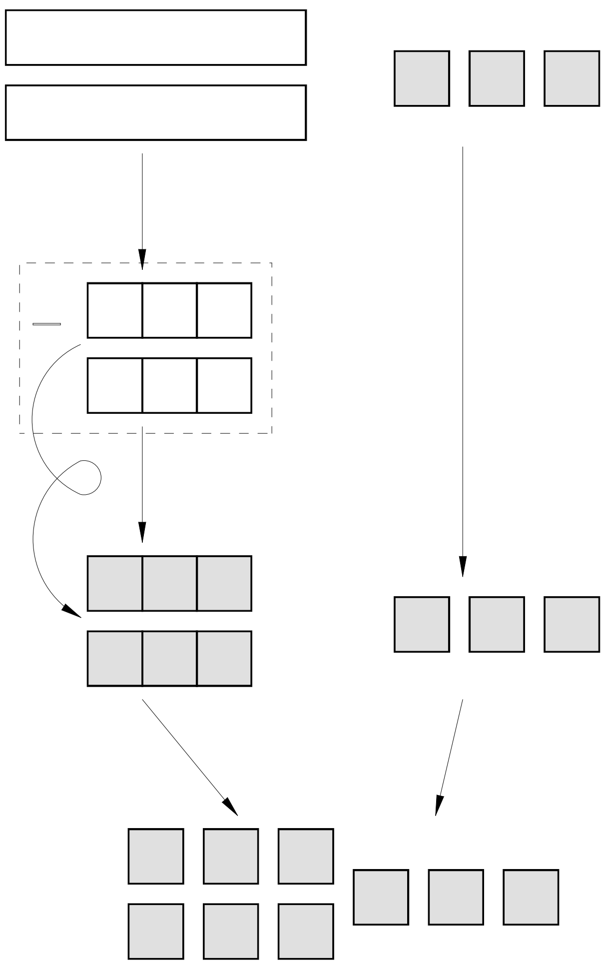
Chapter 6

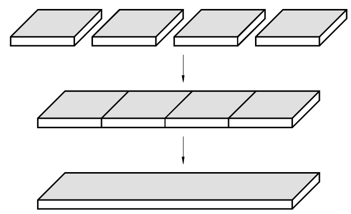
Expressions



Section 1
Simple Expressions

The Meaning of Unknowns

A group of unit chips can be represented by an **unknown** or **variable** like x. We join an unspecified number of chips together to form a bar called x:



Unknowns can be positive, negative, or zero. We use unknowns to represent quantities that will be known at a later time. Because unknowns are actually numbers, we treat them in the same manner as any other numbers.

Expressions

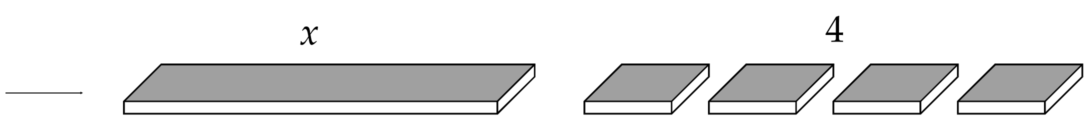
An **expression** is any quantity that stands for a number. Expressions may be as simple as one number or unknown, or they be lengthy statements including many numbers, unknowns, and operations:

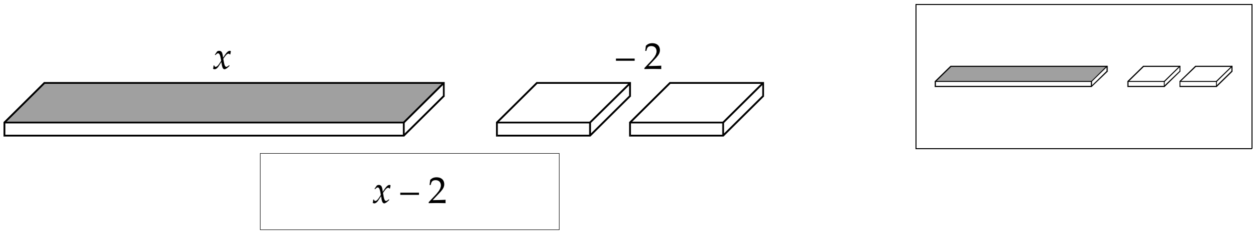
Examples of Expressions
$3x$
$3x + 1$
$^-17$
$3x + 2 + 6x - 2$

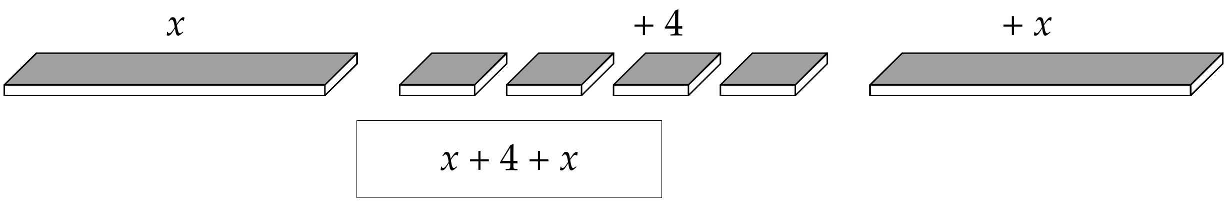
Simple Expressions

It is easy to visualize expressions that include only one or two symbols:

$x + 4$

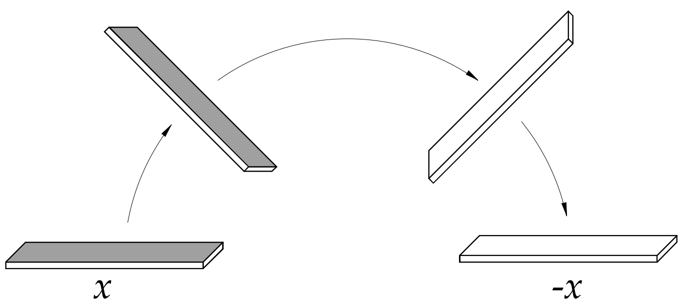






The Opposite of *x*

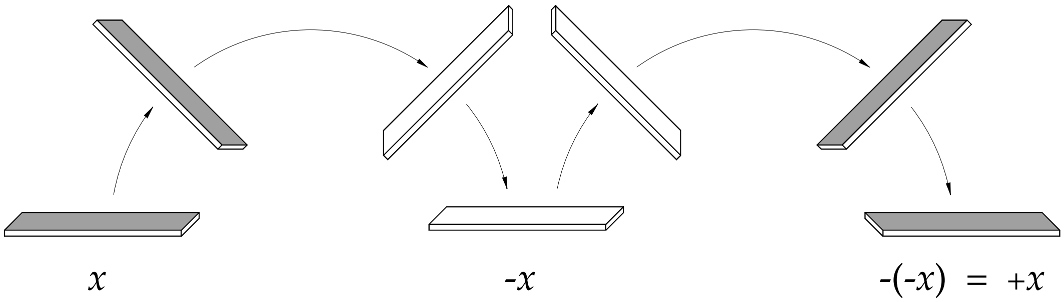
Just as we can find opposites of numbers by flipping the chips, the opposite of x can be shown as the x-bar flipped over:

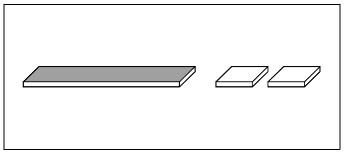


This $-x$ may be called **the opposite of x**, **the additive inverse of x**, or **negative x**. The last term—negative x—should be used with care. Because x may stand for either a positive or negative number, negative x stands for the opposite of x; it is not necessarily a negative number.

