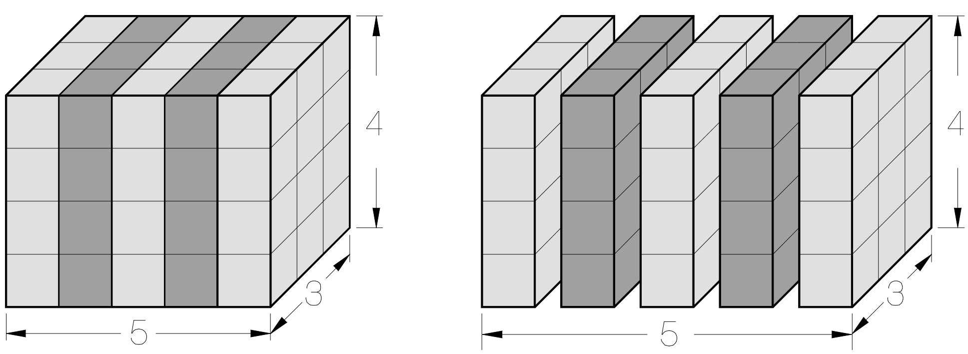
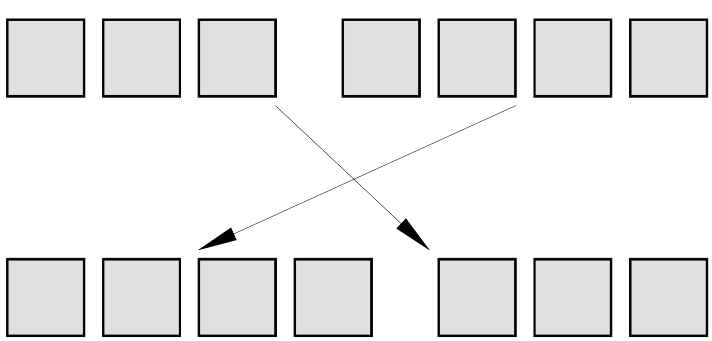
Chapter 5

Properties





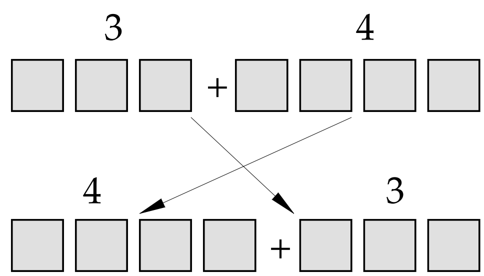
Section 1
Properties of Addition and Multiplication

Commutative Property of Addition

We know that

$$3 + 4 = 4 + 3$$

because



In symbols:

Commutative Property of Addition

$$\mathbf{3 + 4 = 4 + 3}$$

or

$$\boldsymbol{a + b = b + a}$$

for any numbers $\boldsymbol{a}$ and $\boldsymbol{b}$

When we add two numbers, the order does not matter because we get the same total of chips in either case.

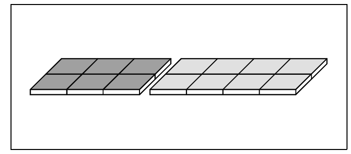
Associative Property of Addition

What is the meaning of **3 + 4 + 5** ?

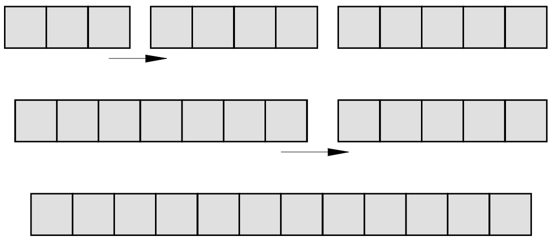
Because we think of adding only two numbers at one time, do we mean

$$(3 + 4) + 5 \quad \text{or} \quad 3 + (4 + 5)\ ?$$

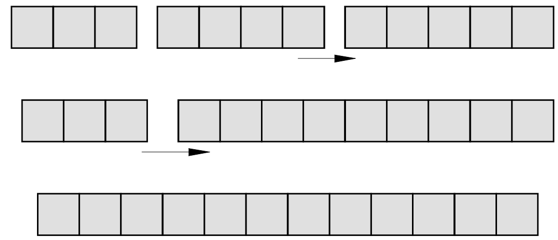
Of course, we can see that it doesn't matter because we will get the same answer in either case:



$(3 + 4) + 5$



$3 + (4 + 5)$



In summary:

Associative Property of Addition

$$\mathbf{(3 + 4) + 5 = 3 + (4 + 5)}$$
or
$$\boldsymbol{(a + b) + c = a + (b + c)}$$
for any three numbers ***a*, *b*, *c***

Commutative Property of Multiplication

We all know that

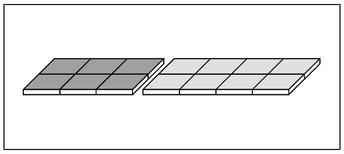
$$3 \cdot 4 = 4 \cdot 3$$

Why is this true? We can check it by doing the addition

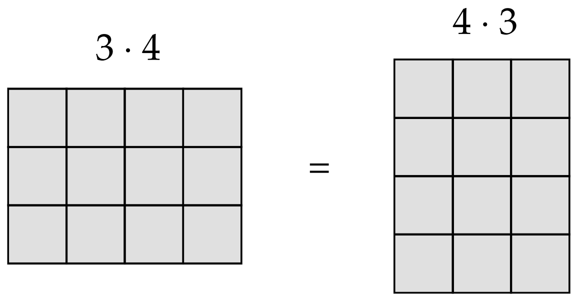
$$3 \cdot 4 = 4 + 4 + 4 = 12$$
$$4 \cdot 3 = 3 + 3 + 3 + 3 = 12$$

but this only confirms that it is true; we still don't know why.

Since multiplication is making rectangles, we can build two rectangles–one will be three by four and the other will be four by three.



Then we have:



With symbols, we have:

Commutative Property of Multiplication

$$3 \cdot 4 = 4 \cdot 3$$

or

$$a \cdot b = b \cdot a$$

for

any two numbers a and b

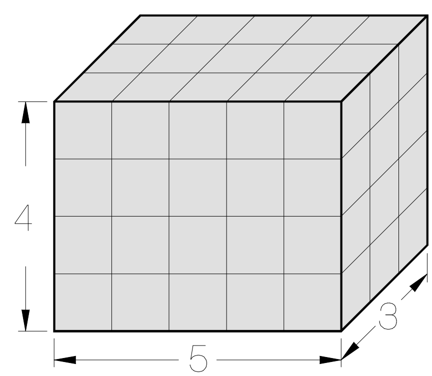
When we multiply two numbers, the order doesn't matter because we are making a rectangle that is the same size in either case.

Many properties are this easy to understand; many properties describe ideas that you already know.

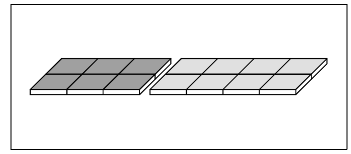
Associative Property of Multiplication

What is the meaning of $3 \cdot 4 \cdot 5$?

