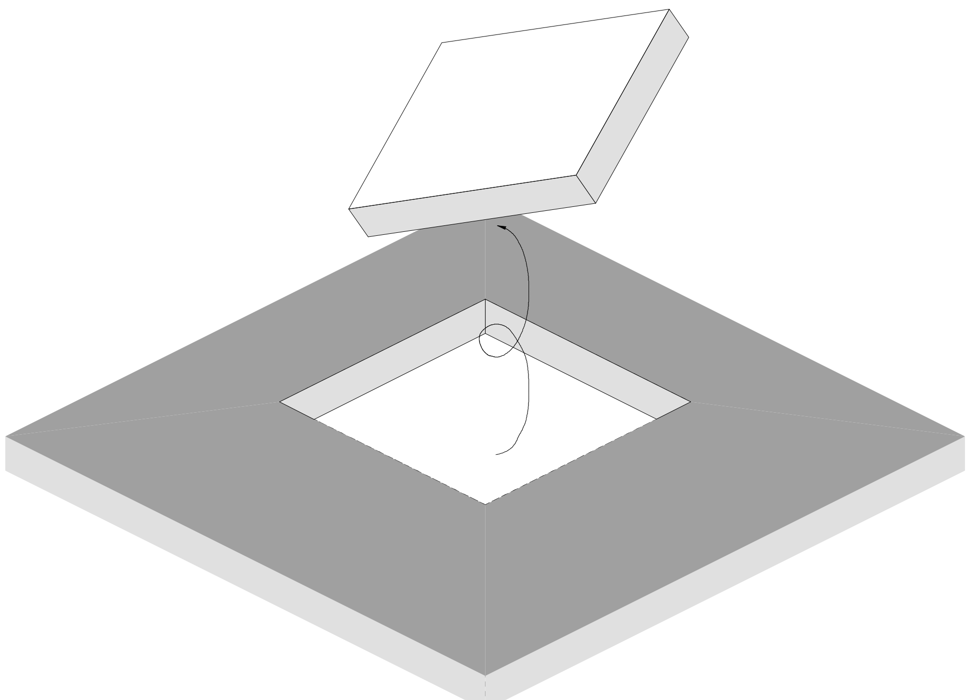
FLiP-CHIP™ ESSENTIALS

INTRODUCTORY LEVEL

Frank Edge — Steven Kant



Second Edition

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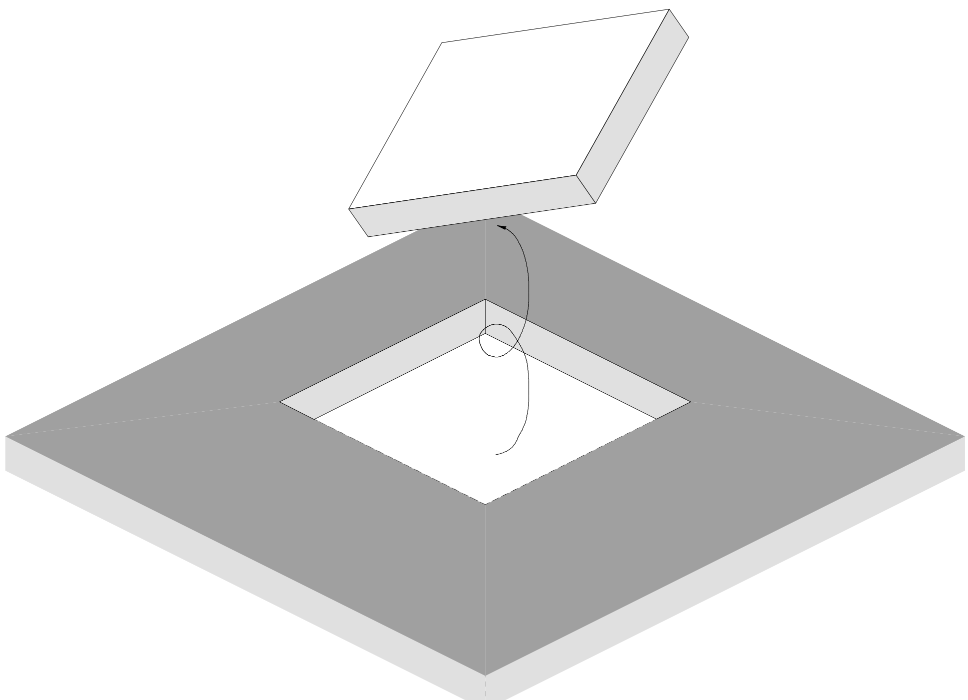

Second Edition

The authors would appreciate corrections and suggestions. For more information, write to Flip-Chip Enterprises, 428 Rogers Street NW, Olympia WA, 98502 or call (360) 943-4535.

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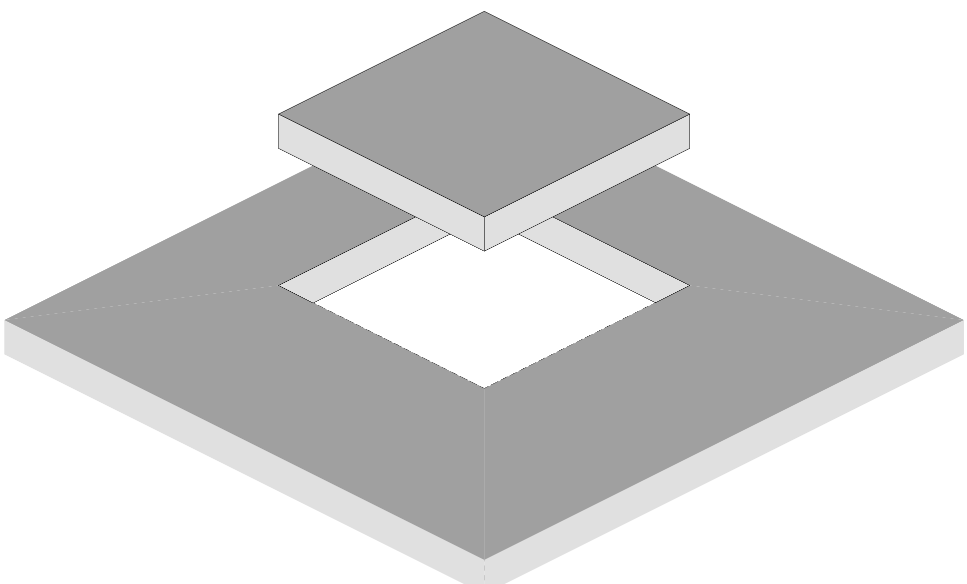
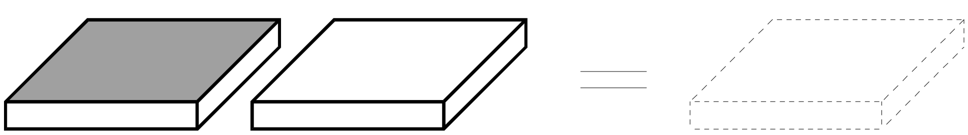
Positive and Negative Numbers

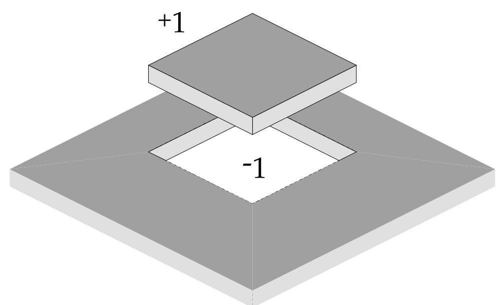




Section 1
Positive and Negative Numbers

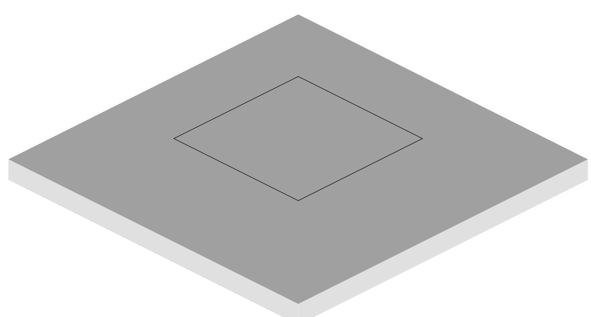
The Meaning of Positive and Negative Numbers

Imagine a slab with a square section removed:

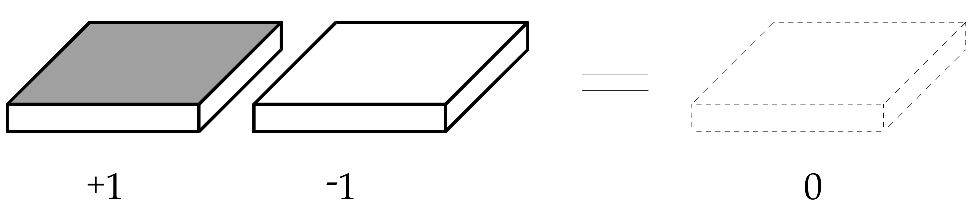


Positive one ($^{+}1$) is the square chip that is cut out of the slab. Negative one ($^{-}1$) is the hole that it came out of.

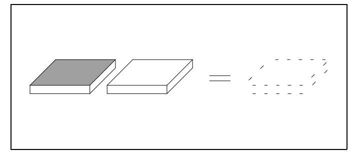
Add $^{+}1$ and $^{-}1$ back together and you fill in the hole; zero is your result:



For practical purposes, it is more convenient to use two chips of different colors to represent $^{+}1$ and $^{-}1$. When they are added together, they cancel each other out, leaving zero.



Signed Numbers and Flip-Chips™

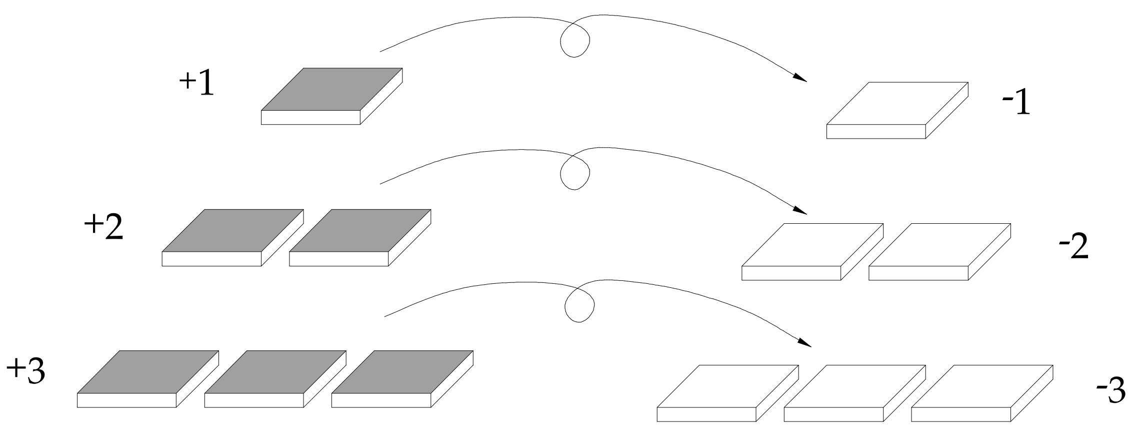


A number with a sign (+ or –) directly to its left (in front of the number when reading from left to right) is called a **signed number**. The positive (+) or negative (–) sign tells *what color* chips the number represents and the number tells *how many* of these chips are represented. Together, all of the positive and negative numbers are called **integers**.

