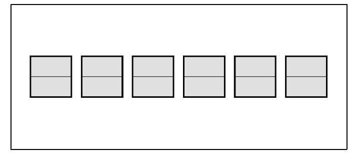
“How many groups of $\frac{2}{3}$ are in 1 whole?”

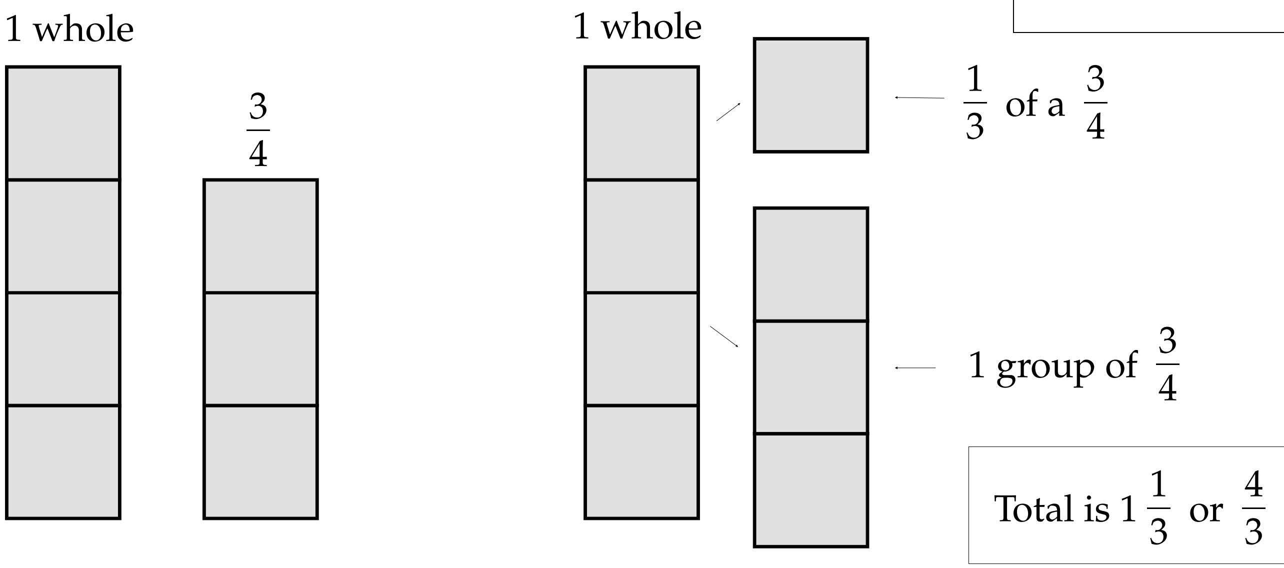
We are asking

“What times $\frac{2}{3}$ is 1?”

The answer is $\frac{3}{2}$ because $\frac{3}{2} \cdot \frac{2}{3} = \frac{6}{6} = 1$.

Here is an illustration of finding how many groups of ¾ can be made out of one whole. The result is the **reciprocal** of ¾ or ⁴⁄₃. There are 1⅓ or ⁴⁄₃ groups:

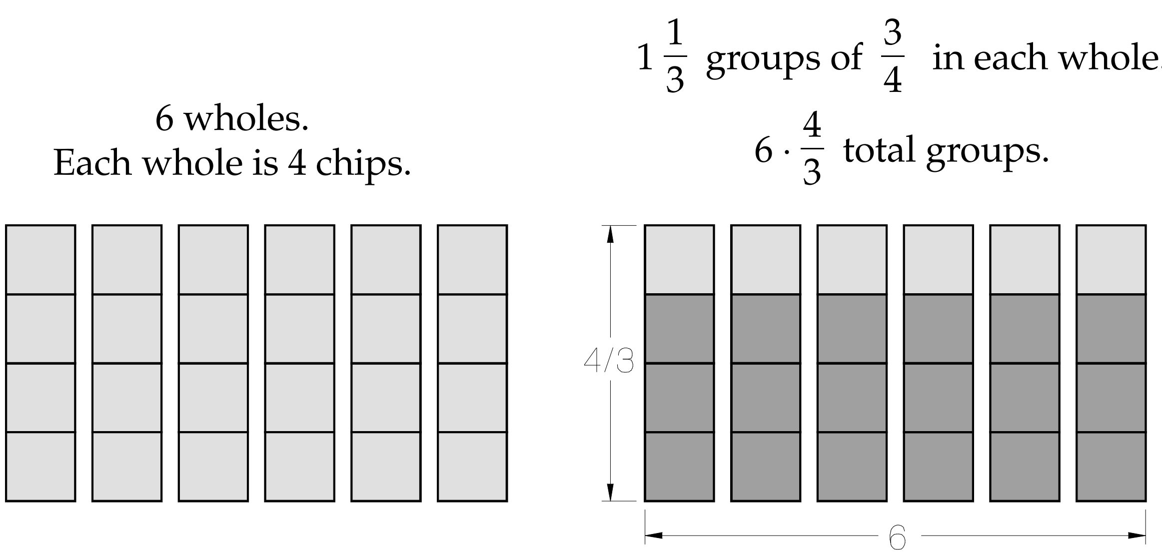




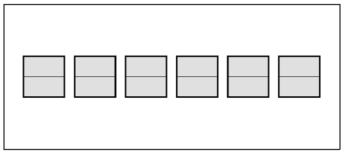
Now we can do the same problem that we did in the previous topic:

$$6 \div \frac{3}{4}$$

There are ⁴⁄₃ groups of ¾ in every whole, and we have 6 wholes, so we have a total of 6 times ⁴⁄₃ or 8 groups as a result:



$$6 \div \frac{3}{4} = 6 \cdot \frac{4}{3} = \frac{6}{1} \cdot \frac{4}{3} = \frac{6 \cdot 4}{1 \cdot 3} = \frac{24}{3} = 8$$



Summary

To divide by a fraction, we have two methods. With Method I, we multiply by the denominator and divide by the numerator of the divisor. With Method II, we multiply by the reciprocal of the divisor. Both methods accomplish the same thing, but they are slightly different ways of visualizing the same process.

Method I

$$a \div \frac{b}{c} = (a \cdot c) \div b = \frac{ac}{b}$$

Method II

$$a \div \frac{b}{c} = a \cdot \frac{c}{b} = \frac{a}{1} \cdot \frac{c}{b} = \frac{a \cdot c}{1 \cdot b} = \frac{ac}{b}$$

Note: We can use the chips to illustrate more complex situations such as the division of two fractions or two mixed numbers. See the APPENDIX for more information.

Exercises

Use your chips to illustrate the solution to these problems. Try both methods.

1. $2 \div \frac{1}{2}$

2. $2 \div \frac{2}{3}$

3. $5 \div \frac{5}{6}$

4. $6 \div \frac{2}{3}$

5. $3 \div \frac{2}{3}$

6. $4 \div \frac{2}{5}$

7. $1 \div \frac{2}{3}$

Section **3**
Compound Fractions

Division and the Meaning of Fractions

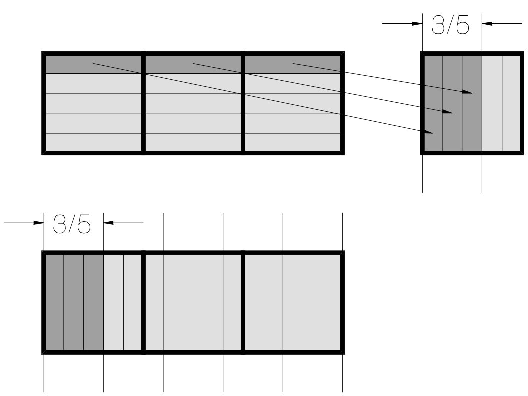
A fraction can be thought of as a division problem. For example:

$$\frac{3}{5} \text{ means } 3 \div 5$$

It is not necessary to take this for granted. If we look at the meaning of the division 3 ÷ 5, we will see that the fraction ⅗ represents an equivalent amount. We have thought of division in several different ways:

- **1. Divide 3 into 5 equal pieces. How large is each piece?**
- **2. How many 5's in 3?**
- **3. Arrange 3 units in a rectangle with a width of 5. How high is the rectangle?**

The first case:

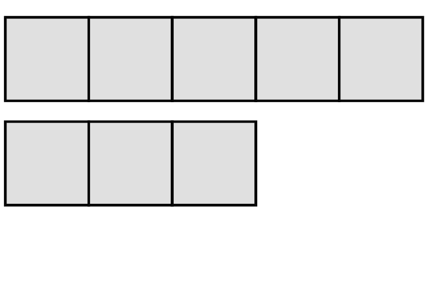


Start with 3 units.
Divide into 5 equal pieces.
Each strip is ⅗.