Since we have looked at multiplication of two numbers as making a rectangle, we will look at the third number as making a three-dimensional box:

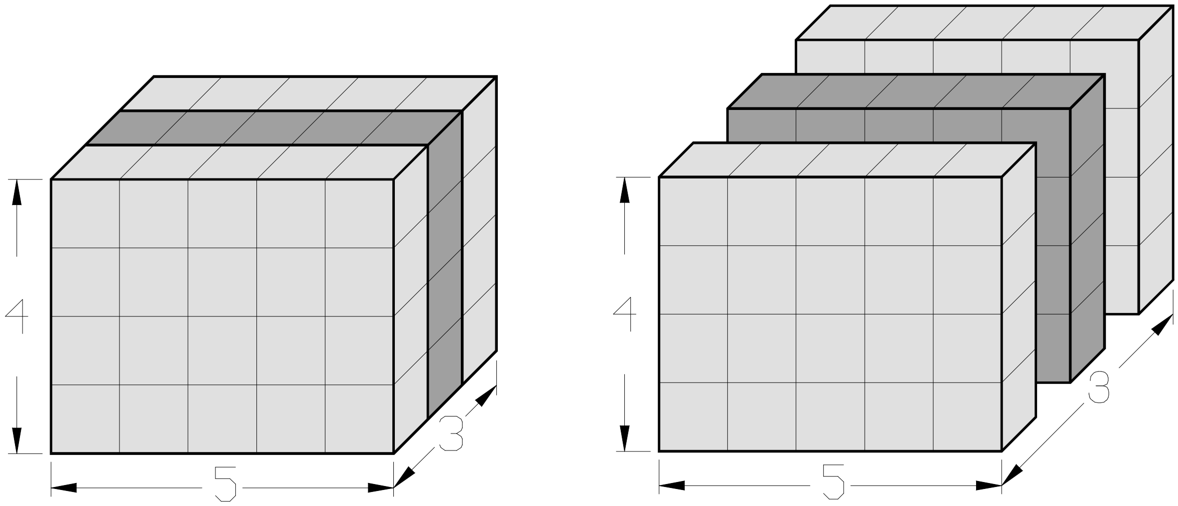


Let's look at



$$3 \cdot (4 \cdot 5)$$

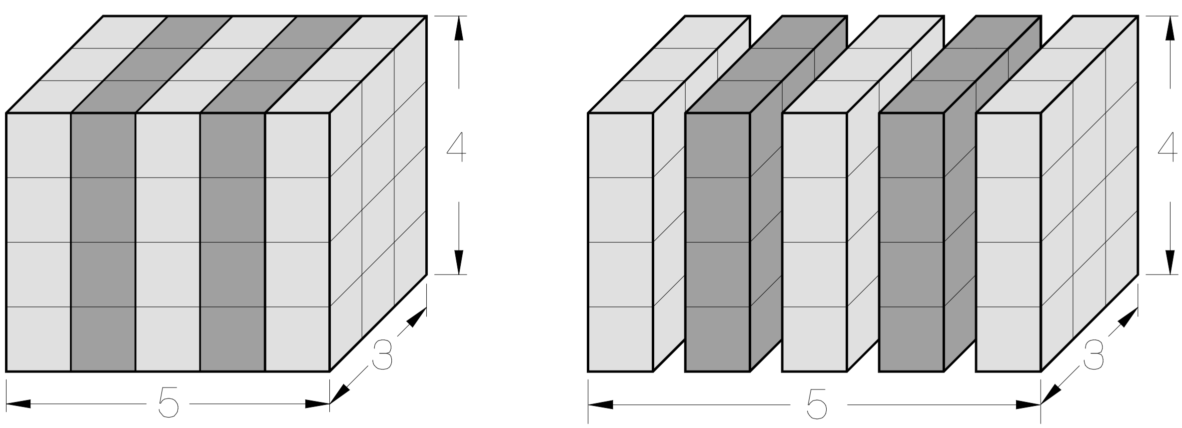
as three (3) groups or slices that are four by five $(4 \cdot 5)$:

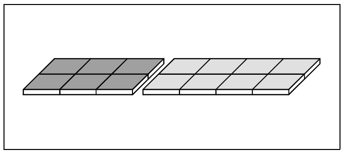


What is

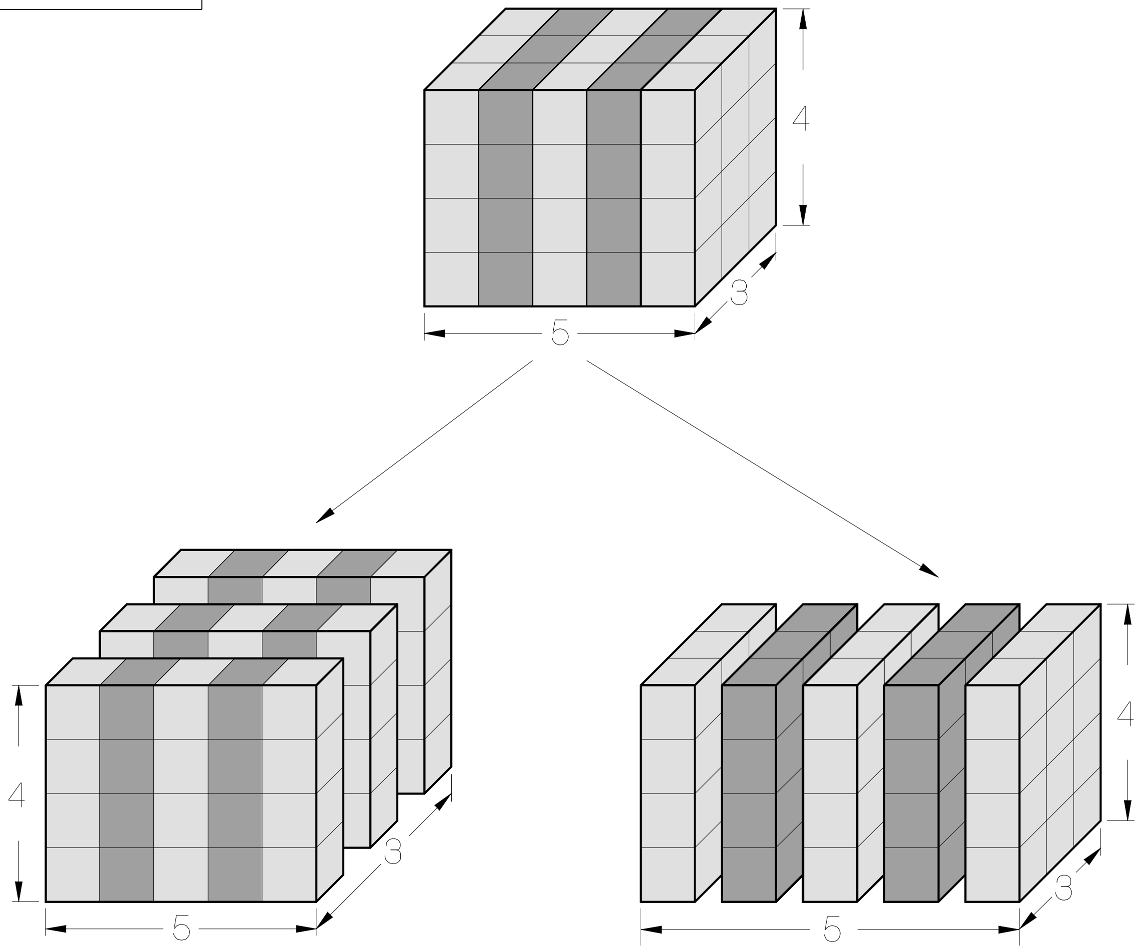
$$(3 \cdot 4) \cdot 5\ ?$$

It is five (5) slices, each three by four $(3 \cdot 4)$. We count the small cubes (60):





These two amounts are obviously equal, because we are simply counting the same number of blocks in a different order.