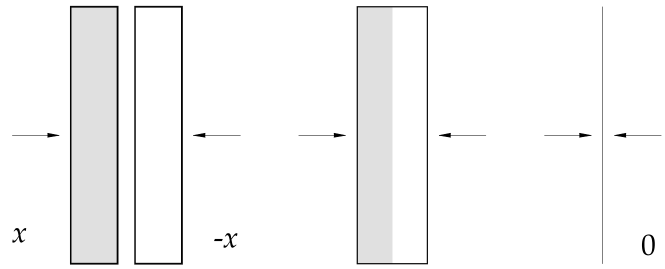
The opposite of the opposite is still the original amount:

$$^{-}(^{-}5) = {}^{+}5$$
$$-(-x) = +x$$





Finally, *x* and -*x* are additive inverses. When added, they "cancel" to zero in the same way that $^{+}3$ and $^{-}3$ cancel:



Evaluating Expressions

An expression that includes unknowns has no exact value because we do not know the value of the unknown. If we choose a value for the unknown, we can then **evaluate the expression** to determine its value.

To evaluate an expression, simply substitute the value of the unknown into the expression and then carry out the indicated operations:

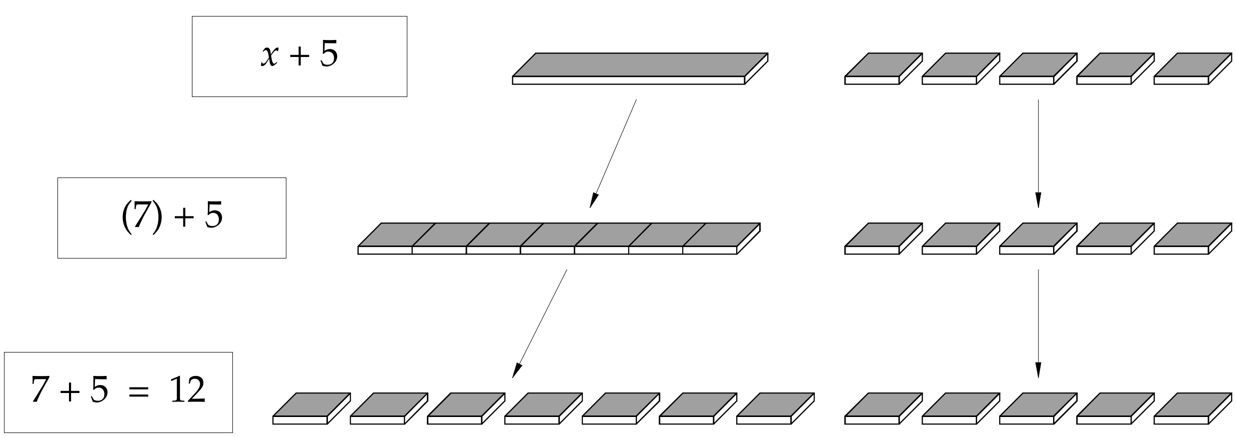
If $x = 7$, to evaluate $x + 5$:

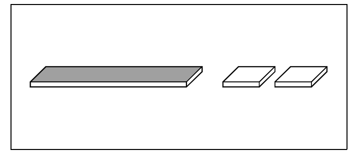
$$x + 5$$
$$(7) + 5$$
$$12$$

If $x = {}^{-}5$, to evaluate $4 - x$:

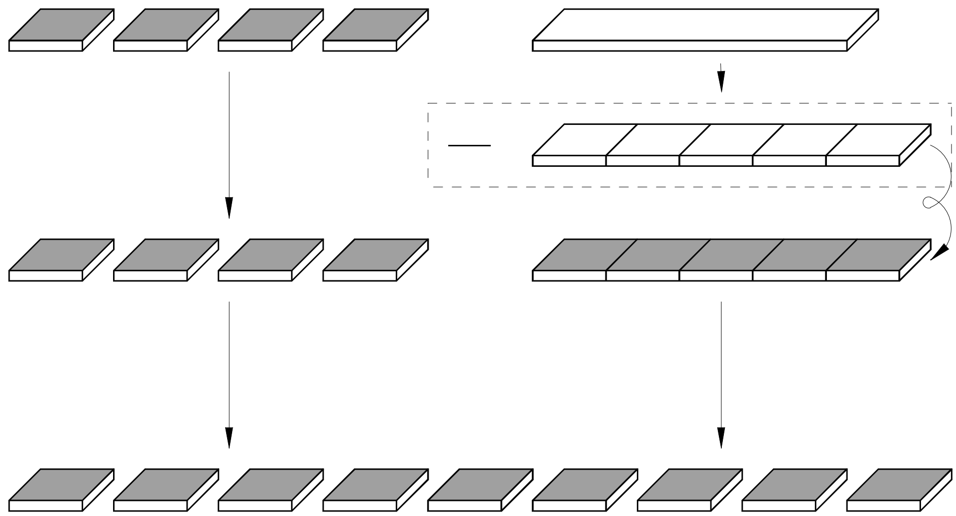
$$4 - x$$
$$4 - ({}^{-}5)$$
$$4 + 5$$
$$9$$

With the chips, we simply substitute the indicated number of units for the *x*-bar and then complete the count of unit chips. For $x + 5$, where $x = 7$:





For the second example, we start with a diagram of $4 - x$, then we figure out the value of $-x$ when x is $^{-}5$, and then we complete the count of unit chips:



The x-bar can stand for ***any*** *number—positive, negative, or zero.*

Exercises

Draw pictures of the following expressions. Evaluate each expression three different times—when x is 3, 0, and $^{-}2$:

1. $x + 6$

2. $2 + -x$

3. $x + 5 + x + x$

4. $^{-}5 + x + 5 + -x$

5. $3 + x + 5 + (-x) - x - 1$

6. $x + x + x - 5$

7. $x + x + x + x + x - 3$

8. $4 + x$

9. $x - x$

10. 5

11. $x + 3 + (^{-}3)$

12. $x - 3 - x - 2 - x - 1$

13. $3 - x$

14. $-x + 3$

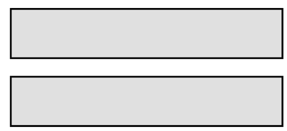
15. $x + (^{-}3)$

Section 2
Multiples of *x*

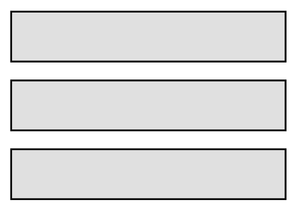
More than One *x*

If x is a certain quantity, the idea of several x's is a natural extension of our idea of one unknown:

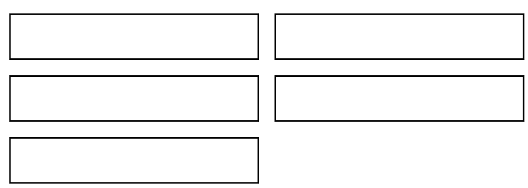
$2x$



$3x$



$^{-}5x$



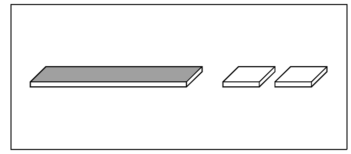
An expression like $\mathbf{2x}$ also can be thought of as a multiplication problem:

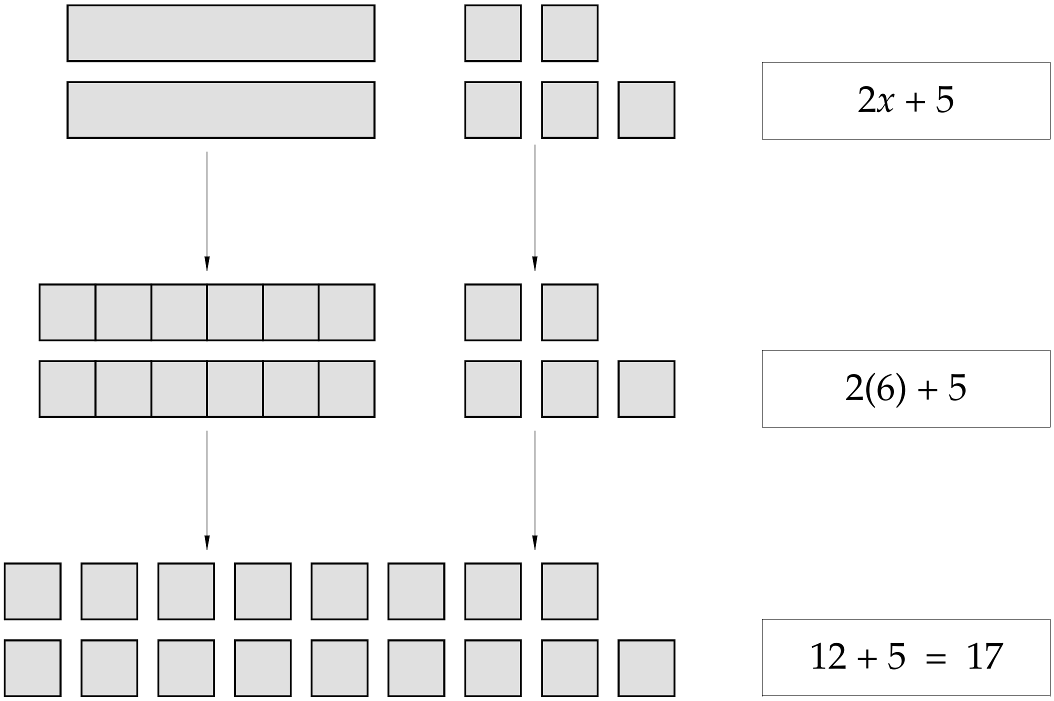
$$2x \;=\; 2 \cdot x \;=\; x + x$$

As we can see, we can call this expression "two x" or " two times x" and the meaning is still the same. Expressions such as $^{-}5x$ will be shown as 5 negative x-bars. In later chapters, we will see that the idea of $(^{-}5)\cdot(x)$ is also appropriate.

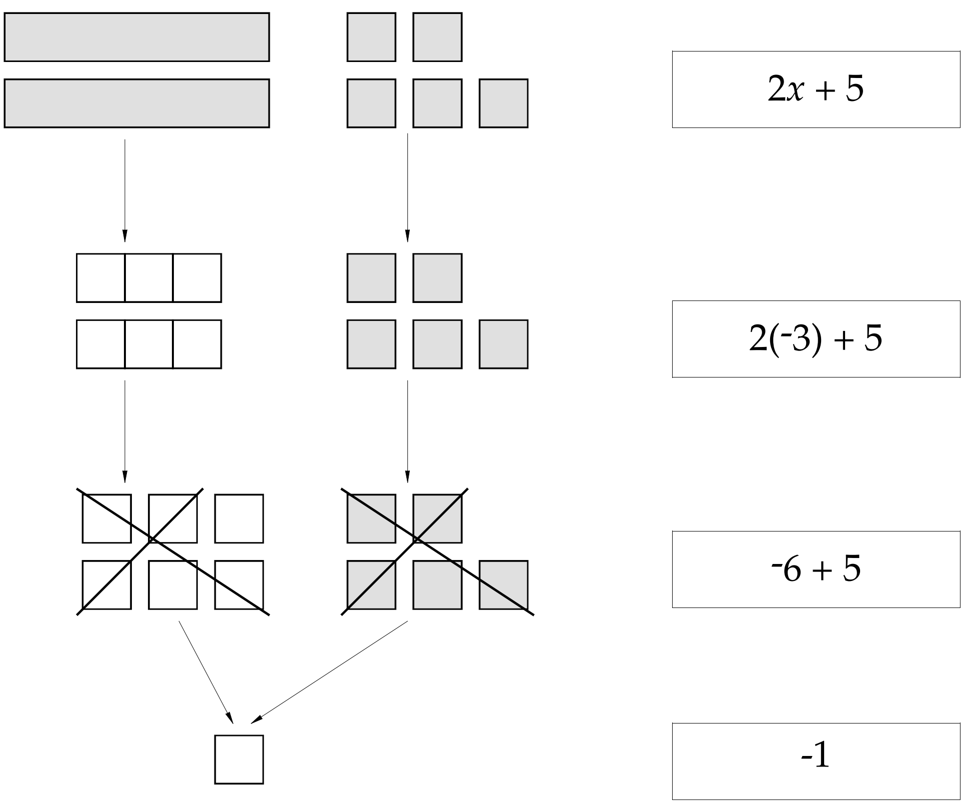
Evaluating Expressions with Multiple *x*'s

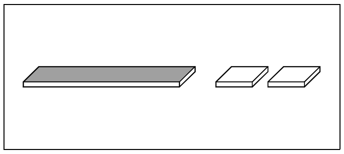
To evaluate an expression like those shown above, we still substitute a certain quantity for x, but we must be careful to carry out the indicated multiplications where a number is multiplied times x. For example, evaluate $2x + 5$, where $x = 6$:



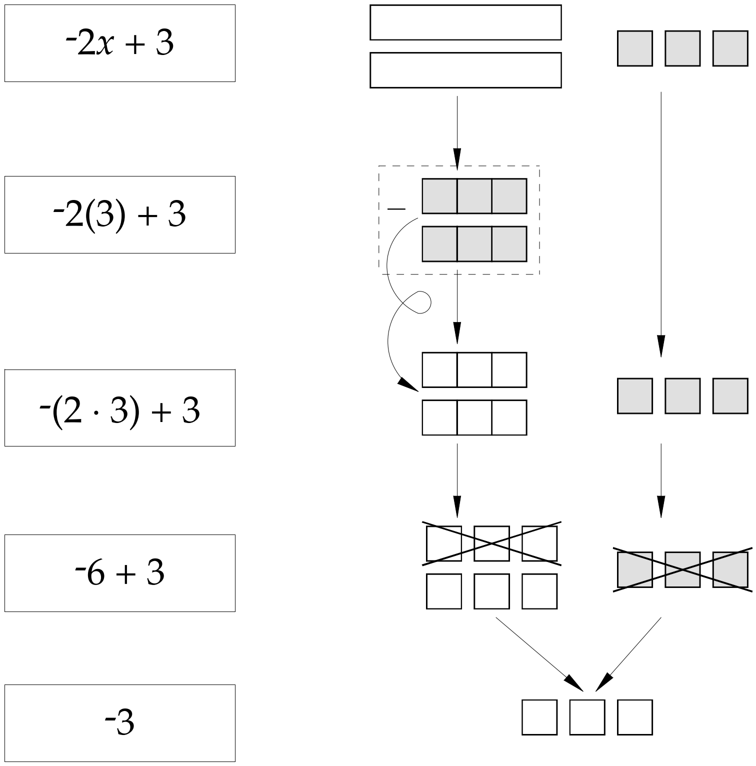


Repeating this example for a different value, where $x = {}^{-}3$:

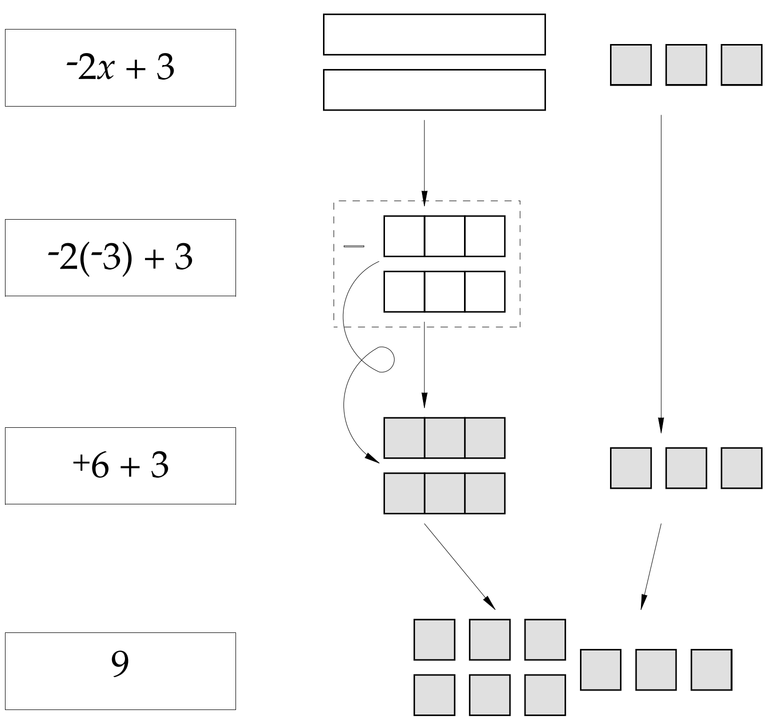




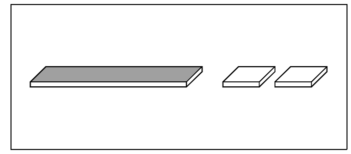
If the expression contains negative x's, we must first substitute the correct value for x; then we flip over the substituted chips to show the opposite of x. For example, to evaluate $^-2x + 3$, where x is 3:



Here is a second evaluation of $^-2x + 3$, where this time x is $^-3$:



Exercises



Use chips to represent the following expressions. Evaluate each expression four different times for $x = 1$, $x = 5$, $x = 0$, and $x = {}^{-}1$.

1. $3x - 15$

2. ${}^{-}3x - 12$

3. $2x + 5 + {}^{-}3x$

4. ${}^{-}3x + 2 + x + ({}^{-}3)$

5. $225x + 1 + {}^{-}225x$ (Do you need the chips?)

6. $5 - x$

7. $2 - 3x$

8. $0 - x + 16$

9. $3x - 2x + 6 - 2x$

10. $2 - ({}^{-}3x)$

11. $2x - 3 + x - 5$

12. ${}^{-}4 - x + 2 + 3x$

13. $5x - 2 + x$

14. $3 + 2x - 1 - x$

15. ${}^{-}({}^{-}2) + 3x + 5$

16. $x - 5 - 3x + 1$

17. ${}^{-}({}^{-}4x) + 3 + x$

18. ${}^{-}3 - 2x - ({}^{-}5)$

19. $x + x + 3x - 5$

20. $2 - 3x - 4x + 1$

Section 3
Combining Similar Terms

Combining Terms

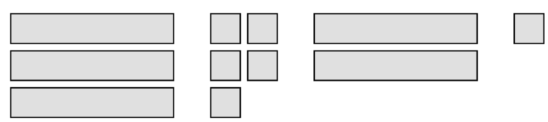
Each group of similar chips in an expression is called a **term**:

Expressions	Terms
$3x + 5 + 2x$	$3x$, 5, $2x$
$17 - 2x$	17, $^{-}2x$

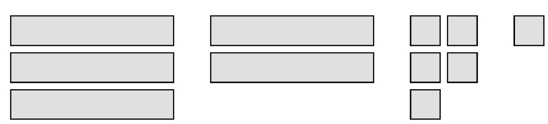
Notice that the **2** in **2*x*** is not a term.

Before we evaluate or use an expression, it is usually best to combine similar terms:

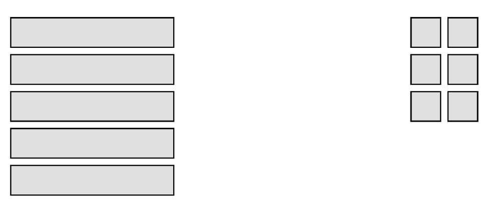
$3x + 5 + 2x + 1$



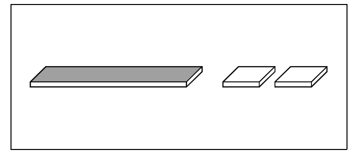
$3x + 2x + 5 + 1$



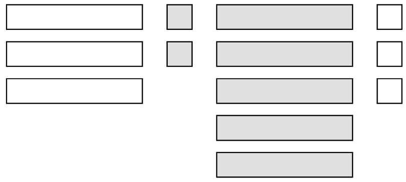
$5x + 6$



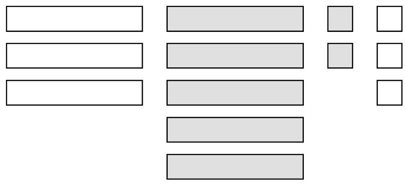
The process of combining similar terms will save time when evaluating expressions and will also be helpful in techniques presented in future chapters.