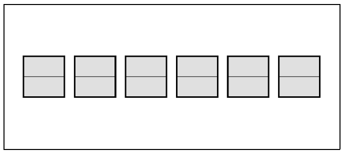
OR

Divide 3 into five equal sections.
Each section is ⅗.

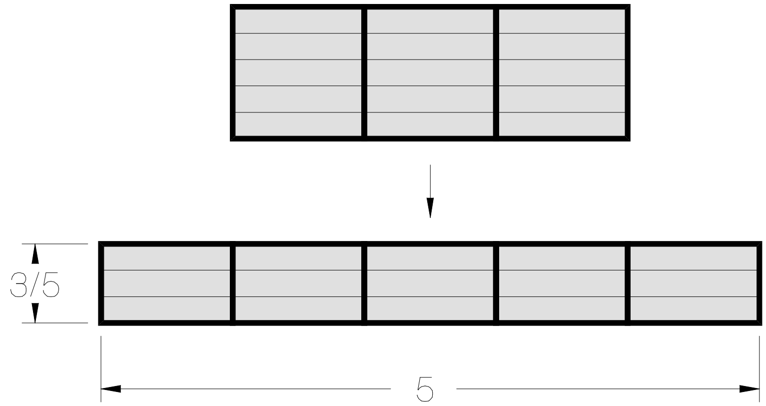
The second case:



How many 5's can we make from 3?
3 is ⅗ of a 5. The result is ⅗.



The third case:



Take 3 units. Divide into fifths. Build a rectangle of width 5. How high is it? The result is $\frac{3}{5}$.

To summarize:

Fractions and Division

$$\frac{a}{b} = a \div b$$

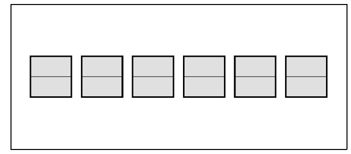
Compound Fractions

We can use this idea of division to simplify more complex fractions that contain fractions. A fraction containing other fractions is called a **compound fraction**. *To simplify compound fractions, think of the larger fraction as a division problem. Then multiply the first number by the reciprocal of the second.* For example:

$$\frac{\frac{1}{2}}{\frac{3}{4}} = \frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \cdot \frac{4}{3} = \frac{1 \cdot 4}{2 \cdot 3} = \frac{4}{6} = \frac{2}{3}$$

Other problems can all be done in a similar manner:

$$\frac{\frac{5}{8}}{\frac{4}{3}} = \frac{5}{8} \div \frac{4}{3} = \frac{5}{8} \cdot \frac{3}{4} = \frac{5 \cdot 3}{8 \cdot 4} = \frac{15}{32}$$



If one number is not a fraction, you may want to write it as fraction before continuing:

$$\frac{3}{\frac{1}{3}} = \frac{\mathbf{\frac{3}{1}}}{\frac{1}{3}} = \frac{3}{1} \div \frac{1}{3} = \frac{3}{1} \cdot \frac{3}{1} = \frac{3 \cdot 3}{1 \cdot 1} = \frac{9}{1} = 9$$

With algebra symbols, we can summarize the process like this:

Compound Fractions

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Exercises

Illustrate these examples with pictures to demonstrate why the fraction and the division problem are equivalent.

1. $\frac{2}{3} = 2 \div 3$

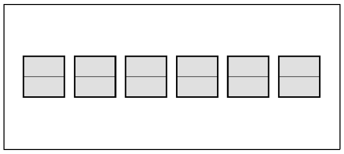
2. $\frac{5}{4} = 5 \div 4$

3. $\frac{1}{2} = 1 \div 2$

Simplify these fractions by rewriting the fraction as a division problem. Complete the division to find the answer.

4. $\dfrac{\frac{7}{16}}{\frac{3}{8}}$

5. $\dfrac{\frac{3}{8}}{\frac{7}{16}}$



6. $\dfrac{8}{\frac{3}{7}}$

7. $\dfrac{6}{\frac{3}{4}}$

8. $\dfrac{\frac{2}{3}}{\frac{3}{4}}$

9. $\dfrac{\frac{3}{4}}{\frac{3}{4}}$

10. $\dfrac{\frac{3}{8}}{\frac{3}{4}}$