3 groups of (4·5)

5 groups of (3·4)

Associative Property of Multiplication

$$3 \cdot (4 \cdot 5) = (3 \cdot 4) \cdot 5$$

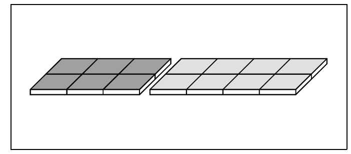
or

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

for any numbers ***a***, ***b***, and ***c***

All of these commutative and associative properties are examples of the same idea: *if you count a group of chips or blocks in two different ways, you get the same answer.*

The words used for these properties—associative and distributive—have a meaning in mathematics that is similar to their meaning in everyday language:



Associative Properties describe the grouping or *association* of numbers.

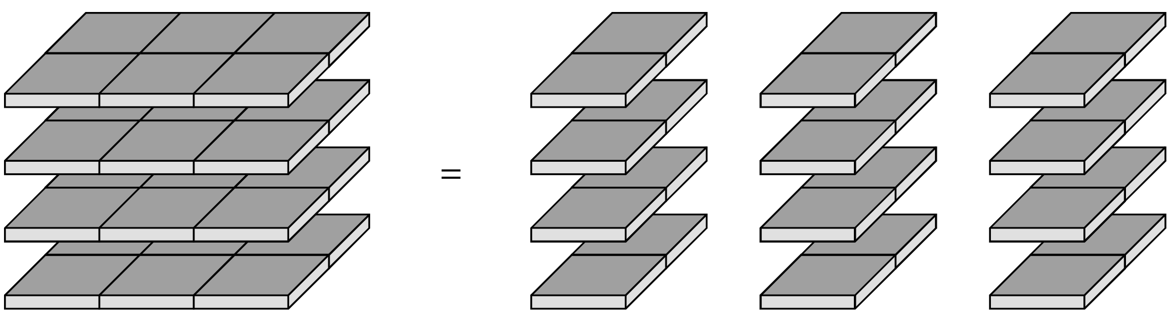
Commutative Properties describe the order or *commuting* of numbers.

Exercises

Use your chips to show why each statement is true, then identify the property (or properties) that you have used. To show the statement is true, arrange chips to represent each side of the equal sign and show why the two pictures are equal.

Example: $4 \cdot (2 \cdot 3) = (4 \cdot 2) \cdot 3$

Solution: $4 \cdot 6 = 8 \cdot 3 = 24$



1. $7 \cdot 4 = 4 \cdot 7$
2. $1 + 2 = 2 + 1$
3. $5 + (6 + 7) = (5 + 6) + 7$
4. $1 \cdot (2 \cdot 3) = (1 \cdot 2) \cdot 3$
5. $^-2 + (^-3 + 4) = (^-2 + ^-3) + 4$
6. $4 + (^-3 + ^-2) = (^-2 + ^-3) + 4$
7. $2 \cdot (3 \cdot 4) = 2 \cdot (4 \cdot 3)$

Section **2**

The Distributive Property

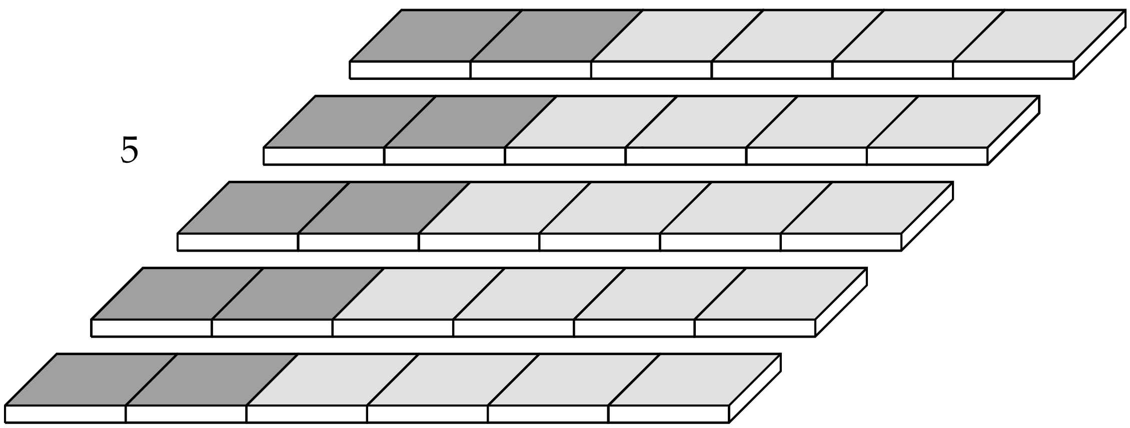
Multiplying a Number Times a Sum

When we multiply a number times a sum (the addition of two other numbers) we discover the **distributive property**. It is a very easy concept; in fact, you may already know it.

5(2 + 4)   means: 5 groups of (2 plus 4):

2 + 4

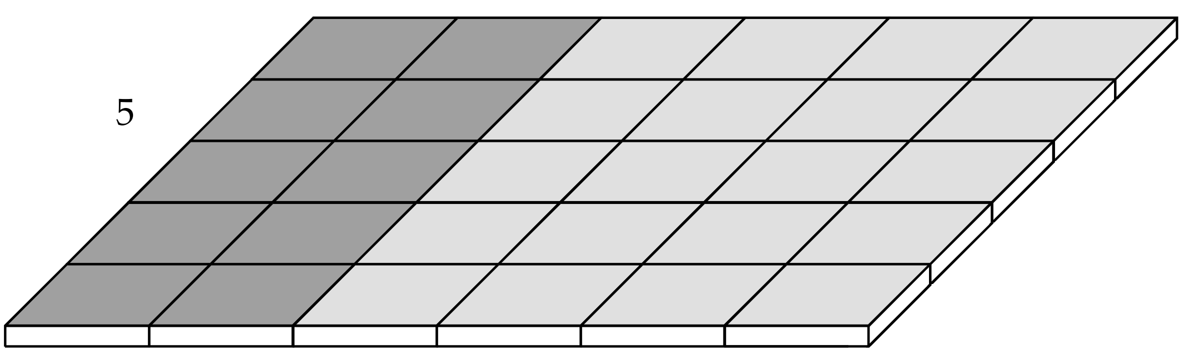
5



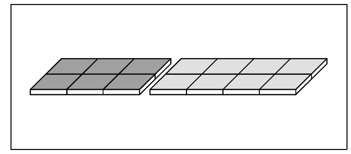
This is a large rectangle composed of two smaller ones:

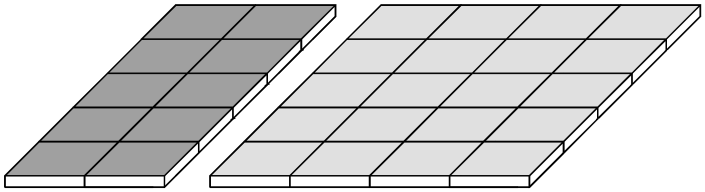
2 + 4

5



If we count the rectangles separtely, the total number of chips will be the same as if we count them together:





$5 \cdot 2$ $\quad$ $5 \cdot 4$

Again, you can add first, then multiply:

$$5(2 + 4) = 5 \cdot 6 = 30$$

or you can multiply first, then add:

$$5(2 + 4) = (5{\cdot}2) + (5{\cdot}4) = 10 + 20 = 30$$

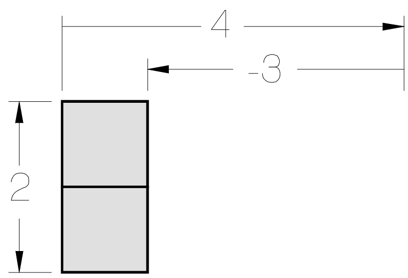
If you multiply first, you must multiply the 5 times both the 2 *and* the 4.