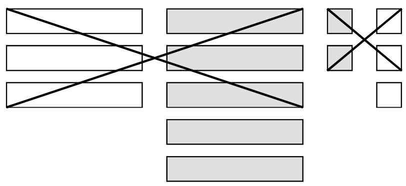
Positive x-bars and negative x-bars are combined in the same way as positive and negative chips:

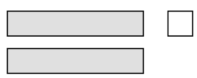


$^{-}3x + 2 + 5x - 3$



$-3x + 5x + 2 - 3$



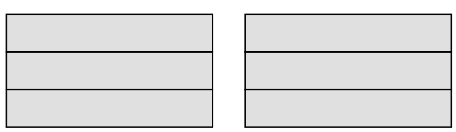


$2x - 1$

Multiplying Numbers and *x*'s

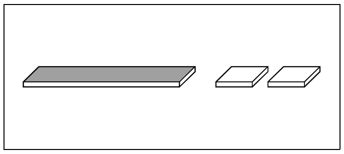
What is the meaning of:

$$2 \cdot 3x \ ?$$



Using symbols, we can write:

$$\begin{aligned} 2 \cdot 3x &= 2 \cdot (3 \cdot x) \\ &= 2 \cdot 3 \cdot x \\ &= (2 \cdot 3) \cdot x \\ &= 6 \cdot x \\ &= 6x \end{aligned}$$



If this seems too formal, think of the answer as 2 groups of $3x$. This is $3x + 3x$ or $6x$. Here are other examples:

$$\begin{aligned} 6x \cdot 5 &= (6 \cdot x) \cdot 5 \\ &= (x \cdot 6) \cdot 5 \\ &= x \cdot 6 \cdot 5 \\ &= x \cdot 30 \\ &= 30 \cdot x \\ &= 30x \end{aligned}$$

$$\begin{aligned} 6x \cdot (^-5) &= (6 \cdot x) \cdot (^-5) \\ &= (x \cdot 6) \cdot (^-5) \\ &= x \cdot 6 \cdot (^-5) \\ &= x \cdot [6 \cdot (^-5)] \\ &= x \cdot (^-30) \\ &= {}^-30 \cdot x \\ &= {}^-30x \end{aligned}$$

Common Errors

Most of the errors made by students can be avoided by paying attention to what the symbols *mean*. For example, consider the following errors made while combining similar terms:

$3x - x = 3$ (*Not* true)

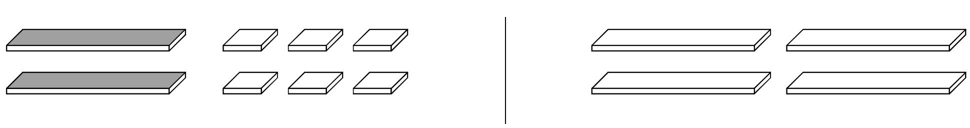
$3x + 6 = 9$ (*Not* true)

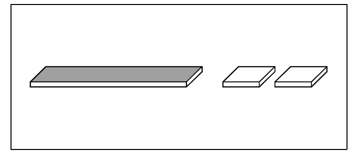
$3x + 6 = 9x$ (*Not* true)

$2x - 6 = {}^-4x$ (*Not* true)

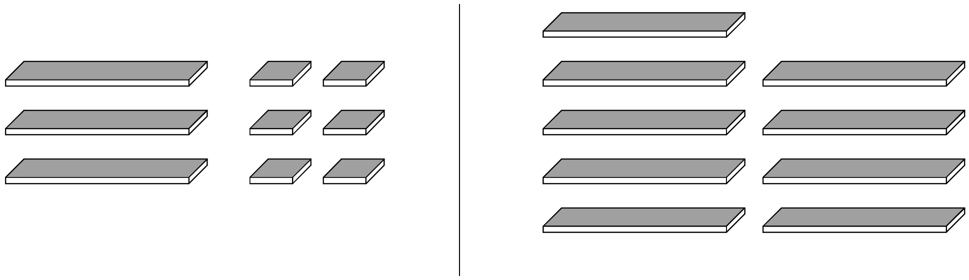
These errors usually occur when students are attempting to manipulate symbols by memorizing rules. When the chips are used, this type of mistake is much more obvious:

$2x - 6$ *is not* ${}^-4x$





$3x + 6$ *is not* $9x$



Exercises

Identify the terms in these expressions:

1. $3x - x + 5$

2. 0

3. $4x + 1 + 1$

4. $1 - 2 + x$

Simplify these expressions by combining similar terms:

5. $3x + (2x)(5) + 1$

6. $4x + 3 + (3)(2x) + 2$

7. $4x + 1 + 6x + 3 + x$

8. $^{-}4x + 3x + 1$

9. $2 + {}^{-}3x + 5x + {}^{-}6$

Evaluate these expressions *before* combining similar terms and *after* combining similar terms. Are the results the same? Do each problem with $x = 4$ and with $x = {}^{-}1$.

10. $3x + 2x + 1$

11. $4x + 3 + 2x + 2$

12. $4x + 1 + 6x + 3 + x$

13. $^{-}4x + 3x + 1$

14. $2 + {}^{-}3x + 5x + {}^{-}6$

Section 4
Expressions and Parentheses

Using Parentheses

Parentheses have the same meaning with unknowns as they do with exact numbers. You will remember that with exact numbers, parentheses indicated an operation or group of operations that were to be done first, before any other operations were done outside of the parentheses.

With unknowns, parentheses still indicate a group of terms that are together, but we cannot always complete the operations indicated because we do not know the value of the unknown term. Three examples are:

$$3(x + 5)$$

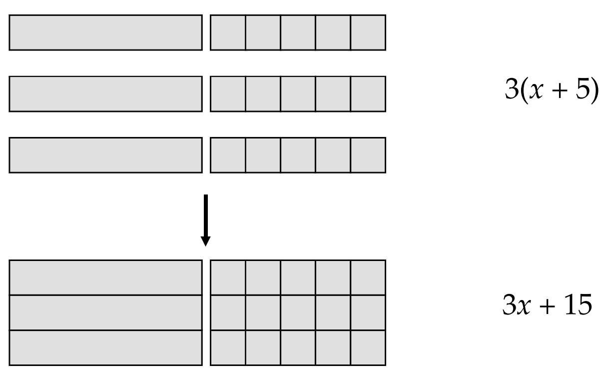
$$5 + 2(x - 3)$$

$$6 + 2(3x + 4) + x$$

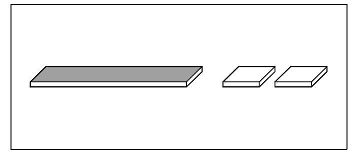
As we can see, the operations inside the parentheses cannot be immediately completed. *The symbols inside the parentheses still represent a group*. We can use the distributive property to finish the multiplication; then we combine similar terms. Here are the same examples worked out:

Example 1:

$$3(x + 5) = 3(x) + 3(5)$$
$$= 3x + 15$$



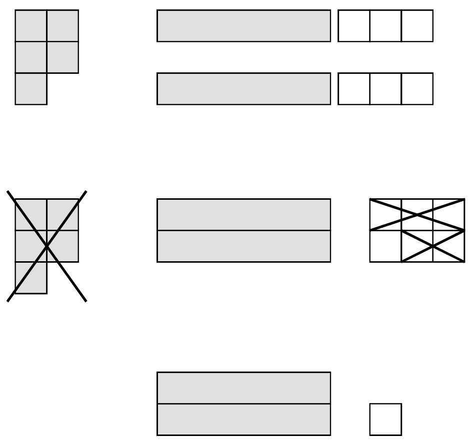
The distributive property shown in this diagram states that three groups of $(x + 5)$ is the same as $3x$ and 15. Three times the whole quantity is the same as three of each term.