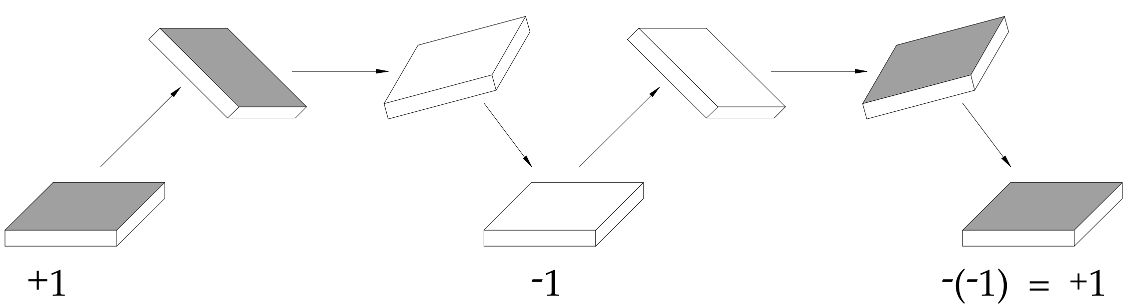
With a piece of material which has a different color on each side it is possible to make a **Flip-Chip**—a piece which represents $^{+}1$ with one side up, and $^{-}1$ with the other side up. *Flipping the chip changes the sign***!**

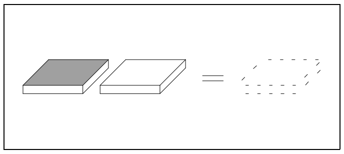
The chips we use are colored on one side and white on the other side, so we call the colored side **positive** or **plus** (+) and the white side **negative** or **minus** (–). This way we always know which side is which.

Flipping the chip changes the sign!



And a second negative sign flips the chip *again*!:





Double or Multiple Signs

A number may be shown having more than one sign in front (to the left) of it. These signs can be written in several ways; parentheses are often used to enclose the number and one sign:

$$^{+}(^{-}3)$$
$$^{-}(^{-}2)$$
$$^{+}(^{+}5)$$
$$^{+}(^{-}4)$$

Thinking of these numbers as chips, remember that each negative (–) sign in front of a number flips the chips one time, so two minus signs flip the chips twice, giving a positive (+) side up. We always begin with the colored (+) side up before we start flipping. Here is the result of four different combinations of signs:

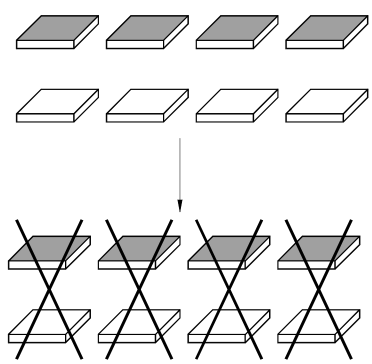
$$^{+}(^{+}3) = {}^{+}3$$
$$^{+}(^{-}3) = {}^{-}3$$
$$^{-}(^{+}3) = {}^{-}3$$
$$^{-}(^{-}3) = {}^{+}3$$

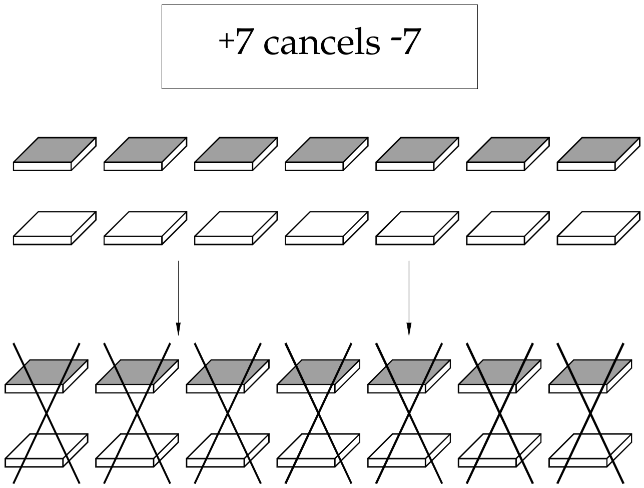
Each negative sign means to flip the chips once; each positive sign means to leave them alone. We always start with the colored (positive) side up.

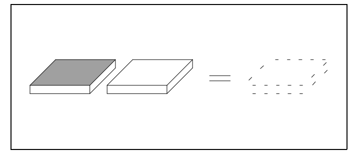
Cancelling of Positives and Negatives

The basic principle of grouping positive and negative chips together is that one positive chip grouped with one negative chip cancels to give zero. This means that if we put an equal number of positive and negative chips together, they will cancel to give zero:

$^{+}4$ cancels $^{-}4$

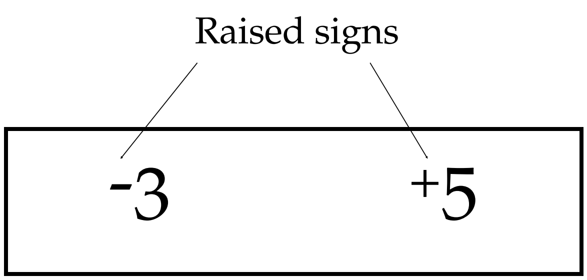




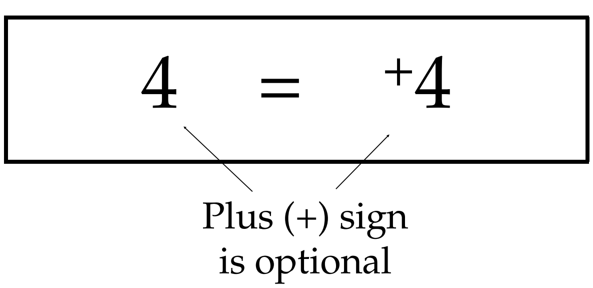


Symbols and Signs

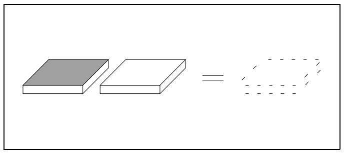
We have been using several symbols that may be unfamiliar. First we have been showing positive and negative numbers with small plus or minus signs that are on the left of the number and raised up slightly.



Positive numbers can be shown *with or without* the positive sign. The familiar number 4 and the new symbol $^{+}4$ have the same meaning:



Although a positive sign is optional, a negative number must always be shown with a minus sign so that we can tell that it is negative.

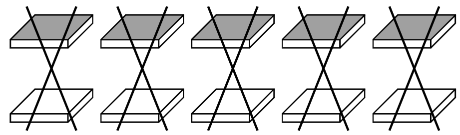


Exercises

Use the chips to illustrate the following results:

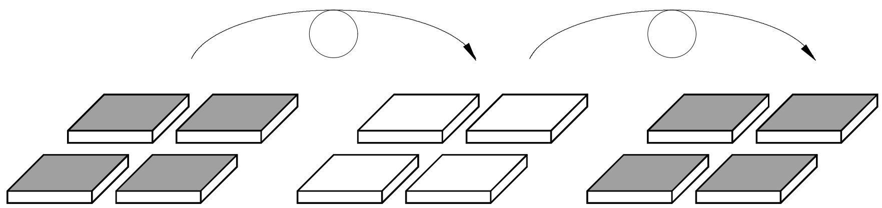
Example: 5 and $^{-}5$ cancel to 0

Solution:



Example: $^{-}(^{-}4) = {}^{+}4$

Solution:



Start with 4 — Flip to $^{-}4$ — Flip again, to $^{-}(^{-}4) = {}^{+}4$

1. $^{-}(^{-}7) = {}^{+}7$
2. $^{-}(^{+}3) = {}^{-}3$
3. $^{-}(^{-}11) =$
4. $^{-}(+3) =$
5. $^{+}(^{-}9) =$
6. $^{-}(^{-}10) =$
7. $^{-}(3) = {}^{-}3$
8. 3 and $^{-}3$ cancel to 0
9. 6 and $^{-}6$ cancel to 0
10. $^{-}6$ and $^{-}(^{-}6)$ cancel to 0
11. $^{-}(11)$ and $^{+}11$ cancel to 0
12. $^{-}(^{-}17) =$
13. $^{+}(^{-}0) =$
14. $^{-}(^{-}0) =$

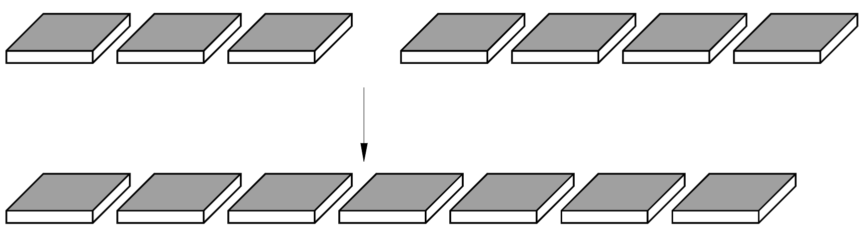
Section 2
Addition of Signed Numbers

The Meaning of Addition

In the past, adding two numbers meant that we took two amounts and combined them. Now that we have invented positive and negative numbers, addition will still have the same basic meaning, as long as we understand the idea that equal groups of positive and negative chips cancel each other out.

Adding Two Positives

If we are adding two positive numbers, we simply combine two groups of positive chips to give one larger group of all positive chips:



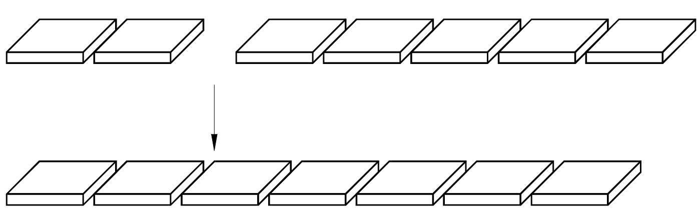
$(^{+}3) + (^{+}4) = 7$

Adding Negatives

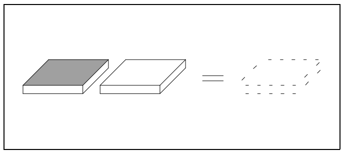
To add negative numbers, we combine the groups of negative chips. For example:

$$(^{-}2) + (^{-}5)$$

This expression means that we should take 2 negative chips and 5 negative chips and group them together. The result is clearly 7 negative chips:



$(^{-}2) + (^{-}5) = {}^{-}7$



As we can see from the last two examples, adding numbers with the same sign is very easy—we simply combine the chips and count the total number:

$$(^{+}6) + (^{+}3) \;=\; {}^{+}9$$

$$(^{+}12) + (^{+}3) \;=\; {}^{+}15$$

$$(^{-}3) + (^{-}5) \;=\; {}^{-}8$$

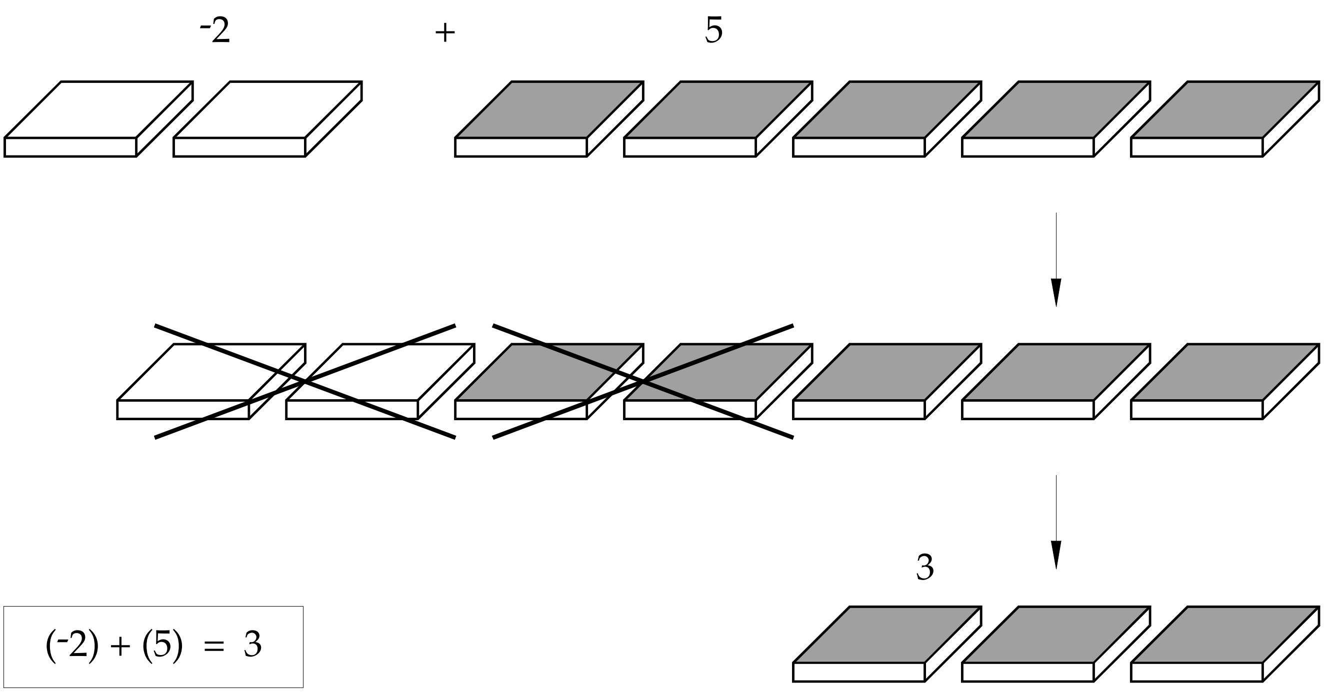
$$(^{-}6) + (^{-}4) \;=\; {}^{-}10$$

The parentheses shown above are not required but can be helpful. We use them to separate the number from the addition sign; if you leave them out, make sure to keep the negative signs raised and close to the numbers:

$${}^{-}6 + {}^{-}4 \;=\; {}^{-}10$$

Adding Negative and Positive Numbers

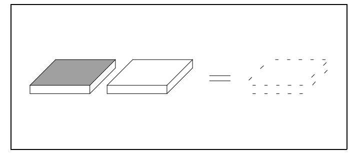
If we need to add a negative number and a positive number, we combine the two groups of chips and cancel out pairs of negatives and positives:

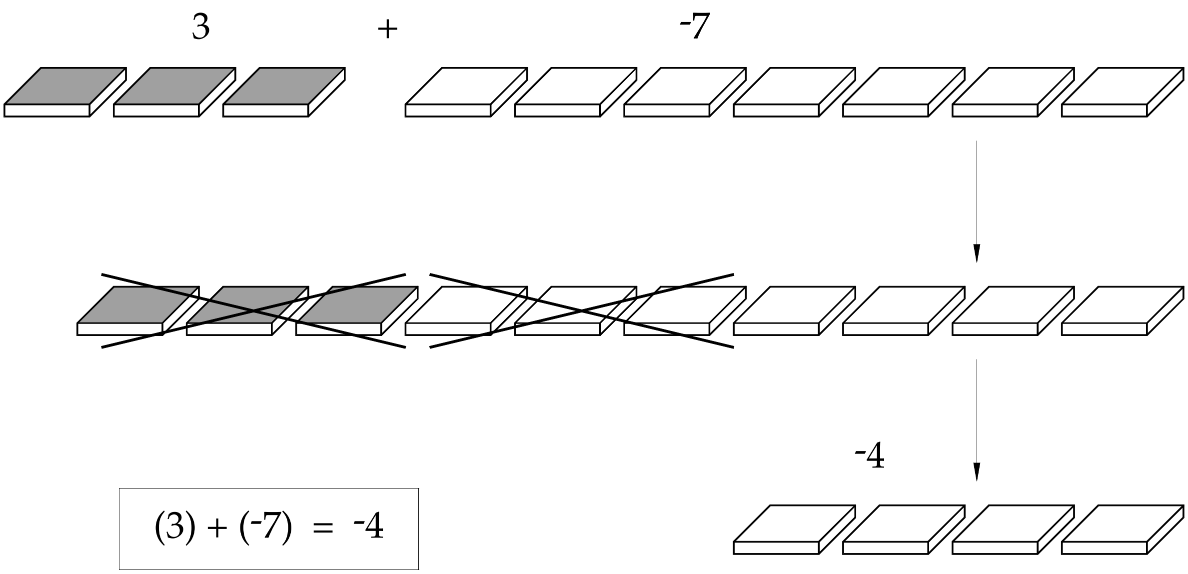


$$(^{-}2) + (5) \;=\; 3$$

Did you notice that there were more positives than negatives? Because of this, when the cancelling is done, we are left with positives.

Here is an example of adding a positive number and a negative number where there are more negative chips:





As you would expect, the positive chips cancelled out some of the negatives, but there are still negatives remaining.

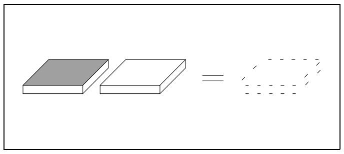
Summary

To add two numbers, we combine the chips, cancelling if we have a mixed group of positives and negatives:

- **Adding two positives—Combine the groups of chips for a total of more positives.**

- **Adding two negatives—Combine the groups of chips for a total of more negatives.**

- **Adding a positive and a negative—Combine the groups of chips and let positive and negative chips cancel out in pairs. The chips which remain will have the same color (sign) as the larger original group.**



Exercises

Use your chips to set up and solve the following addition problems:

1. $(^{-}5) + (^{+}5) =$

2. $(^{+}3) + (^{+}11) =$

3. $(^{-}5) + (^{-}1) =$

4. $(^{-}3) + (^{-}3) =$

5. $(^{-}1) + (^{-}1) =$

6. $(^{+}8) + (^{+}4) =$

7. $(^{-}4) + (^{-}3) =$

8. $(^{-}3) + (^{-}4) =$

9. $(^{-}6) + (^{-}7) =$

10. $(^{-}12) + (^{-}1) =$

11. $(^{-}7) + (^{+}6) =$

12. $(^{+}7) + (^{-}6) =$

13. $(^{-}11) + (^{+}2) =$

14. $11 + (^{-}2) =$

15. $4 + {}^{-}5 =$

16. ${}^{-}4 + 5 =$

17. $1 + (^{-}2) =$

18. ${}^{-}1 + (^{-}3) =$

19. ${}^{-}1 + 3 =$

20. ${}^{-}2 + {}^{-}3 =$

21. $7 + {}^{-}5 =$

22. ${}^{-}3 + {}^{-}5 =$

23. ${}^{-}3 + 5 =$

24. $6 + 2 =$

25. $6 + (^{-}2) =$

26. ${}^{-}6 + 2 =$

27. ${}^{-}6 + (^{-}2) =$

28. $4 + 5 =$

29. $4 + {}^{-}5 =$

30. ${}^{-}4 + 5 =$

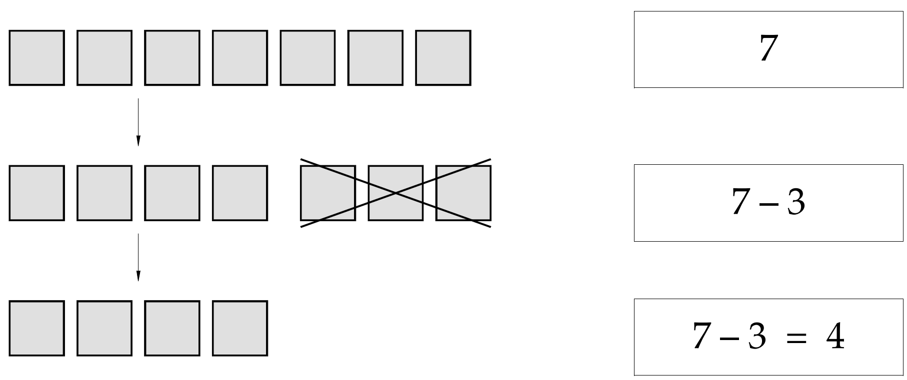
Section 3
Subtraction of Signed Numbers

The Meaning of Subtraction

We were able to easily extend our old idea of addition to cover signed numbers, but we will have to do a little more work to invent a new definition of subtraction. By subtraction, we have always meant the concept of taking away part of what we have. For example:

$$7 - 3$$

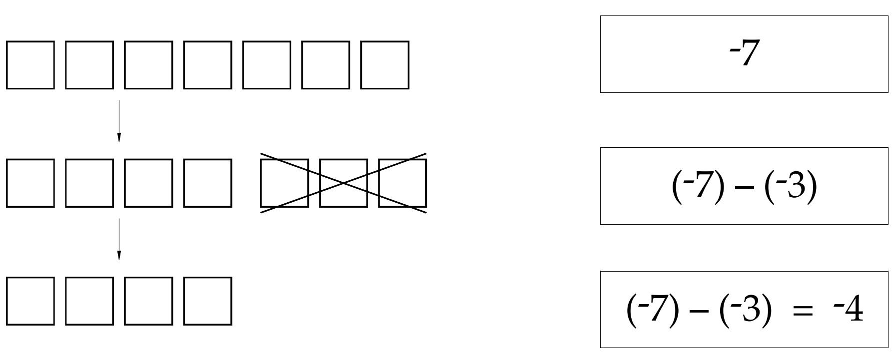
With the chips, this means that we start with 7 chips and then take away 3 chips. The result is 4:

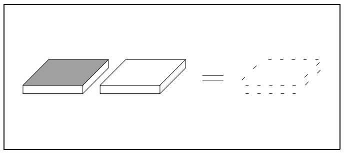


This standard idea of subtraction will also work well with the following example:

$$(^{-}7) - (^{-}3)$$

We start with 7 negative chips and take away 3 negative chips:





Although these examples work well with our idea of "taking away," subtraction is not always that easy. What if we are asked to subtract more chips than we start with?

$$5 - 7$$
$$3 - 18$$
$$9 - 10$$

$$(^{-}5) - (^{-}6)$$
$$(^{-}2) - (^{-}8)$$

Our system of subtraction needs to make sense when given these types of problems. We also need to know what to do if we start with one color chips, but we are asked to take away or subtract the *other* color of chips:

$$3 - (^{-}2)$$
$$^{-}4 - 5$$
$$0 - (^{-}6)$$

The old idea of "take away" is clearly not going to work for subtraction of signed numbers.

Subtraction of Signed Numbers: Method I

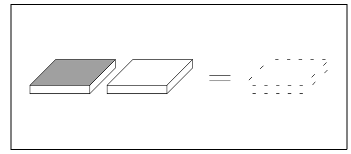
Our first new method of doing subtraction will be very simple—in a given expression, each number will tell us how many chips are in one group, and the sign in front (to the left) of each number will tell us what color chips are in that group. We will then add the groups together. If the chips are different colors, let them cancel in pairs.

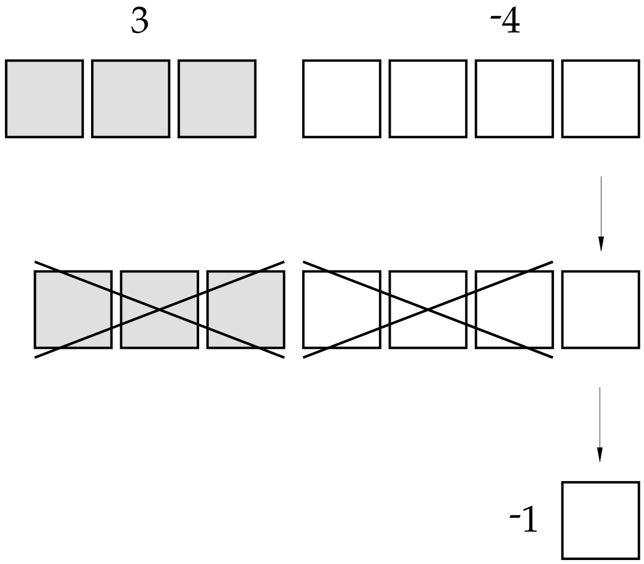
$$3 - 4$$

$+3$ (3 colored chips) $^{-}4$ (4 white chips)

Instead of subtraction, we think of the problem as adding groups of chips which are sometimes different colors. Look at each number and the sign to its left. Since 3 has no sign, it is positive; since 4 has a minus sign (–) it is negative. In this situation, *the subtraction sign is considered to be the same as a negative sign.*

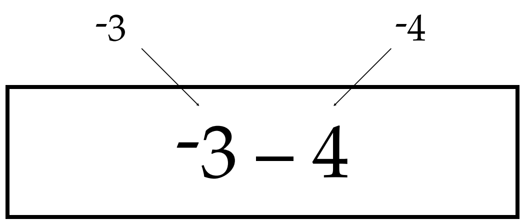
When we add 3 and $^-4$, the result is $^-1$:



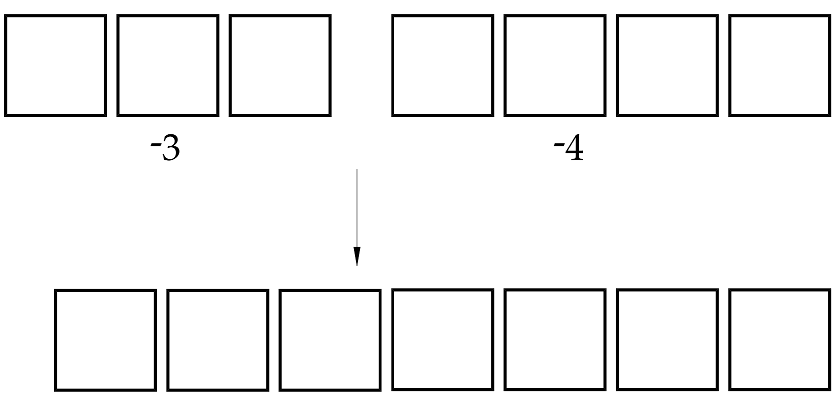


$$3 - 4 = {}^-1$$

Here is another example:

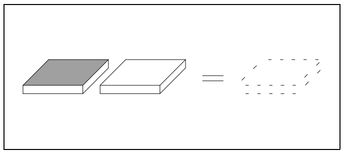


We think of the problem as starting with $^-3$ and adding $^-4$:



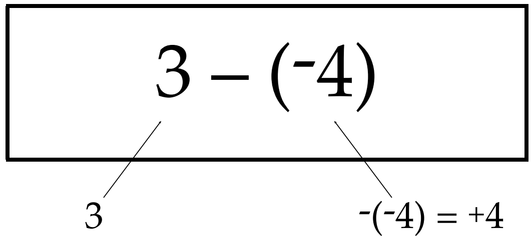
$$^-3 - 4 = {}^-7$$

The result is $^-7$.



Subtraction and Double Signs: Method I

If two signs appear next to each other with no number in between, think of them as double signs. Flip the chips for *each* negative or subtraction sign. If there are two negative signs, we flip the chips twice and the result is positive. We then add: For example:



This gives:

$$3 - ({}^-4) = 3 + 4 = 7$$

Summary: Method I

To add or subtract:

- **Identify each number as positive or negative by the sign in front of it. Choose the correct color chips for each group, then add the groups together.**

- **If there are double signs in front of any number, flip that group of chips the proper number of times, then add the groups together.**

- **Think of all addition and subtraction as *addition.***

Subtraction: Method II

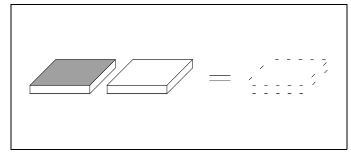
We will now look at another way of subtracting. Method II is very much like Method I; you should use whatever method is most comfortable. It is best to understand both methods—they are simply two different ways to illustrate the same idea.

First, let's look at some examples of adding and subtracting where two different problems have the same answer:

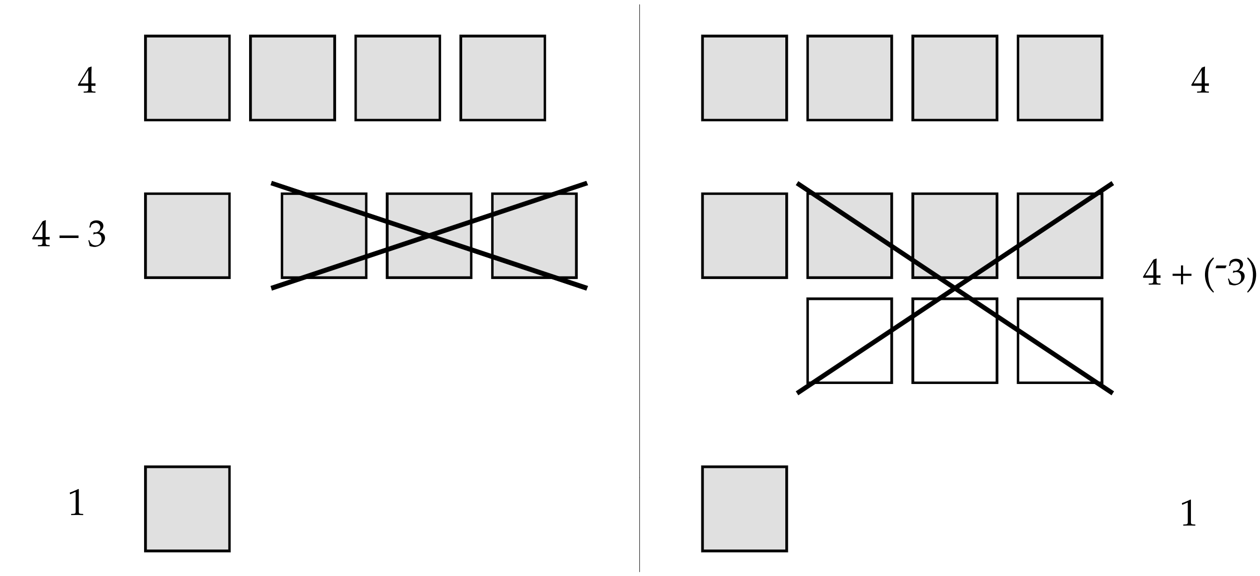
$$4 - 3 = 1$$

$$4 + ({}^-3) = 1$$

In the diagram below, you can see that *adding* $^-3$ is the same as s*ubtracting* 3:



$$4 - 3 \quad = \quad 4 + (^-3)$$



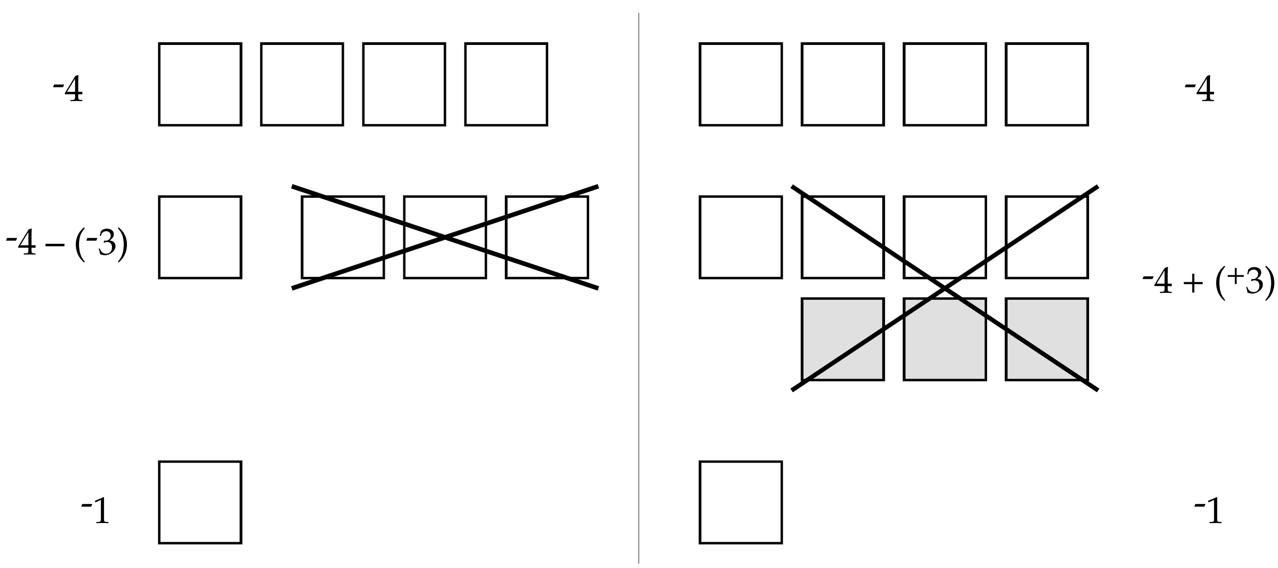
We can see that the following two examples also have the same result:

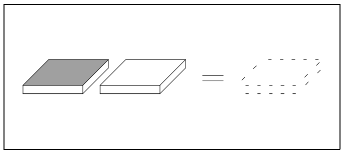
$$^-4 - (^-3) \ = \ ^-1$$

$$^-4 + (^+3) \ = \ ^-1$$

The diagram shows why this is true:

$$^-4 - (^-3) \quad = \quad ^-4 + (^+3)$$





We can see that:

- **Subtracting a positive number is the same as adding a negative number.**
- **Subtracting a negative number is the same as adding a positive number.**
- **In general, subtracting any number is the same as adding its opposite.**

$$4 - 3 = 4 + ({}^{-}3)$$

$$^{-}4 - ({}^{-}3) = {}^{-}4 + ({}^{+}3) = {}^{-}4 + 3$$

Here are some examples of how to use Method II with subtraction:

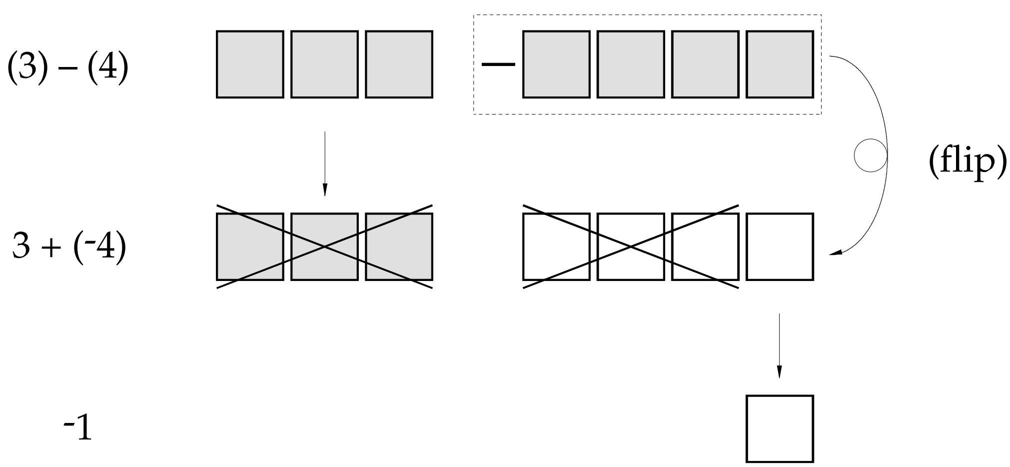
$$7 - 2 = 7 + ({}^{-}2) = 5$$

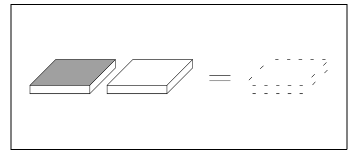
$$8 - ({}^{-}3) = 8 + (3) = 11$$

$$^{-}6 - 3 = {}^{-}6 + ({}^{-}3) = {}^{-}9$$

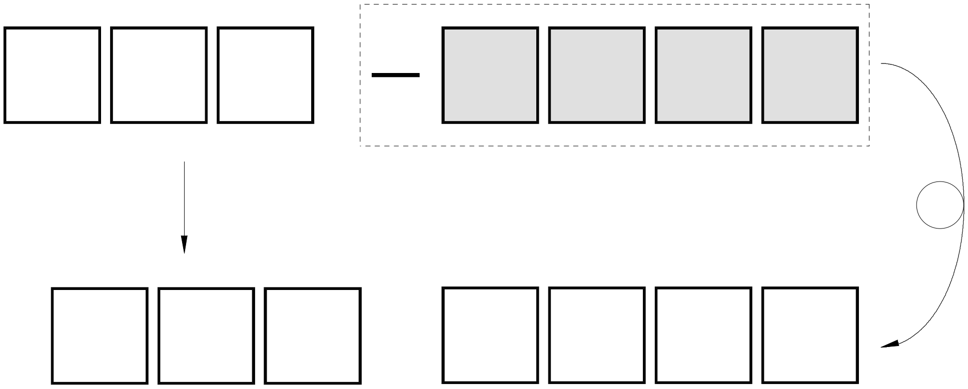
With the chips, we set up a subtraction with Method II by taking out the two groups of chips indicated. We than flip the subtracted group of chips and combine the two groups. Here are three examples:

$$3 - 4 = {}^{-}1$$





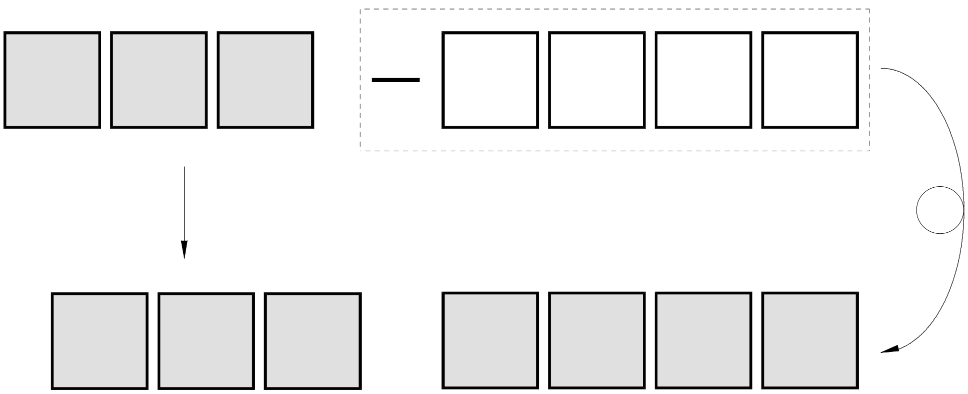
$^{-}3 - 4 = {}^{-}7$



$(^{-}3) - (4)$

$(^{-}3) + (^{-}4) = {}^{-}7$

$3 - (^{-}4) = 7$



$(3) - (^{-}4)$

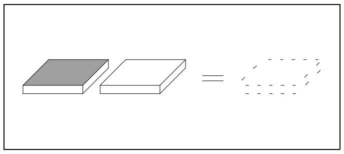
$(3) + (^{+}4) = 7$

Summary: Method II

To subtract (*a* and *b* stand for any numbers):

- $\boldsymbol{a - b = a + (^{-}b)}$
- $\boldsymbol{a - (^{-}b) = a + (^{+}b)}$
- **To subtract any number of chips, flip the subtracted chips and add.**

Summary: Method I and Method II



We have looked at two methods for doing subtraction. With both methods, we think of subtraction as adding. With Method I, we just look at the signs in front of each number to see what color chips to add; with Method II, we look at every subtraction as adding the opposite.

To Subtract:

Method I: Choose the color of chips by looking at the signs in front of each number, then add.

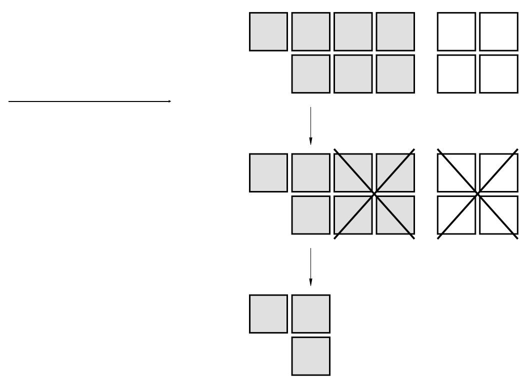
Method II: Instead of subtracting the second number, flip the chips and add the opposite.

Exercises

Use the chips to do the following subtractions:

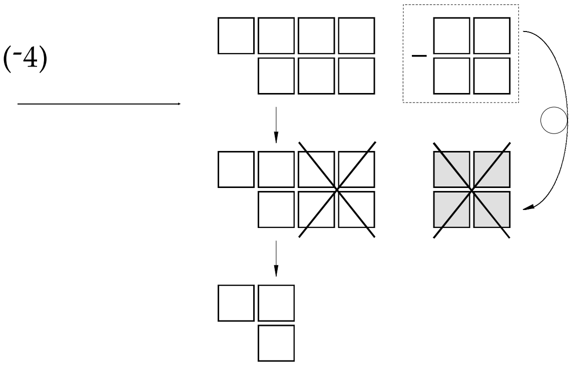
Example: 7 – 4
Solution: 3

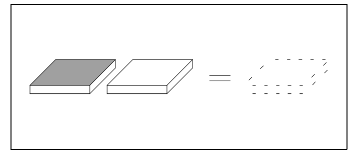
(Method I)



Example: (⁻7) – (⁻4)
Solution: ⁻3

(Method II)



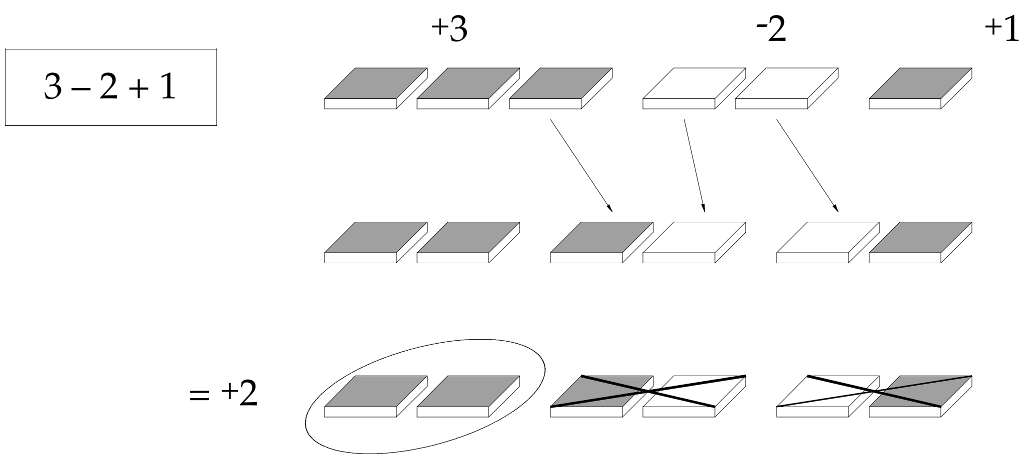


1. $5 - ({}^{-}3)$
2. ${}^{-}5 - ({}^{+}3)$
3. ${}^{-}5 - ({}^{-}3)$
4. ${}^{-}6 - ({}^{-}3)$
5. $3 - 5$
6. $3 - ({}^{-}5)$
7. ${}^{-}3 - 5$
8. ${}^{-}3 - ({}^{-}5)$
9. $0 - ({}^{-}17)$
10. $4 - 0$
11. $6 - ({}^{-}0)$
12. $1 - ({}^{-}1)$
13. $12 - ({}^{-}5)$
14. ${}^{-}12 - ({}^{-}5)$
15. ${}^{-}7 - 9$
16. ${}^{-}7 - ({}^{-}9)$
17. ${}^{-}4 - 4$
18. ${}^{-}4 - ({}^{-}4)$
19. $4 - ({}^{-}4)$
20. ${}^{-}7 - 3$
21. ${}^{-}7 - ({}^{-}3)$
22. $7 - 3$
23. $7 - ({}^{-}3)$
24. $5 - 2$
25. ${}^{-}5 - ({}^{-}2)$
26. $2 - 5$
27. ${}^{-}2 - ({}^{-}5)$
28. ${}^{-}2 - 5$
29. $2 - ({}^{-}5)$
30. $8 - ({}^{-}4)$

Section 4
Addition and Subtraction

Combining Addition and Subtraction

In a math sentence, if several signed numbers are written in a row with plus or minus signs in between the numbers, the sentence means that we should add the numbers by sliding the chips together and letting chips of different colors cancel out. The simplified answer is given by the sign and number of chips that are left when you're done. For example:



When you have three or more numbers together, we still think of them as being added. When subtraction is indicated, you may want to rewrite it as addition of the opposite kind of chips:

$$3 - 2 + 1 \;=\; 3 + (^{-}2) + 1$$

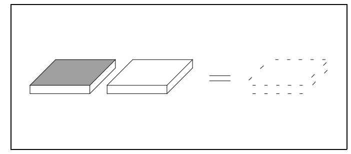
Then combine the chips to get the result. You can combine them in order from left to right:

$$3 + (^{-}2) + 1$$

$$1 \quad + \quad 1$$

$$2$$

Or you can rearrange the chips to add up the positives and negatives separately, and then cancel:



$$6 - 3 - 2 + 5$$

$$6 + (^{-}3) + (^{-}2) + 5$$

$$6 + 5 + (^{-}3) + (^{-}2)$$

$$11 \quad + \quad ^{-}5$$

$$6$$

Summary

When we have to add and subtract more than 2 numbers in a row, we use either method from the previous section and we consider all addition and subtraction as combining groups of chips:

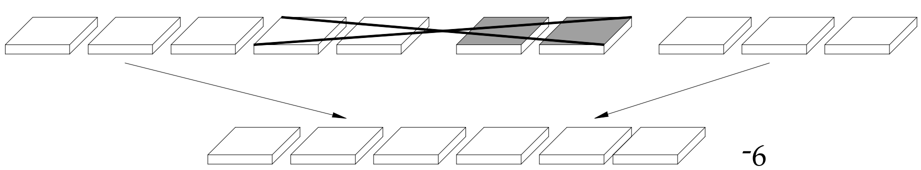
- **Combine the numbers in pairs**
- ***Or,* rearrange all of the positive numbers in one group and the negative numbers in another. Find the total negatives and total positives, then combine the totals.**

Exercises

Use the chips to find the answer and to illustrate the following problems:

Example: $^{-}5 + 2 - 3 = {}^{-}6$

Solution:

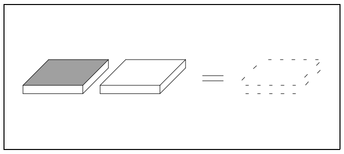


1. $^{+}1 - 4 + 3 = 0$

2. $^{-}2 + 1 - 4 =$

3. $^{+}2 + 3 - 1 - 2 =$

4. $^{+}5 - 6 - 3 - 1 =$



5. $^{+}2 - 7 + 5 - 1 =$

6. $^{+}6 + 4 + 3 + 3 =$

7. $^{-}1 + 5 - 6 + 2 =$

Use chips to show the following:

8. $^{+}(^{-}3) = {}^{-}3$

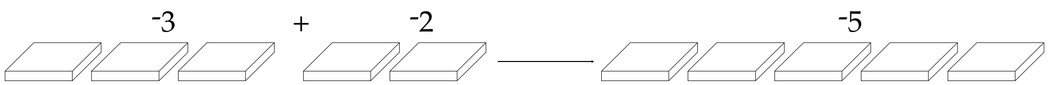
9. $^{-}(^{-}2) = {}^{+}2$

10. $^{+}(^{+}5) = {}^{+}5$

Use chips to do the following problems:

Example: $^{-}3 + (^{-}2) = {}^{-}5$

Solution:



11. $^{-}2 - {}^{-}2 = 0$

12. $^{+}2 - {}^{-}2 = {}^{+}4$

13. $^{-}1 - {}^{-}5 = {}^{+}4$

14. $^{-}(^{-}2) + 3 =$

15. $^{+}(^{-}5) - (^{-}2) =$

16. $^{-}(^{+}2) + 6 =$

17. $^{-}3 - {}^{-}7 =$

18. $^{-}3 + 5 =$

19. $3 + {}^{-}5 =$

20. $^{-}3 + {}^{-}5 =$

21. $3 - {}^{-}5 =$

22. $^{-}6 - 2 =$

23. $^{-}7 - {}^{-}3 =$

24. $^{-}7 - 3 =$

25. $7 - 3 =$

26. $7 - {}^{-}3 =$

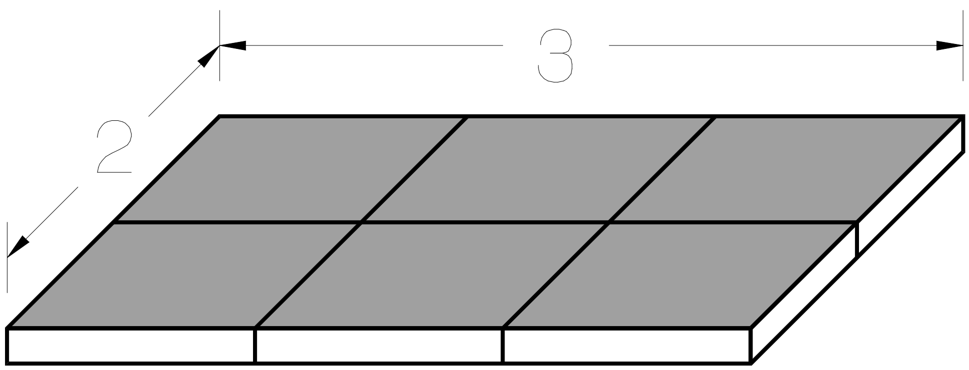
27. $^{-}8 + {}^{-}6 =$

Section 5
Multiplication

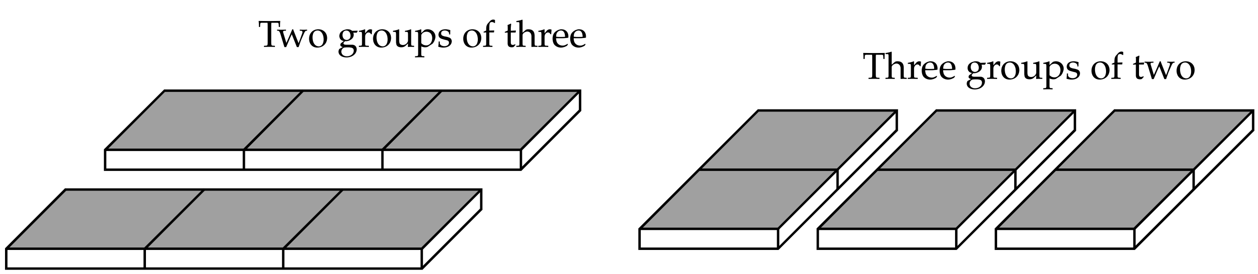
The Meaning of Multiplication

To multiply the numbers 3 and 2 using chips, make a rectangle 3 chips long and 2 chips wide, using six chips in all. We use a raised dot to indicate multiplication:

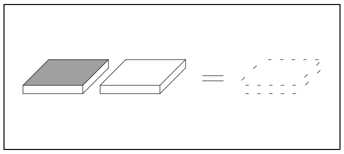
$$3 \cdot 2 = 6$$

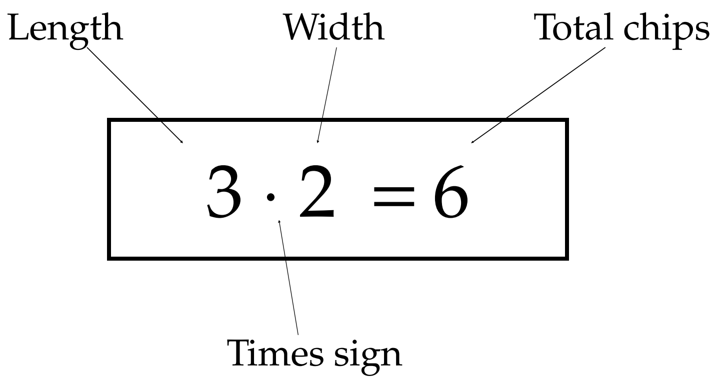


This shows either 3 groups of 2, or 2 groups of 3.



Multiplying any two numbers using chips means making a rectangle of chips with the numbers being the length and width. *Multiplying is making rectangles.* The answer to the multiplication—the product—is the total number of chips in the rectangle.



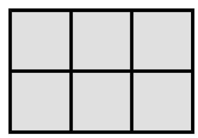


Multiplying with Signed Numbers

When multiplying signed numbers using chips we will still make a rectangle of chips, but we flip the chips once for each negative (–) sign used in the multiplication. Remember that we start with colored side up.

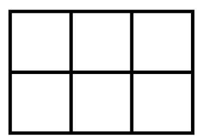
$(^{+}3) \cdot (^{+}2) = 6$

(No Flips)



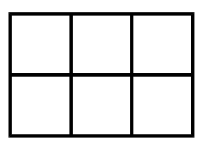
$(^{+}3) \cdot (^{-}2) = {}^{-}6$

(One Flip)



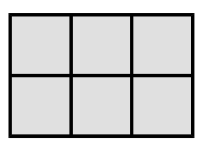
$(^{-}3) \cdot (^{+}2) = {}^{-}6$

(One Flip)



$(^{-}3) \cdot (^{-}2) = {}^{+}6$

(Two Flips)



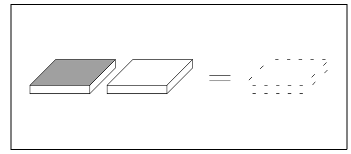
Here are some more examples:

$9 \cdot (-8) = -72$ (1 flip)

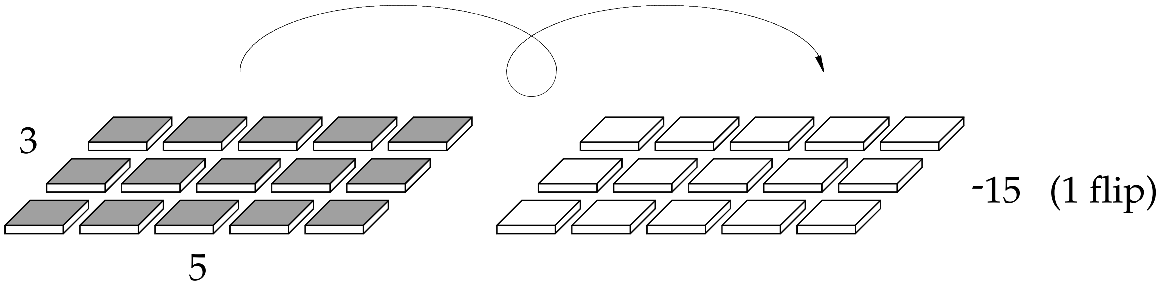
${}^{-}6 \cdot 3 = {}^{-}18$ (1 flip)

$(^{-}6) \cdot (^{-}3) = 18$ (2 flips)

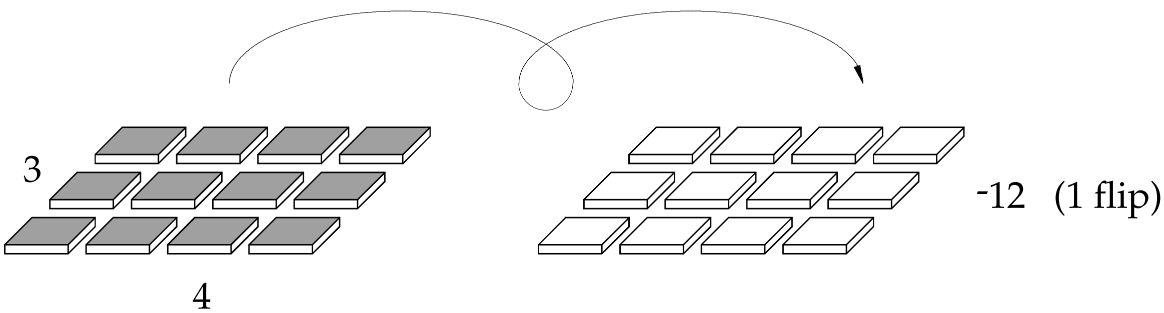
Here is how to use the chips for multiplying signed numbers:



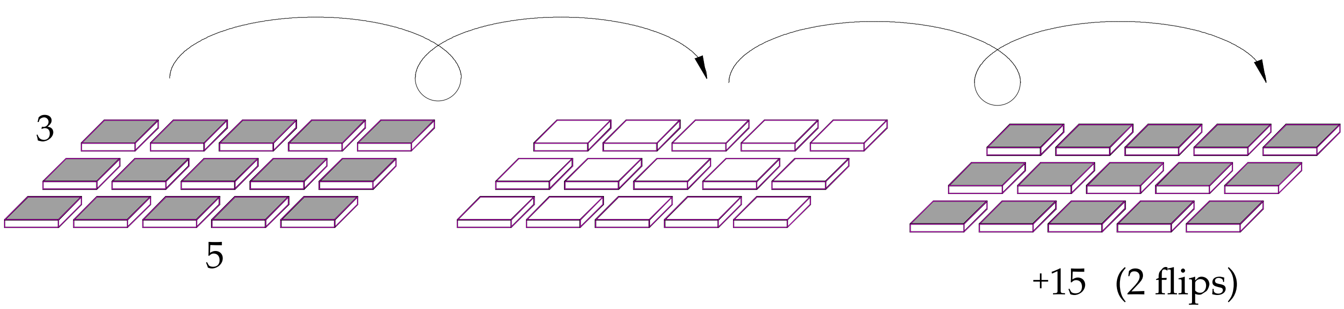
$5 \cdot (^{-}3) \ = \ ^{-}15$

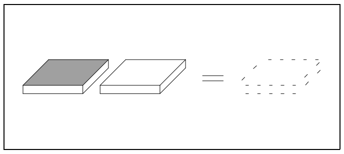


$^{-}4 \cdot 3 \ = \ ^{-}12$



$(^{-}3) \cdot (^{-}5) \ = \ +15$





We can now state the procedure for multiplying:

Multiplication of Two Numbers:

Make a rectangle with one number as the length and the other as the width.

Flip all the chips once for each negative sign.

The area and the color give the result.

We can see that there is an obvious method for finding the sign of the answer in a multiplication problem:

The Sign of the Result:

If one side of the rectangle is negative and the other side is positive, the rectangle is *negative*.

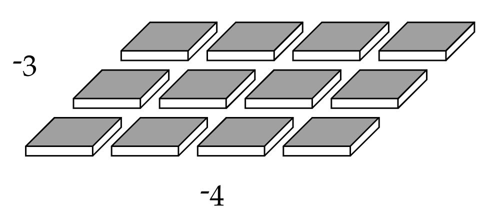
If both sides of the rectangle are positive, or both sides are negative, then the rectangle is positive.

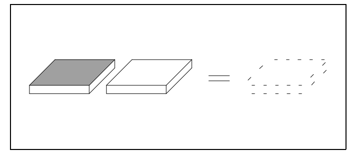
Exercises

Use chips to perform the following multiplications:

Example: $(^{-}3)\cdot(^{-}4) = {}^{+}12$

Solution:





1. $(^{-}3)\cdot(^{+}3) = {}^{-}9$

2. $(^{-}2)\cdot(^{-}5) = {}^{+}10$

3. $(^{-}2)\cdot(^{+}5) =$

4. $(^{+}5)\cdot(^{+}3) =$

5. $(^{-}4)\cdot(^{-}3) =$

6. $(^{+}3)\cdot(^{-}1) =$

7. $(^{-}2)\cdot(^{-}2) =$

8. $(^{-}2)\cdot(^{+}2) =$

9. $4 \cdot 7 =$

10. $(^{-}4)\cdot(^{-}7) =$

11. $(^{-}4) \cdot 7 =$

12. $1 \cdot 1 =$

13. $1 \cdot (^{-}1) =$

14. $(^{-}1)\cdot(^{-}1) =$

15. $(1)\cdot(17) =$

16. $(^{-}1)\cdot(17) =$

17. $(0)\cdot(^{-}17) =$

18. $(^{-}5)\cdot(^{-}6) =$

19. ${}^{-}3\cdot(2) =$

20. $(^{-}5)\cdot(^{-}3) =$

21. ${}^{-}4\cdot 3 =$

22. $2\cdot(^{-}7) =$

23. ${}^{-}2\cdot(7) =$

24. ${}^{-}2\cdot(^{-}7) =$

25. $(6)\cdot(^{-}3) =$

26. $(^{-}6)\cdot(3) =$

27. $(^{-}6)\cdot(^{-}3) =$

28. ${}^{-}1\cdot(^{-}12) =$

29. ${}^{-}3\cdot(^{-}3) =$

30. ${}^{-}5\cdot(^{+}5) =$

Section 6
Division

The Meaning of Division

Division is often described as backwards multiplication. For example, if we want to know:

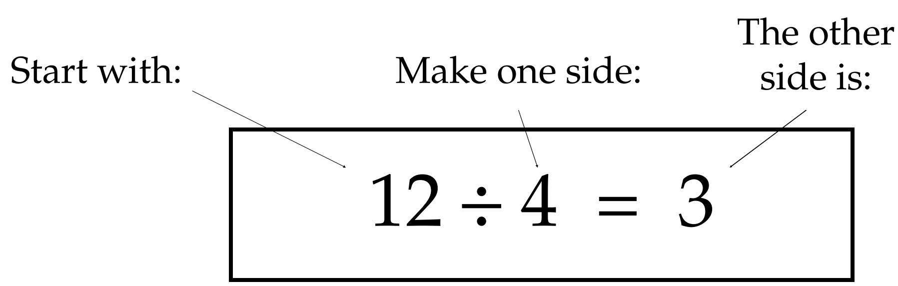
$$12 \div 4 = ?$$

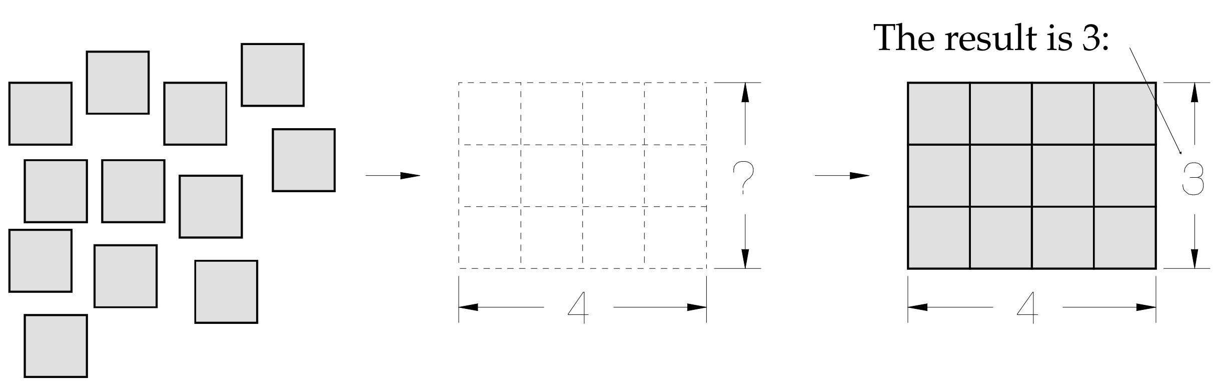
We usually think of this as:

"How many fours are in 12?"

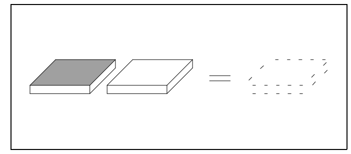
Using chips, this is also the opposite of multiplication. Since multiplication is making rectangles and counting the result, division also involves rectangles. The problem above becomes:

"Take 12 unit chips and form a rectangle with side 4.
What is the other side?"





Division with Signed Numbers



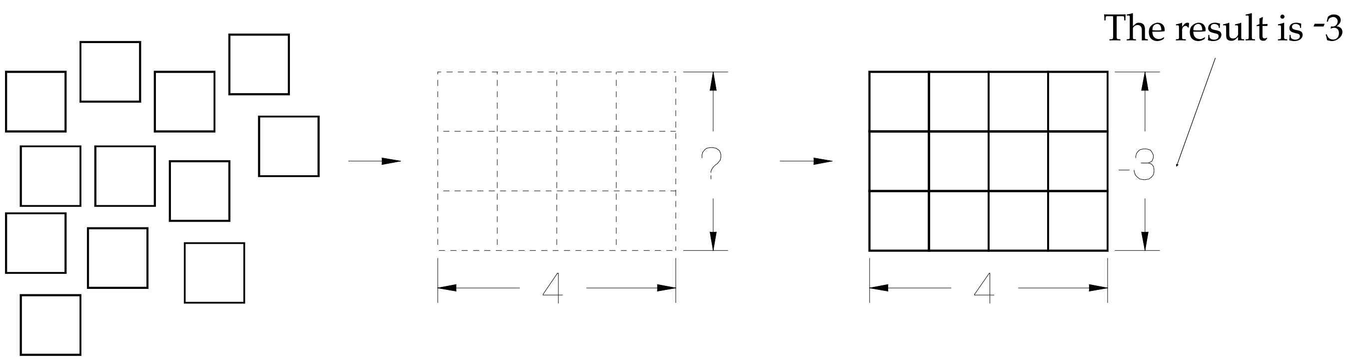
If we have a division problem with one or two negative numbers, we continue to think backwards:

$$^{-}12 \div 4 \;=\; ?$$

becomes

"What times 4 is equal to $^{-}12$?"

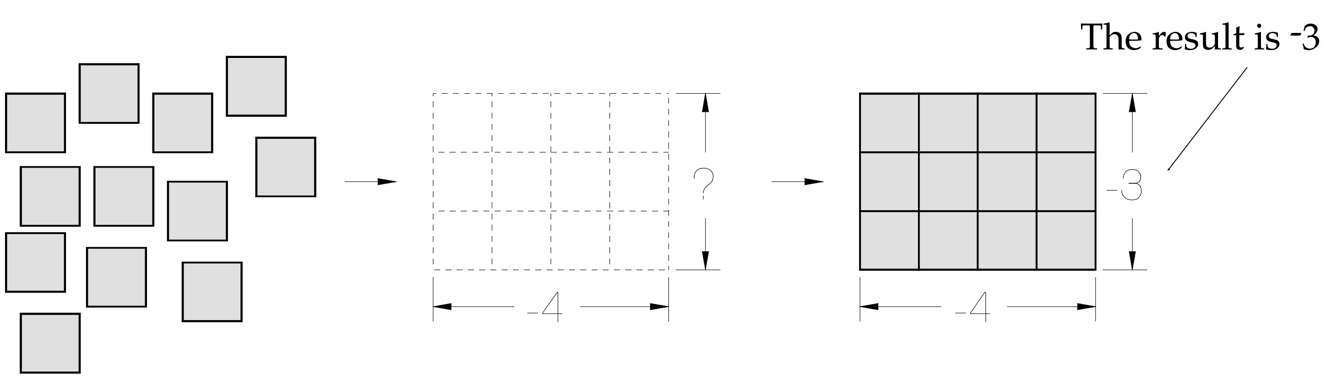
The answer is $^{-}3$ because -3 times 4 is $^{-}12$. To do this with chips, we start with $^{-}12$ unit chips and build a rectangle that is 4 on one side. The other side is 3 units. Because the answer needs to be $^{-}12$, we can see that the chips have been flipped once, so the answer—the missing side—must be negative.

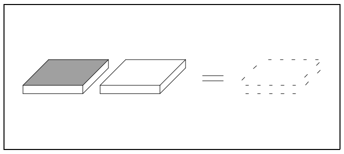


We can do other division problems in the same way. For example, what is:

$$12 \div (^{-}4)?$$

We start with 12 chips and build a rectangle with one side of $^{-}4$. The given side ($^{-}4$) is negative and acoounts for one flip. To get back to an area of $^{+}12$, we need another flip, so the other side must be negative. The answer is -3.

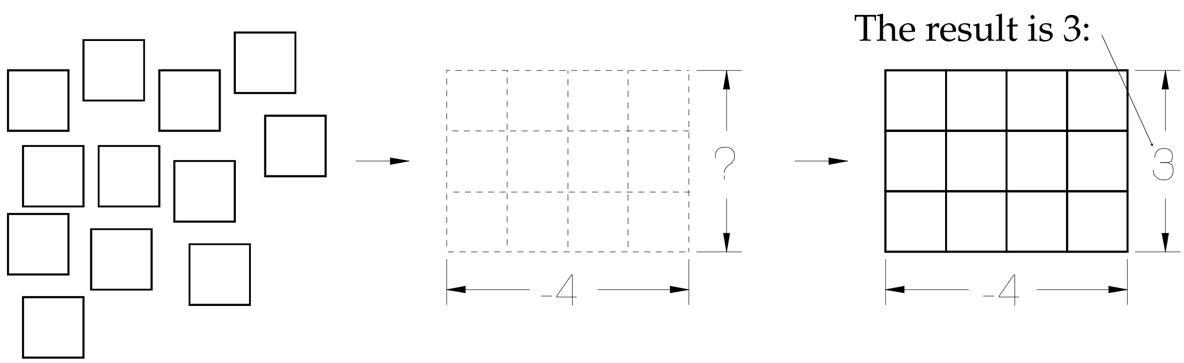




Finally, how would we illustrate:

$$^{-}12 \div (^{-}4)?$$

As we did above, we start with $^{-}12$ chips and a side of $^{-}4$ and then we can see that the other side is 3. We flip the chips once for $^{-}4$, giving the negative



sign that $^{-}12$ requires, so the other side is positive 3.
Division problems in algebra are most often written as fractions; instead of writing

$$12 \div 4 = 3$$

we will commonly write

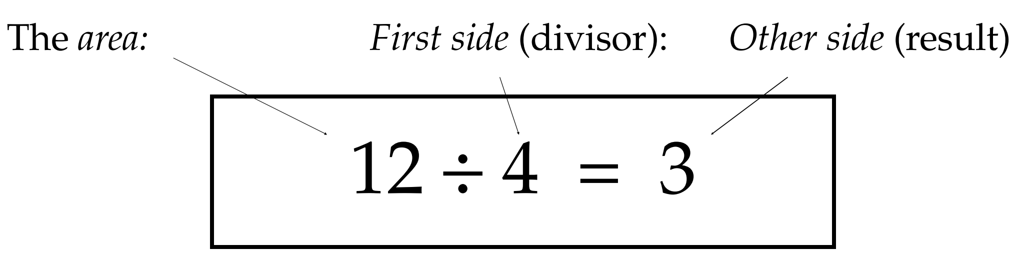
$$\frac{12}{4} = 3$$

You are probably aware that we can think of fractions as division problems and we can rewrite division problems as fractions. When writing division problems as fractions, we normally will reduce all fractions and we will write "improper" fractions as mixed numbers.

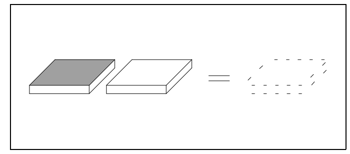
For an explanation of why a division problem can be rewritten as a fraction, please see Section 3 (Compound Fractions) of the FRACTIONS *chapter.*

Summary

Division is the opposite of multiplication. Since multiplication is making



rectangles, division is making rectangles in reverse:



Division:

1. **Start with unit chips (the *area*).**
2. **Build a rectangle with the divisor for the *first side*.**
3. **How long is the *other side*?**
4. **The color of the *area* and the sign of the *first side* will tell you the sign needed for the *other side* (*result*).**

Division: The Sign of the Result

1. **If the area is positive:**
   Both sides are positive,
   or both sides are negative.

2. **If the area is negative:**
   One side is negative,
   and the other side is positive.