Multiplying Times a Difference

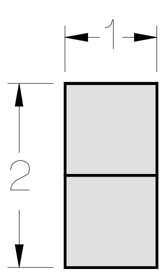
Is there a property that will tell us about

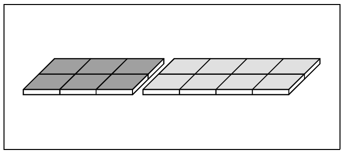
$$2 \cdot (4 - 3)\ ?$$

First, let's think of it as a rectangle that is two (2) on one side and four minus three (4 – 3) on the other:

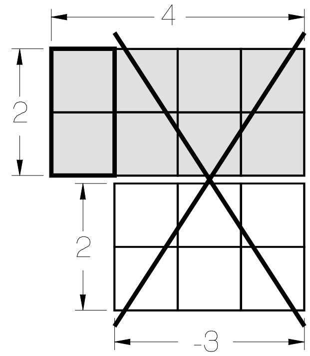


This is really a two by one rectangle equal to two (2):





To try another way, we can think of it as a two by four (2 · 4) rectangle and a two by negative three (2 · ⁻3) rectangle:



This is also equal to two (2).

To summarize, we can subtract first and then multiply, or we can multiply first and then subtract. The answer is the same.

Distributive Property

3(4 + 5) = (3 · 4) + (3 · 5)
or
a(b + c) = (a · b) + (a · c)
a(b – c) = (a · b) – (a · c)

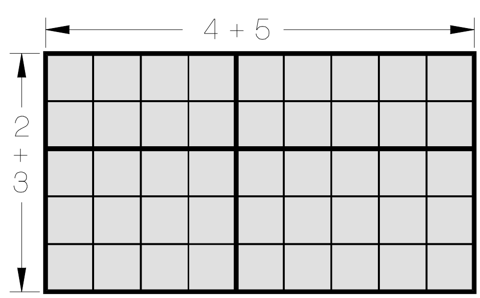
for any three numbers a, **b**, and **c**

Multiplying Two Sums

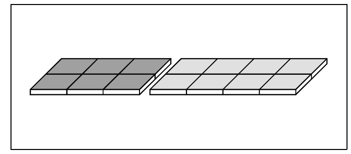
The next case we will look at is the idea of multiplying two sums. For example, consider:

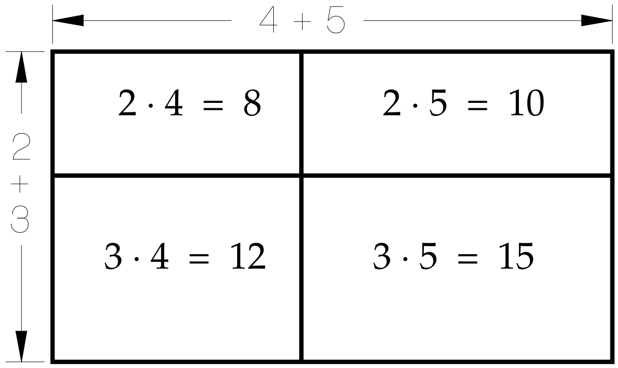
(2 + 3) · (4 + 5)

Since multiplication is making rectangles, we need to make a rectangle that is **(2 + 3)** on one side and **(4 + 5)** on the other side.



This large rectangle is made up of *four* smaller rectangles, and the result is the sum of these four products:





$(2 + 3)(4 + 5) = (2 \cdot 4) + (2 \cdot 5) + (3 \cdot 4) + (3 \cdot 5)$

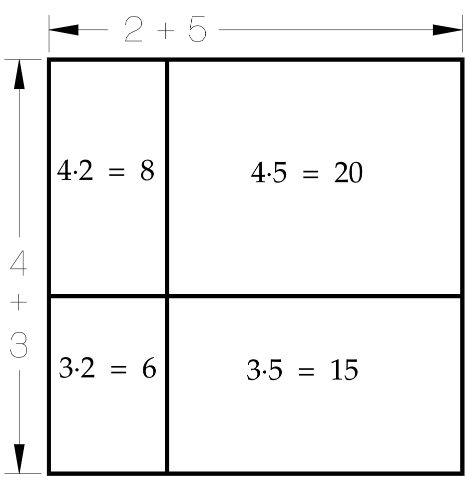
The four rectangles come from the four possible products of each length and each width. In symbols, here is how we do it:

$$(2 + 3)(4 + 5) = (2 \cdot 4) + (2 \cdot 5) + (3 \cdot 4) + (3 \cdot 5)$$

$$= 8 + 10 + 12 + 15$$

$$= 45$$

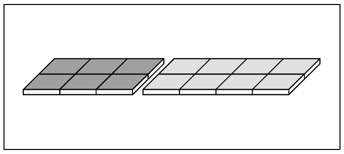
Again, it is important to work on the idea of these problems rather than attempting to memorize the pattern. Here is another example:



$$(4 + 3)(2 + 5) = 4{\cdot}2 + 4{\cdot}5 + 3{\cdot}2 + 3{\cdot}5$$

$$= 8 + 20 + 6 + 15$$

$$= 49$$



A Familiar Example

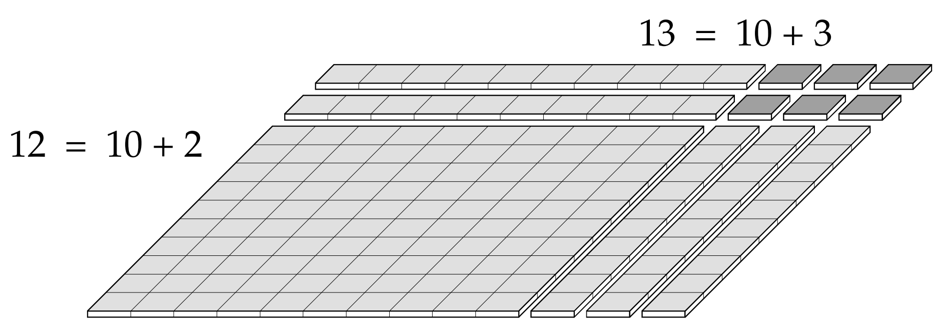
When we multiply two-digit numbers, we have a familiar process that is actually an example of the use of the distributive property. Here is the way it usually works:

$$\begin{array}{r} 1\ 3 \\ \times\ 1\ 2 \\ \hline 2\ 6 \\ 1\ 3\ \ \ \\ \hline 1\ 5\ 6 \end{array}$$