Example 2: $5 + 2(x - 3) = 5 + 2(x + {}^-3)$

$= 5 + 2x + {}^-6$

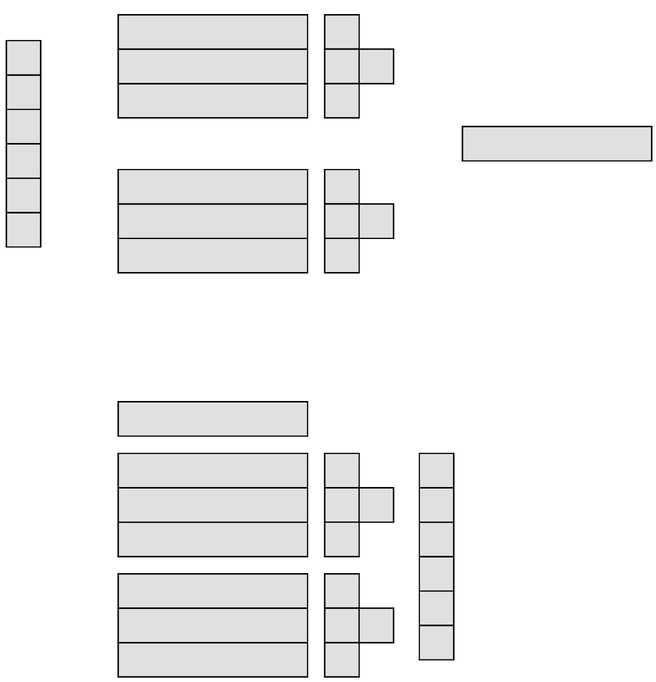
$= 2x - 1$

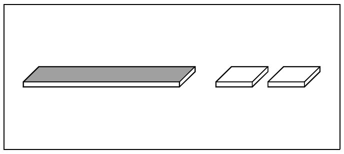


Example 3: $6 + 2(3x + 4) + x = 6 + 2(3x + 4) + x$

$= 6 + 6x + 8 + x$

$= 7x + 14$





Negative Signs and Multiplication

Consider

$$3 - 6(x + 3)$$

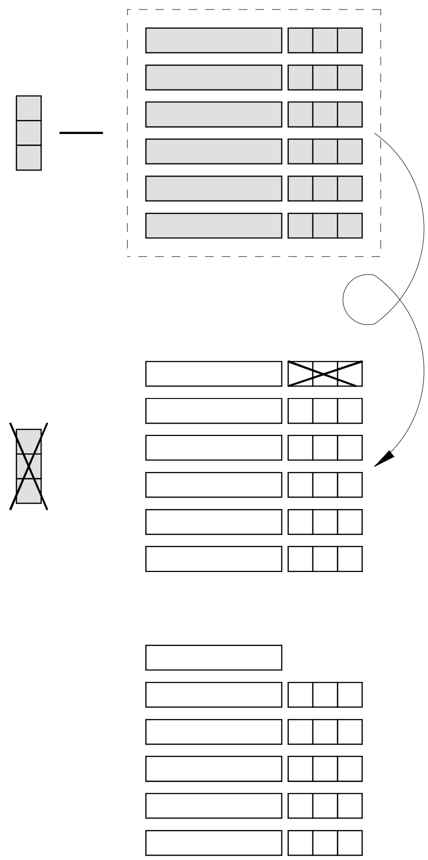
The best way to work with this expression is to rewrite the subtraction as an addition:

$$\begin{aligned} 3 - 6(x + 3) &= 3 + {}^{-}6(x + 3) \\ &= 3 + {}^{-}6x + {}^{-}18 \\ &= {}^{-}6x + {}^{-}15 \end{aligned}$$

$3 - 6(x + 3)$

$3 + {}^{-}6x + {}^{-}18$

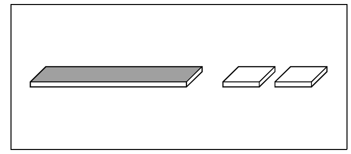
${}^{-}6x + {}^{-}15$



This technique is especially helpful where multiple negative signs are present:

$$\begin{aligned} 6 - 2(x - 3) &= 6 + ({}^{-}2)(x + {}^{-}3) \\ &= 6 + {}^{-}2x + 6 \\ &= {}^{-}2x + 12 \end{aligned}$$

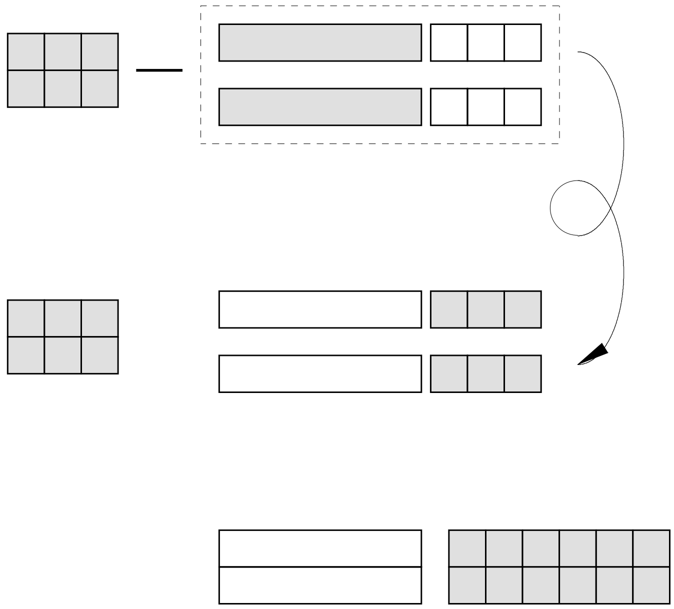
Here is how this process is shown with the chips:



$6 - 2(x - 3)$

$6 + {}^-2x + 6$

${}^-2x + 12$



Summary

To simplify expressions containing parentheses:

- **Rewrite subtractions as additions.**
- **Carry out multiplications using the distributive property.**
- **Combine similar terms.**

Exercises

Simplify the following expressions:

1. $5(x - 3) + 2(3x + 1)$
2. $3 - (x + 5)$
3. $3 - 2(x - 5)$
4. $3 - 2(-x - 5)$
5. $-x - 3x + 4(5 + x)$

Simplify the expressions, then evaluate:

6.	$6x - 2(x + 4)$	$(x = 1)$
7.	$6 - (x - 5) - 3x$	$(x = {}^-1)$
8.	$x + 0(196x - 235)$	$(x = 256)$
9.	$2(3x + 2) - 5(3 + x) + 11$	$(x = {}^-17)$