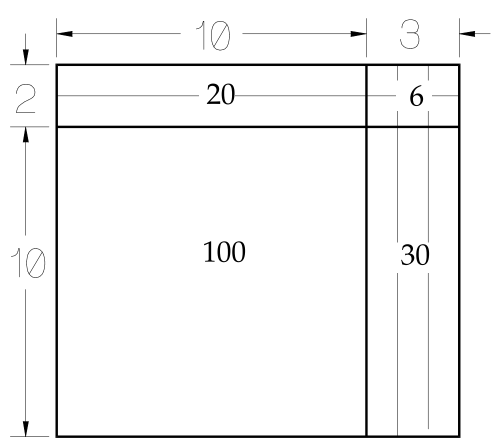
With pictures, think of the problem as

$$13 \cdot 12 = (10 + 3) \cdot (10 + 2)$$

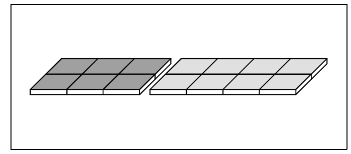
This is a rectangle 13 wide and 12 deep



which is made up of four smaller rectangles:



With symbols, here is what we are doing:



$$\begin{array}{r} 13 \\ \times\ 12 \\ \hline 6 \\ 20 \\ 30 \\ 100 \\ \hline 156 \end{array}$$

2	·	3	=	6
2	·	10	=	20
10	·	3	=	30
10	·	10	=	100

Distributive Property

(2 + 3)(4 + 5) = (2·4) + (2·5) + (3·4) + (3·5)

or

(a + b)(c + d) = (ac) + (ad) + (bc) + (bd)

for any four numbers **a**, **b**, **c**, and **d**

Division and the Distributive Property

Without using pencil and paper, calculate half of $2.50.

Did you need to do long division, or did you think of it as

Half of $2.00 is $1.00. Half of .50 is .25

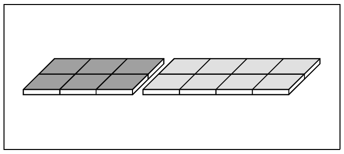
The answer is $1.00 plus .25 or $1.25

This is another example of the distributive property. The new idea is that the property works when you are taking *prt* of something (dividing by two or multiplying by one-half) as well as when you are multiplying by whole numbers.

Next, think about making a recipe smaller. If your recipe calls for four and two-thirds cups of milk, and you want to make half, how do you do it?

Half of $4\frac{2}{3}$ is half of 4 plus half of $\frac{2}{3}$ is

$$2 \text{ plus } \frac{1}{3} = 2\frac{1}{3}$$



The procedure that you naturally use is to take half of the parts and then add these two halves together. This is much easier than:

$$4 + \frac{2}{3} = \frac{12}{3} + \frac{2}{3} = \frac{14}{3}$$

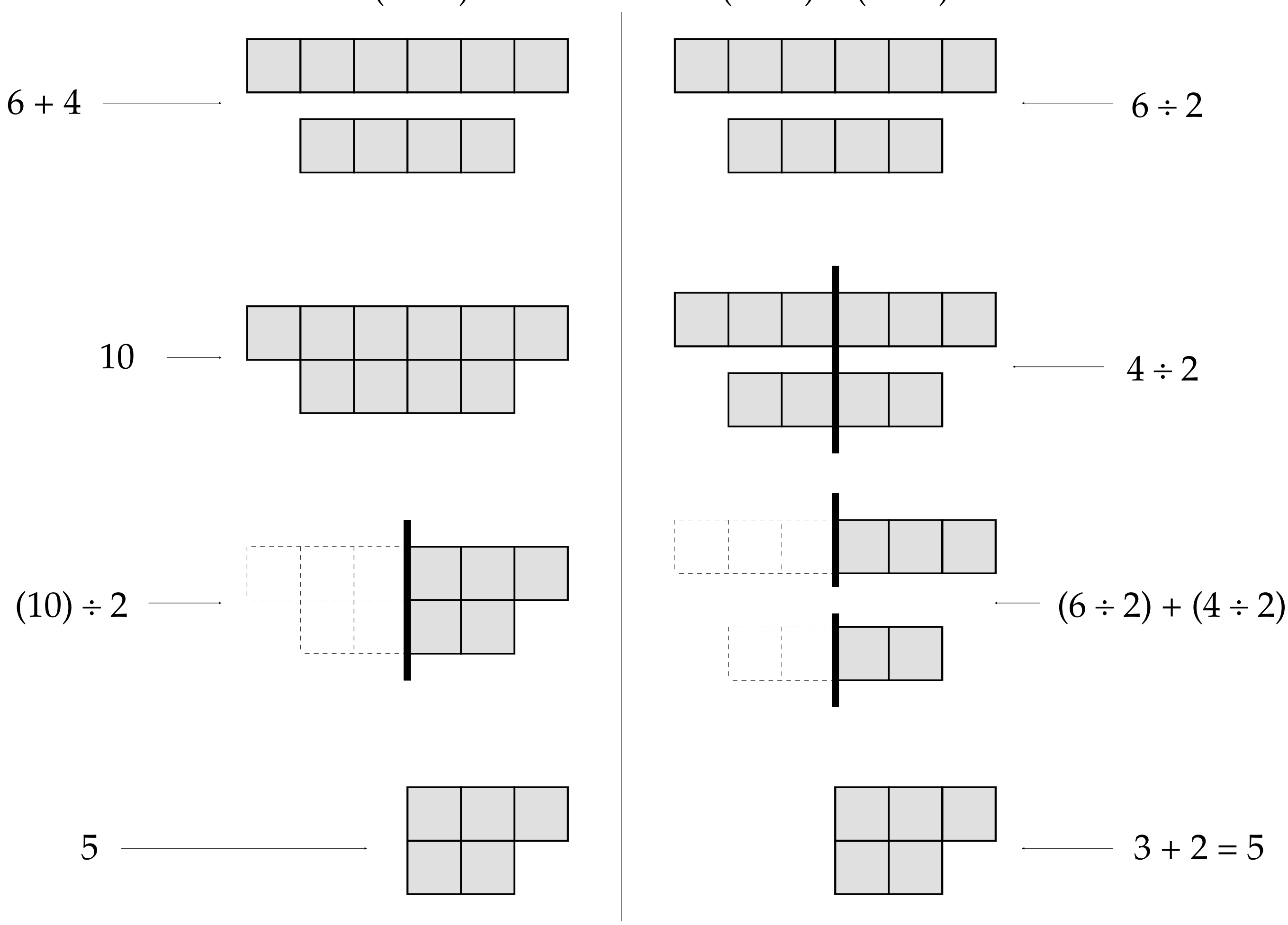
$$\frac{1}{2} \cdot \frac{14}{3} = \frac{14}{6} = \frac{7}{3} = 2\frac{1}{3}$$

OR

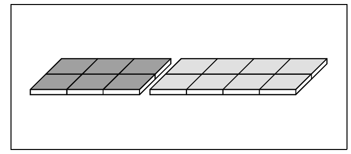
$$\frac{1}{2}\left(4 + \frac{2}{3}\right) = \frac{1}{2}\left(\frac{12}{3} + \frac{2}{3}\right) = \frac{1}{2}\left(\frac{14}{3}\right) = \frac{14}{6} = \frac{7}{3} = 2\frac{1}{3}$$

The next example illustrates this property with pictures. The two sets of pictures demonstrate that the two sides of the equation are equal.

$$(6 + 4) \div 2 \quad = \quad (6 \div 2) + (4 \div 2)$$



With symbols alone, here is how it looks:



Distributive Property of Division over Addition

$$(6 + 4) \div 2 = (6 \div 2) + (4 \div 2)$$

or

$$(a + b) \div c = (a \div c) + (b \div c)$$

for any numbers **a**, **b**, and **c**

Exercises

Use pictures or chips to showeach problem and calculate the answers. *Multiply or divide first, using the distributive property.* Check by adding or subtracting the quantity in parenthese first.