Section 5
Expressions Containing Fractions

Fractions and Unknowns

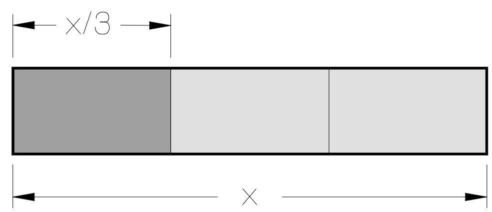
When we wish to represent part of an unknown, we use the same fractional notation that we use with everyday numbers:

$$\frac{1}{3} \text{ of } 6 \text{ means } \frac{1}{3} \cdot 6$$

$$\frac{1}{2} \text{ of } 7 \text{ means } \frac{1}{2} \cdot 7$$

$$\text{So } \frac{1}{2} \text{ of } x \text{ means } \frac{1}{2} \cdot x = \frac{1}{2} \cdot \frac{x}{1} = \frac{1 \cdot x}{2} = \frac{x}{2}$$

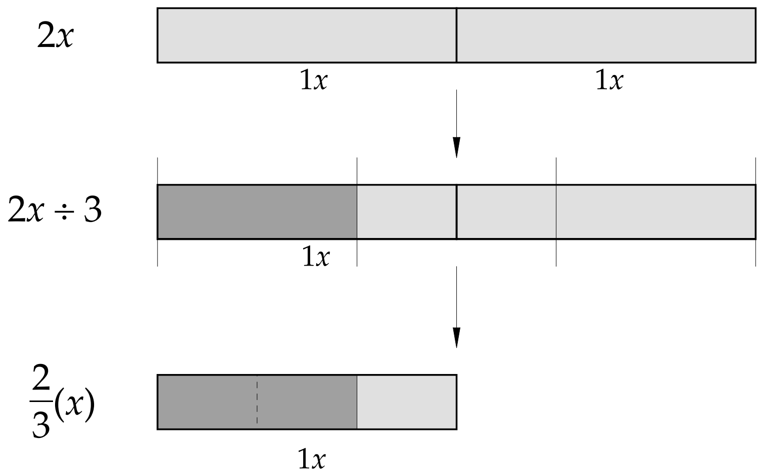
One-third of x would look like this:

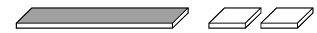


There are often several ways to show these fractional unknowns:

Meaning	Alternative Notations		
³⁄₅ of x	$\frac{3}{5} \cdot x$	$\frac{3}{5}x$	$\frac{3x}{5}$
²⁄₃ of x	$\frac{2}{3} \cdot x$	$\frac{2}{3}x$	$\frac{2x}{3}$
–²⁄₃ of x	$-\frac{2}{3} \cdot x$	$-\frac{2}{3}x$	$\frac{-2x}{3}$

If the alternatives seem to represent different quantities, here is a demonstration of the reasons why $^{(2x)}\!/_3$ is equal to $^2\!/_3(x)$:





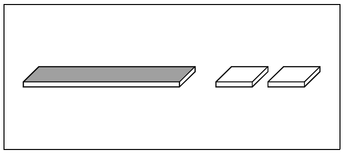
Simplifying Expressions with Fractional Unknowns

Fractional unknowns are simplified in the same way as numbers. When adding, find common denominators and then combine:

$$\begin{aligned}
\frac{2x}{3} + \frac{3x}{4} &= \frac{2x}{3} \cdot \frac{4}{4} + \frac{3x}{4} \cdot \frac{3}{3} \\
&= \frac{2x \cdot 4}{3 \cdot 4} + \frac{3x \cdot 3}{4 \cdot 3} \\
&= \frac{8x}{12} + \frac{9x}{12} \\
&= \frac{8x + 9x}{12} \\
&= \frac{17x}{12}
\end{aligned}$$

When multiplying, use the same technique that we used with regular fractions:

$$\begin{aligned}
\frac{2x}{3} \cdot \frac{3}{4} &= \frac{2x \cdot 3}{3 \cdot 4} \\
&= \frac{6x}{12} \\
&= \frac{6}{12} \cdot x \\
&= \frac{1}{2}x \\
&= \frac{x}{2}
\end{aligned}$$



Exercises

Simplify these expressions. Find common denominators as needed.

1. $\frac{x}{2}+\frac{x}{4}$

2. $\frac{x}{2}-\frac{3x}{4}$

3. $\frac{1}{2}\cdot\frac{x}{3}\cdot\frac{1}{4}$

4. $\frac{1}{2}\div\frac{3}{x}$

5. $\frac{1}{2}(2x + 4)$

6. $\frac{1}{2}\left(2x+\frac{1}{2}\right)$

7. $\frac{1}{2}(6x)$

8. $\frac{2}{3}\cdot\frac{3x}{2}$

9. $\frac{3}{4}\cdot\frac{4x}{3}$

10. $12\left(\frac{x}{2}+\frac{x}{3}+\frac{5x}{6}\right)$

11. $\frac{2x}{3}+\frac{x}{5}$

12. $\frac{3x}{2}+\frac{2x}{3}$

13. $\frac{5x}{2}-x$

14. $\frac{5x}{2}-\frac{x}{3}$

15. $\frac{2}{3}\left(\frac{x}{6}\right)$

16. $\frac{3}{5}\left(\frac{2x}{9}\right)$

17. $\frac{1}{3}(6x-5)$

18. $\frac{3}{5}(10x+5)$

Section 6
Properties of Expressions

Properties of Numbers

All of the associative, commutative, and distributive properties described in the previous chapter are true for expressions as well as for numbers. Because the unknowns represent numbers, there is usually no need to state separate properties for expressions.

Most of the following ideas are so intuitive that we have already used them without noticing anything new. It may be helpful, however, to restate some of the properties in a more formal manner.

Commutative and Associative Properties

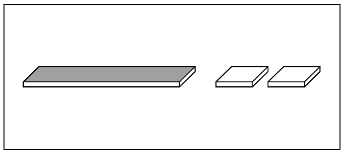
These properties concern the *order* and *grouping* of numbers or terms and are most useful in combining similar terms. For example, consider this illustration from a previous section:

$6x \cdot (^-5)$	$= (6 \cdot x) \cdot (^-5)$	
	$= (x \cdot 6) \cdot (^-5)$	Commutative Prop of Mult
	$= x \cdot 6 \cdot (^-5)$	
	$= x \cdot [6 \cdot (^-5)]$	Assoc. Prop. of Mult
	$= x \cdot (^-30)$	
	$= {^-30} \cdot x$	Comm. Prop. of Mult.
	$= {^-30}x$	

Because the parts of an expression that are being added or multiplied can be rearranged in different orders and groupings, we can more easily combine similar terms.

Commutative and Associative Properties

When *adding* or *multiplying* terms in an expression, the *order* or *grouping* of terms may be changed without affecting the value of the expression.



Distributive Properties

We have easily decided that

$$3x + 4x = 7x$$

More formally, this can be justified by the distributive property:

$$\begin{aligned} 3x + 4x &= 3(x) + 4(x) \\ &= (3 + 4)(x) \\ &= (7)(x) = 7x \end{aligned}$$

We have also been using this property in subtraction problems:

$$\begin{aligned} 3x - 4x &= (3 - 4)x \\ &= ({}^-1)x = -x \end{aligned}$$

We can use the idea in division problems and with fractions:

$$\begin{aligned} \frac{6x + 4}{2} &= \frac{6x}{2} + \frac{4}{2} \\ &= 3x + 2 \end{aligned}$$

$$\begin{aligned} \frac{6x - 4}{2} &= \frac{6x}{2} - \frac{4}{2} \\ &= 3x - 2 \end{aligned}$$

Expressions have the same properties as numbers because expressions *represent* numbers.