1. $3(5 + 6)$

2. $2(1 + 4)$

3. $(4 - 1) \cdot 3$

4. $4(5 - 2)$

5. $(1 + 2)(3 + 4)$

6. $(3 + 2)(1 + 1)$

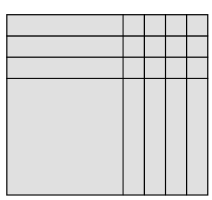
7. $(8 + 10) \div 2$

8. $(6 + 12) \div 3$

For the following problems, use the large square in your kit to represent 100 (10 by 10), the long bar to represent 10 (1 by 10), and the small chips to represent 1 (1 by 1).

Example: $14 \cdot 13 = (10 + 4)(10 + 3)$

Solution: $100 + 30 + 40 + 12 = 182$



$$\begin{array}{r} 100 \\ 30 \\ 40 \\ +\ 12 \\ \hline 182 \end{array}$$

9. $13 \cdot 15$

10. $14 \cdot 16$

11. $21 \cdot 16$

Section 3
Identities and Inverses

Operations

Addition, subtraction, multiplication, and division are called **operations**. These are **binary** operations because *two* numbers are required to find the result.

The previous sections of this chapter have dealt with some basic ideas about operations:

- **The *order* of the numbers (Commutative Properties)**
- **The *grouping* of three numbers into pairs (Associative Properties)**
- ***Combinations* of operations (Distributive Properties)**

This section will discuss certain special numbers that are called **identities** and **inverses**.

Identity Elements

When we add zero to a number, the number is unchanged. This i certainly not a great mystery, because adding zero means taking a group of chips and doing nothing to it; since the original number has an *identical* value, zero is called the **identity element for addition**.

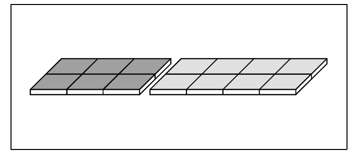
The identity element for an operation is the number that has no effect for that operation.

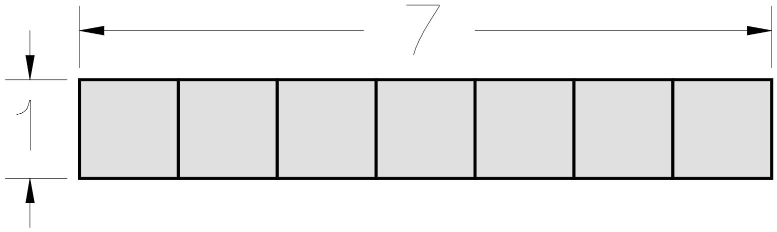
For subtraction, zero also has no effect; taking away zero chips leaves the identical number with which we started.

Multiplication is somewhat different. The identity element is the number which, when multiplying any other number, leaves it unchanged:

$$7 \cdot 1 = 7$$
$$8 \cdot 1 = 8$$
$$1 \cdot 7 = 7$$
$$1 \cdot (^{-}3) = {}^{-}3$$

*The identity element for multiplication is **one***. With chips, this means that to multiply **7 · 1** we make a rectangle that is seven chips long and one chip wide. The result is obviously seven:



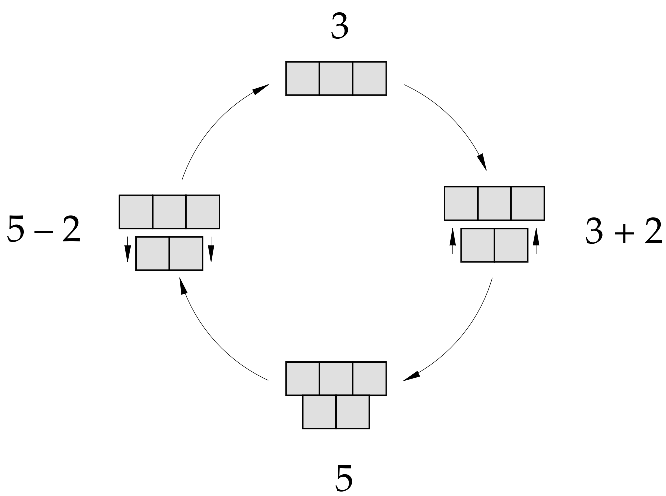


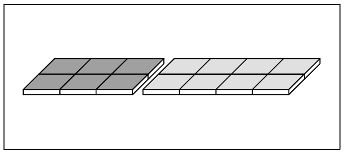
The identity element for division is also one. How many ones are there in seven? The answer is seven. How many ones in negative five? Negative five. Here is a summary of identity elements:

Operation	Identity	Example
Addition	0	$32 + 0 = 32$
Subtraction	0	$53.6 - 0 = 53.6$
Multiplication	1	$23 \cdot 1 = 23$
Division	1	$53 \div 1 = 53$

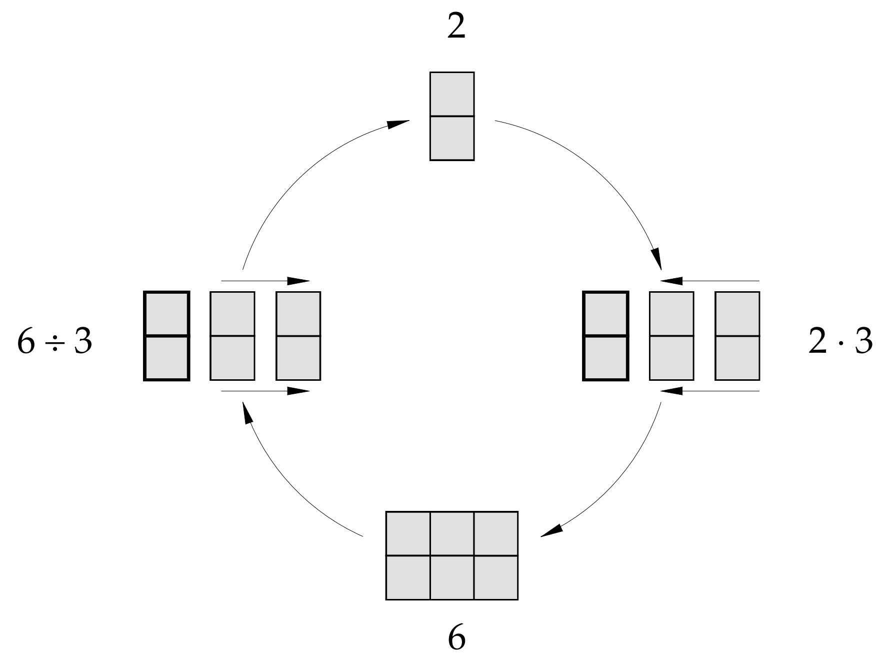
Inverse Operations

Addition and subtraction are called **inverse operations** because they represent opposite actions with numbers and chips. Addition is putting chips together; subtraction is taking chips away:





Multiplication and division have a similar relationship. Multiplication is building a number of rows; division starts with the finished rectangle and counts the number of rows:



Can you explain why the operations in each pair (Multiplication/Division and Addition/Subtraction) have the same identity element?

Inverses of Numbers

The effect of adding 4 to a number can be cancelled out by adding $^-4$:

$$5 + 4 + {}^-4 = 5$$

This occurs because

$$5 + 4 + {}^-4 = 5 + (4 + {}^-4) = 5 + 0 = 5$$

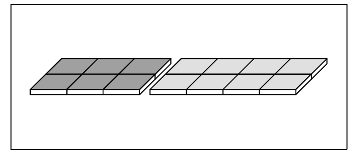
As we can see, 4 and $^-4$ cancel out the effects of each other because

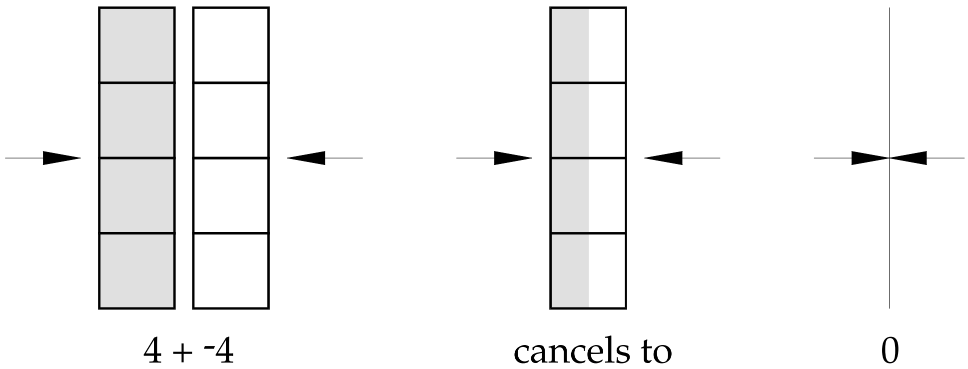
$$4 + {}^-4 = 0$$

Since zero is the identity element, there is no effect on the total. Numbers like these that have opposite effects are called **inverses**. With addition, opposites of positives are negatives and opposites of negatives are positives. The opposite of zero is zero.

Notice that inverses exist with respect to a particular operation only; when we multiply $4 \cdot {}^-4$ one does *not* cancel out the effect of the other as they did when we added.

Here is an example of **additive inverses**:





For multiplication, we can find a similar property of some familiar numbers:

$$23 \cdot 2 \cdot \frac{1}{2} = 23$$

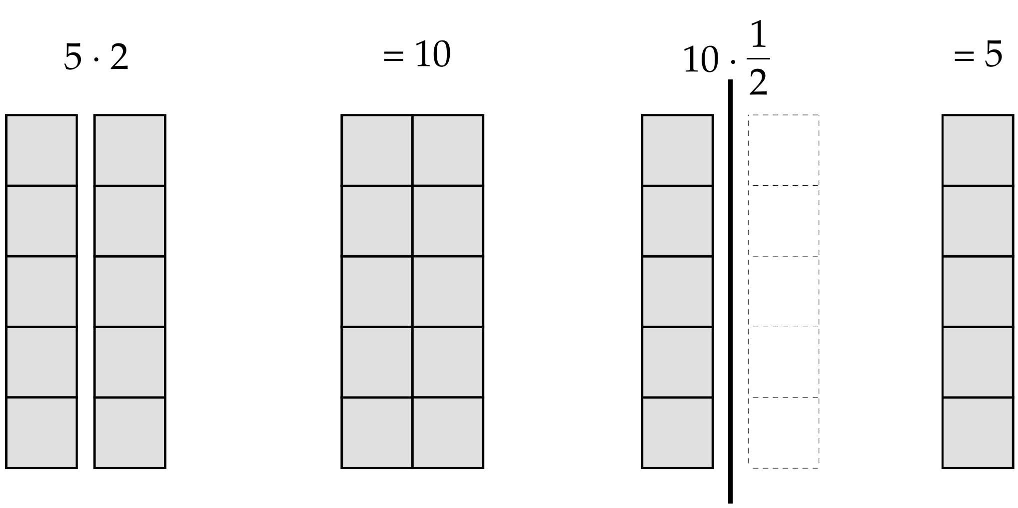
$$8 \cdot \frac{2}{3} \cdot \frac{3}{2} = 8$$

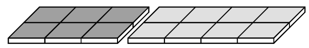
This occurs because

$$23 \cdot 2 \cdot \frac{1}{2} = 23 \cdot \left(2 \cdot \frac{1}{2}\right) = 23 \cdot 1 = 23$$

Pairs of numbers such as 2 and ½ are called **multiplicative inverses** or **reciprocals**. Each cancels out the effect of the other because their product is one—the identity element for multiplication.

Here is an example of multiplicative inverses:





Exercises

Perform the operations using chips and with symbols.

1. $3 + 5 + {}^{-}5$

2. $3 + 21 + {}^{-}3$

3. $14 \cdot 2 \cdot \frac{1}{2}$

4. $8 \cdot \frac{1}{3} \cdot 3$

5. $(3 + {}^{-}3) + \left(\frac{1}{5} \cdot 5\right)$

6. $7 \cdot ({}^{-}7)$

7. $6 + {}^{-}({}^{-}6)$

8. ${}^{-}6 + {}^{-}({}^{-}6)$

9. $12{,}345.8 + {}^{-}12{,}345.8$

10. $17 \cdot 45 \cdot \frac{1}{45}$

Section 4
Properties of Zero

Multiplication by Zero

Zero times any number is zero. This is an obvious fact and does not need to be memorized or practiced; it is clearly true.

Five times zero is zero because five *groups* of zero are zero. Zero times five is zero because if you have no five dollar bills, you have no money at all. With chips, it looks like this:

Here, you can see a rectangle with sides of zero and five representing $\mathbf{0 \cdot 5}$. How many chips do you see?

Dividing with Zero

Consider two interesting questions:

- **What is the meaning of** $\mathbf{0 \div 5}$ **or** $\frac{0}{5}$ **?**

- **What is the meaning of** $\mathbf{5 \div 0}$ **or** $\frac{5}{0}$ **?**

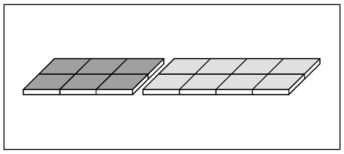
Dividing zero by five or any other number (except zero) is quite straightforward. We are asking

"How many fives in zero?"

or

"What is a fifth of zero?"

In each case, the answer is clearly zero.



Dividing *by* zero is much more troublesome. Here we are asking:

"How many zeroes in five?"

or

"Divide 5 into 0 equal parts. How much is in each part?"

It is clear that there is no sensible answer to either question. First, you can add as many zeroes as you wish, but you will never reach five.