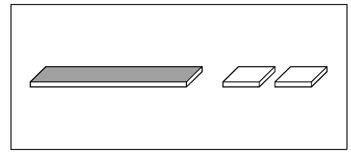
Distributive Properties

$$\mathbf{3x + 4x = (3 + 4)x = 7x}$$
$$\mathbf{3x - 4x = (3 - 4)x = {}^-1x}$$
$$\mathbf{\frac{6x - 4}{2} = \frac{6x}{2} - \frac{4}{2}}$$

or for any numbers a, b, and c, (c not zero)

$$\mathbf{ax + bx = (a + b)x}$$
$$\mathbf{ax - bx = (a - b)x}$$
$$\mathbf{\frac{ax - b}{c} = \frac{ax}{c} - \frac{b}{c}}$$

Properties of One and Negative One



For multiplication, one is the identity element. One times any number is the original number. With expressions, the property is of course the same:

$$1 \cdot 3x = 3x$$

$$1 \cdot ({}^{-}5x) = {}^{-}5x$$

$$(7x + 2) \cdot 1 = 7x + 2$$

$$1 \cdot x = 1x = x$$

We have also seen from the POSITIVE AND NEGATIVE NUMBERS chapter that multiplying ${}^{-}1$ times any number results in the opposite of that number:

$$\begin{aligned} {}^{-}1 \cdot 3x &= {}^{-}1 \cdot 3 \cdot x \\ &= ({}^{-}1 \cdot 3)x \\ &= ({}^{-}3)x \\ &= {}^{-}3x \end{aligned}$$

Finally, it is most important to understand that multiplying ${}^{-}1$ times x is $-x$:

$${}^{-}1 \cdot x = -x$$

Properties of One

$$\mathbf{(1)(3x) = 3x}$$
$$\mathbf{(1)(x) = x}$$
$$\mathbf{({}^{-}1)(x) = -x}$$

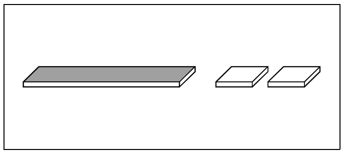
Properties of Zero

When zero is multiplied times an unknown quantity, we are taking an unknown number of zeros. The result is always zero:

$$0 \cdot 3x = 0$$

$${}^{-}3x \cdot 0 = 0$$

$$(3x + 6) \cdot 0 = 0$$



Adding zero to any expression does not change the value of the expression:

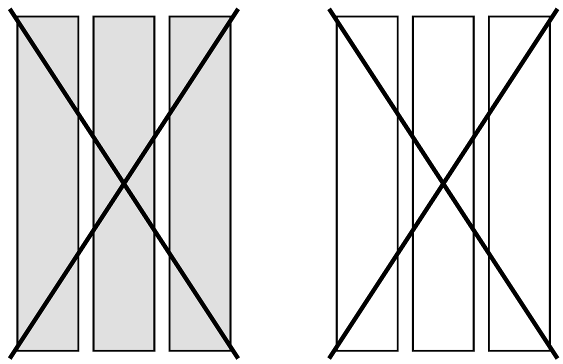
$$0 + 3x = 3x$$

$$^{-}3x + 0 = {}^{-}3x$$

$$0 + (4x - 73) = 4x - 73$$

If we add opposites together, they will cancel to zero:

$$3x + {}^{-}3x = 0$$



A more formal proof of this fact uses the distributive property:

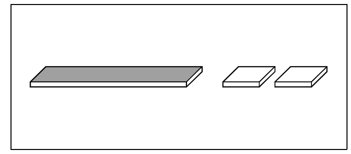
$$\begin{aligned} 3x + {}^{-}3x &= (3)x + ({}^{-}3)x \\ &= (3 + {}^{-}3)x \\ &= (0)x \\ &= 0 \end{aligned}$$

The opposite of any number of x-bars is the same number of -x bars.

Properties of Zero

$$\mathbf{0 + x = x}$$
$$\mathbf{(0)(x) = 0}$$
$$\mathbf{ax + {}^{-}ax = 0}$$

Order of Operations With Multiple Parentheses



When expressions have multiple levels of parentheses then, as before, we begin simplifying starting from the innermost parentheses and working our way out. For example, let's simplify

$$7x + 3[2x - 5(x - 3 - 2x)]$$

$7x + 3[2x - 5(-x - 3)]$ — First we *combine like terms* inside the innermost

rentheses.

$7x + 3[2x + 5x + 15]$ — Then we *multiply through* the innermost parentheses. Now the inner parentheses are gone.

$7x + 3(7x + 15)$ — Next we combine like terms inside the remaining parentheses.

$7x + 21x + 45$ — Then we multiply through the remaining parentheses.

$28x + 45$ — When all parentheses are gone, combine like terms (if necessary) in the remaining expression.

As with other expressions having unknowns and numbers, the final result usually has two terms (one with a letter and one without) which cannot be combined.

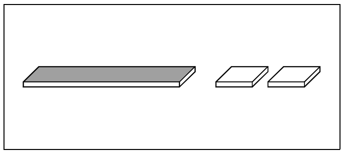
Exercises

Simplify the following expressions. Justify each step by referring to the appropriate property.

Example: $2(3x + 4) - 6x$

Solution:

$2(3x + 4) - 6x$	$= 2(3x) + 2(4) - 6x$	Distributive
	$= 6x + 8 - 6x$	
	$= 8 + 6x - 6x$	Commutative
	$= 8 + 0$	Inverses
	$= 8$	



1. $5(x + 6) + x$
2. $2 - (3 + 2x)$
3. $-x(3) + 3x + 22 + x$
4. $9y - 3y + (6)(-y)$
5. $\frac{1}{3}x + \frac{2}{3}x$
6. $2 - 5(3 + x)$
7. $^{-}2 - 6(3 - x)$
8. $4x + 2x + {}^{-}2x + {}^{-}4x$
9. $\frac{6x + 2}{2} - 1$
10. $\frac{^{-}3x - 6}{^{-}3} + x$
11. $\frac{8x}{2} + \frac{9x}{3}$
12. $\frac{12x + 6}{6} + x + 1$
13. $(2 - 1)(3x)$
14. $-x + 4(x - 3)$
15. $\frac{4x + 4}{4}$
16. $\frac{6x - 6}{6}$
17. $\frac{6x + 1}{6}$
18. $(^{-}1)(5x)$
19. $(5)(7x)$
20. $(7x)(5)$
21. $(^{-}7x)(^{-}5)$
22. $(7x)(^{-}5)$
23. $(0)(3)(12)(6x)$
24. $24(3 - 3)(6x)$
25. $\frac{(6x)(0)}{6}$
26. $5[3x - 2(x + 7)]$
27. $4[3(2 - 3x) + 6x]$
28. $7 - [2x + 5(6 - 3x)]$
29. $^{-}3x + 4[6 + 3(2x - 9)]$
30. $4x + 2[5 - 2(3x - 7)]$