0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+
0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+
0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+
0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+
0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+
0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0
+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+

Secondly, the idea of dividing something into zero parts (a zeroeth?) does not make much sense. We conclude that **division by zero is not defined**. If you are tempted to guess that the answer should be zero, consider these examples:

$$12 \div 3 = 4 \text{ because } 3 \cdot 4 = 12$$

$$16 \div 8 = 2 \text{ because } 8 \cdot 2 = 16$$

If 5 divided by zero were equal to zero, then

$$5 \div 0 = 0 \text{ would mean that } 0 \cdot 0 = 5?$$

$$\text{so} \quad 5 \div 0 \quad \text{is undefined}$$

This illustrates the difficulty—if we ever define division by zero, we will have to create new concepts for our old ideas about multiplying, dividing, and even numbers themselves.

Exercises

Find the answers or determine if the problem is not defined:

1. $(5 \cdot 0) \div 6$

2. $0 \div 0$

3. $(0 \div 3) \div 3$

4. $(3 \div 3) \div 0$

5. $(3 \div 0) \cdot 0$

6. $14 \div (^-23 - {}^-23)$

Section 5: (Optional) Properties or Rules?

Introduction: An Example

This section is for people who have had previous difficulties with the traditional system of memorizing rules. The purpose is to help you think about the best way to learn mathematics.

The Distributive Law of Multiplication over Addition is traditionally explained as a pattern in this way:

Distributive Law

$$\mathbf{a(b + c) = ab + ac}$$

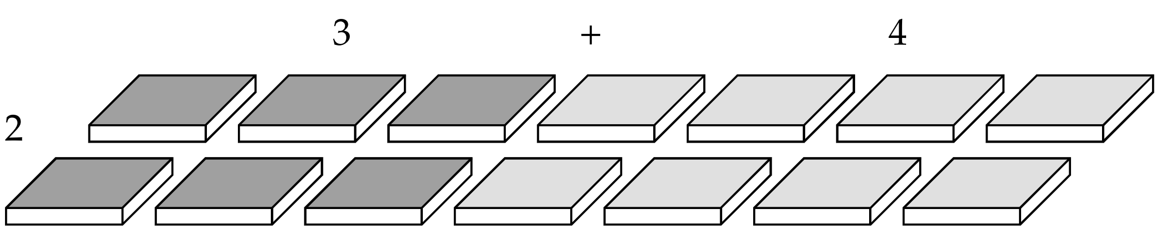
The instructions are: To multiply one number times a sum of two numbers, you multiply the first number times each of the other numbers and then add the products.

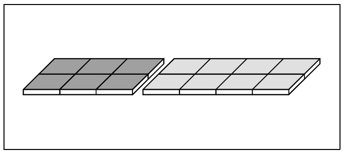
This type of explanation is often very confusing. We don't know *when* to use this property and we don't know *why*. Many students continue to have difficulty with this property even at the college level.

Here is how we have done it in this chapter:

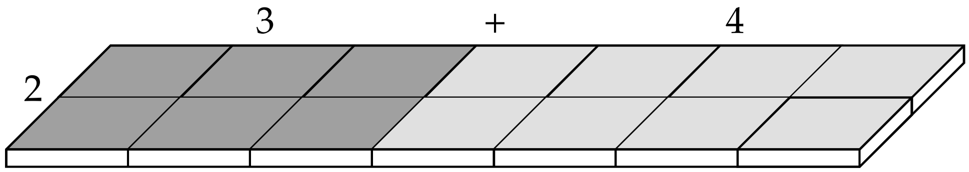
Consider (2)(3 + 4)

Multiplying **(2)** times **(3 + 4)** means making a rectangle that has dimensions of **(2)** and **(3 + 4)**:



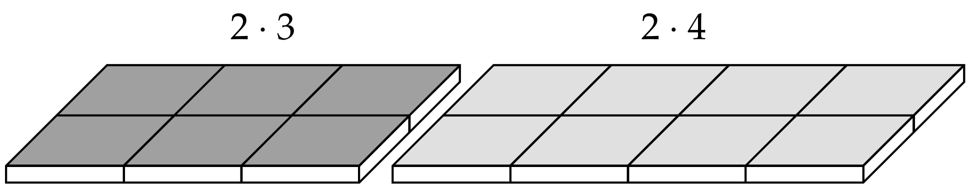


This is a large rectangle made up of two smaller rectangles; one is **(2)** by **(3)** and the other is **(2)** by **(4)**:

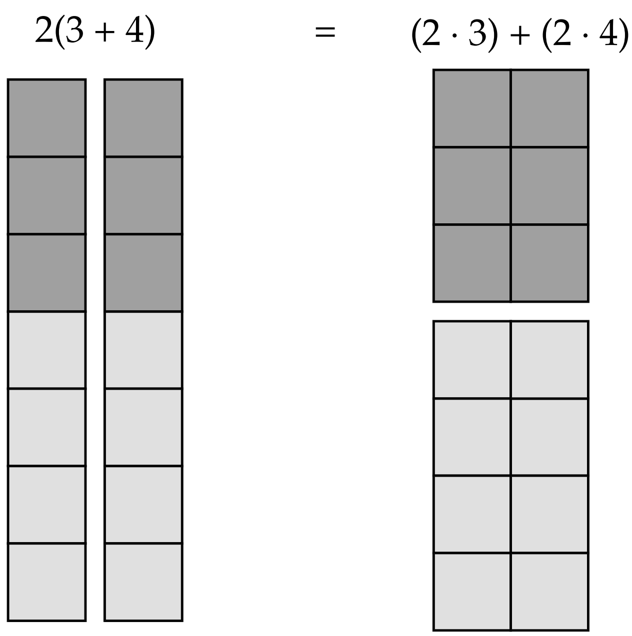


If we separate the two rectangles, we have:

$$(2 \cdot 3) + (2 \cdot 4)$$

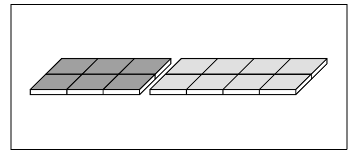


Here is the summary of the property:



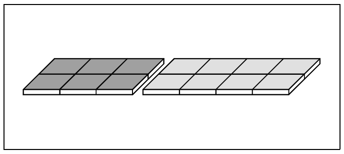
This has been an example of explaining a property with objects instead of symbols. Can you see the difference?

Rules Versus Properties



In this book, we will usually talk about **properties** instead of **rules or laws**. *A property is something real that you can discover and understand*. Properties are easy to learn and and easy to use precisely because they are understood, not memorized. On the other hand, rules are unexplained instructions that must be memorized through lengthy practice. Here are some differences between these two different ways of learning algebra:

	Rules or Laws	**Properties**
Invented by:	Someone else	You
Learned by:	Memorization	Discovery
Practiced by:	Repetition	Investigation
Believed because of:	Authority of others	Your own knowledge and experience
Enjoyable?	Not usually	Yes
Time it takes to learn	Varies, but usually a lot	Varies, but usually less
Length of time re-tained	A month, an hour, or until the next test.	A long time!



We will always concentrate on why things work , not on memorizing what to do. If you try this method, you will find that algebra will be more interesting and much easier to learn.

Here are some of the advantages of learning *why* instead of memorizing *what to do*:

- **Learning each topic will take less time. When you know *why*, you don't have to do as much practice.**

- **Learning will be more fun. The boredom of repetition will be replaced by the excitement of real investigation and problem solving.**

- **You will retain the material for a long time. Memorized, meaningless rules are often forgotten in a few days. Material that is truly understood will never really be lost.**

- **Mathematics will seem less complex. Many properties and techniques are really very similar; many properties are already known to you.**

- **You will be in control. You will know when you understand the material and when you need more work.**