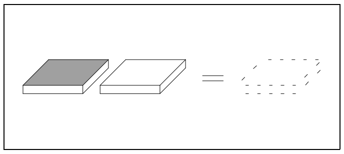
Positive	**divided by**	**Positive**	**is**	**Positive**
Positive	divided by	**Negative**	**is**	**Negative**
Negative	**divided by**	**Positive**	**is**	**Negative**
Negative	**divided by**	**Negative**	**is**	**Positive**

Exercises

Complete the following division problems using the chips:

1. $12 \div (^{-}2)$

2. $^{-}12 \div (^{+}2)$

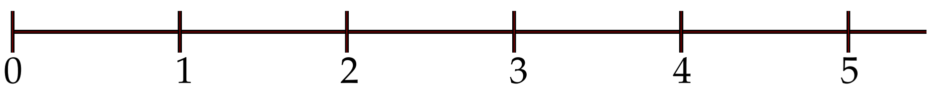


3. $^{-}12 \div (^{-}2)$
4. $16 \div (-8)$
5. $^{-}16 \div (^{-}4)$
6. $4 \div (4)$
7. $4 \div (^{-}4)$
8. $^{-}4 \div (4)$
9. $1 \div (^{-}1)$
10. $^{-}1 \div (^{-}1)$
11. $0 \div 17$
12. $0 \div (^{-}17)$
13. $14 \div (^{-}7)$
14. $^{-}16 \div (^{-}2)$
15. $18 \div (^{-}3)$
16. $^{-}22 \div (^{-}11)$
17. $20 \div (^{-}5)$
18. $^{-}20 \div 5$
19. $^{-}20 \div {}^{-}5$
20. $^{-}5 \div (\ ^{-}5)$
21. $\frac{12}{^{-}3}$
22. $\frac{15}{5}$
23. $\frac{^{-}14}{7}$
24. $\frac{^{-}8}{^{-}2}$
25. $\frac{^{-}20}{4}$
26. $\frac{^{-}20}{^{-}4}$
27. $\frac{^{-}24}{9}$
28. $\frac{^{-}24}{^{-}9}$
29. $\frac{9}{6}$
30. $\frac{^{-}12}{5}$

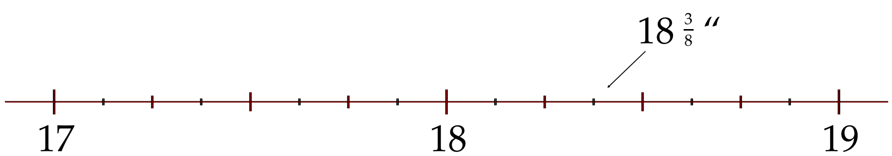
Section 7
The Number Line

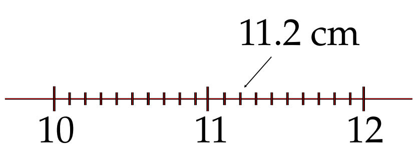
Numbers as Distance

A number line is a useful method of representing positive and negative numbers and their relationships. A number line is similar to a measuring tape; distances from the end of the tape (zero) are marked out in equal divisions along the tape. (Most measuring tapes use units of inches or centimeters.)



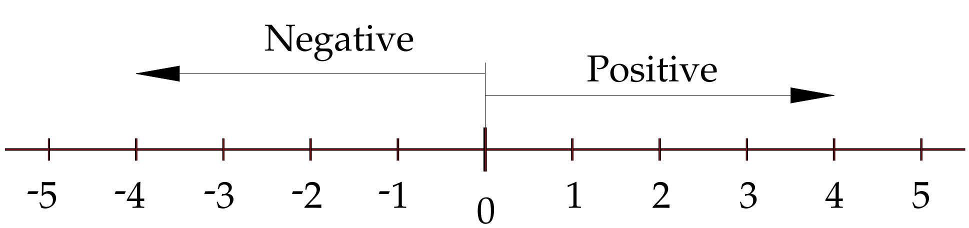
The farther you move along the tape the higher the numbers get. Between the whole numbers units are parts of units, marked off in fractions or decimals.

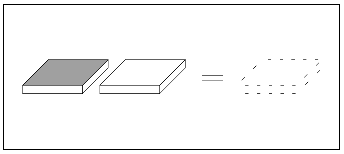




Even between the closest marks on the measuring tape, we know that *any* small fraction or decimal part of a unit could be represented if we used a magnifying glass or a micrometer. In these ways a number line is again just like a measuring tape.

A number line is different from a tape measure in that the number line marks off both *positive* and *negative* distances from zero by defining one direction as positive and the opposite direction as negative, with zero in the middle.





Generally, distances to the right of zero along the number line are called positive, and distances to the left of zero are called negative. Notice from the picture that the large, more positive numbers lie farther to the right, and the more negative numbers lie farther to the left. Since negative numbers are like being *below zero* or *in the hole*, we say that any number on the number line is greater than (more positive than) any number lying to its left.

8 is greater than (more positive than) 3.

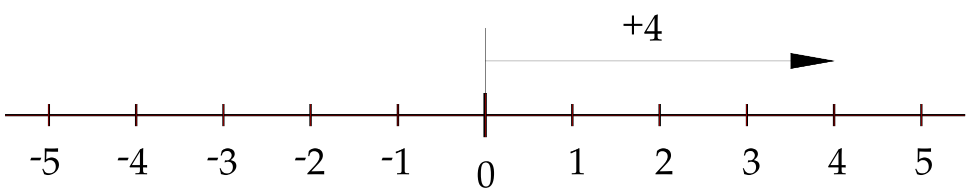
-2 is greater than (more positive than) -5.

A number line also differs from a measuring tape because the units on the number line don't actually represent distances like inches or centimeters. The number line is made up of what are called **pure numbers**, which don't necessarily represent any lengths or objects, but are just numerical values.

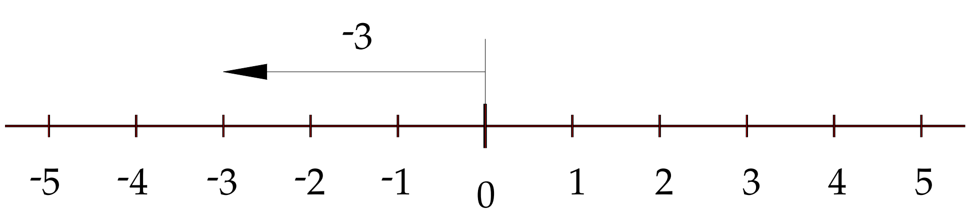
Of course numerical values might be used to represent numbers of objects, etc., but these representations are not necessary to use a number line.

Adding on a Number Line

Positive numbers are represented on a number line as arrows pointing to the right and having a length showing the number of units.



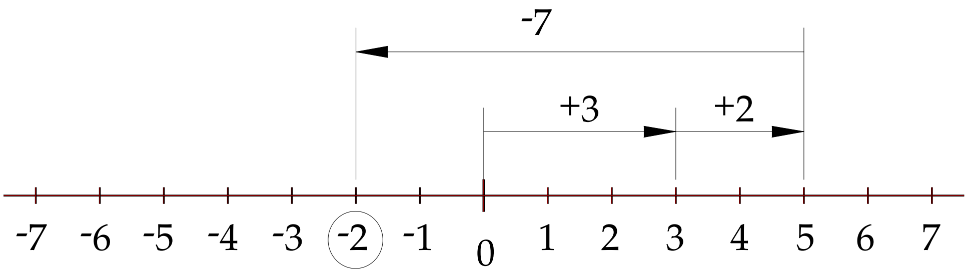
Negative numbers are represented as arrows pointing to the left and also having length equaling the number of units.



To add several numbers on the number line we represent each number as an arrow. Beginning with the tail of the first arrow at zero, we place the tail of each succeeding arrow at the tip-point of the previous arrow. The sum of the numbers is the position on the number line of the tip of the final arrow.

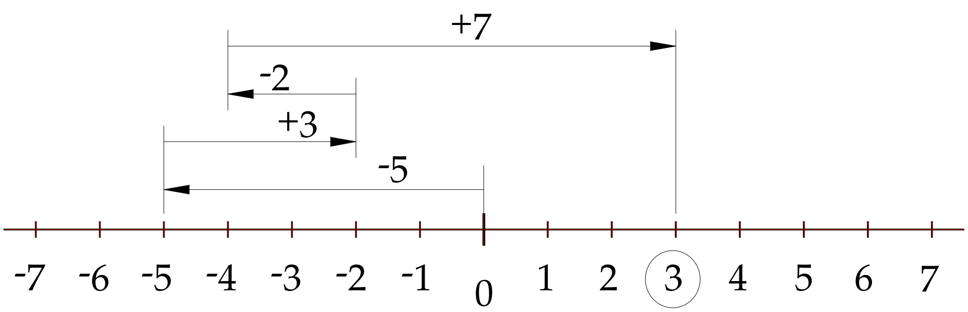
For example:

3 + 2 – 7



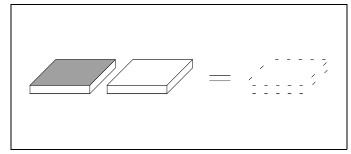
The sum is $^{-}2$. Another example:

$^{-}5 + 3 - 2 + 7$



The sum is $^{+}3$.

Before adding on a number line, you must simplify all double negatives to positives. The answers we get from adding on a number line will always be exactly the same as the answers we get by adding positive and negative chips; only the representation is different.



Exercises

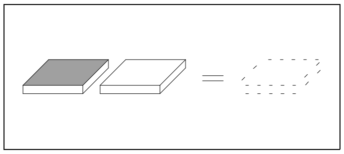
Draw number lines and arrows to complete these additions. Circle the resulting sum. (Remember, the spaces between the units on the number line must all be the same.)

1. 3 + 5 – 2

2. $^{-}2 + 4 - 6$

3. $^{-}3 - 2 + 4$

4. $2 - (^{-}5) - 3$



Make a number line and complete the following additions by counting with your pencil point. Start with your pencil point at zero, and count steps to the right for each positive number and steps to the left for each negative number added. Get your result from the number line without drawing arrows.

5. $2 - 5 + 3 - 1$

6. $^{-}3 - 5 + 2 + 1$

7. $7 + 1 - 5 - 3$

8. $^{-}2 + 5 - 6 + 1$

9. $4 + 3 - (^{-}2) + (^{-}5)$

10. $^{-}5 - (^{-}3) + (^{-}2) - 4$

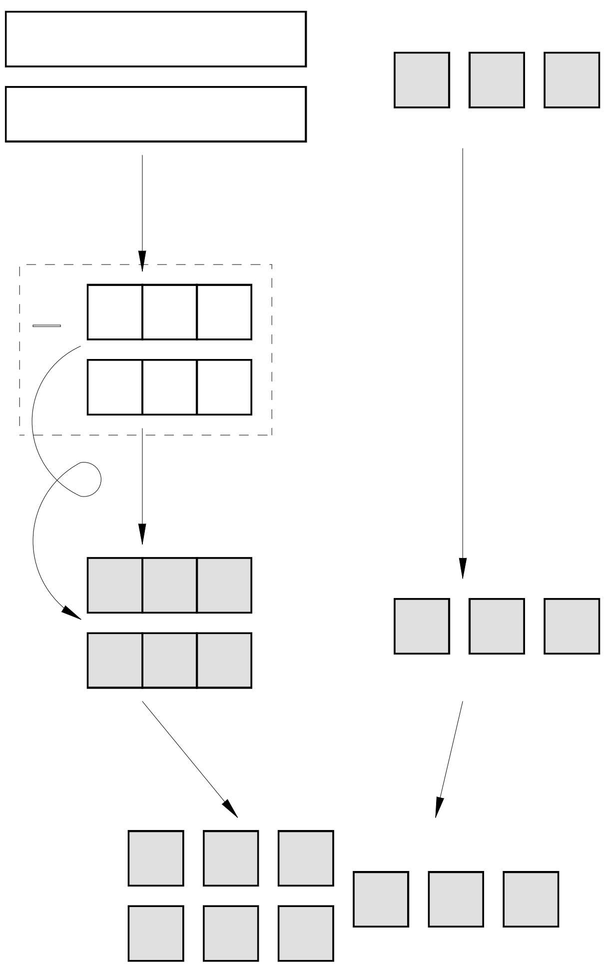
For discussion:

11. If a tape measure is going to work, why must the separation of all the units be the same?

12. How would you multiply using a number line?

Chapter 2

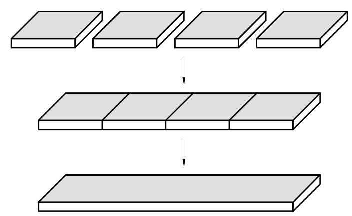
Expressions



Section 1
Simple Expressions

The Meaning of Unknowns

A group of unit chips can be represented by an **unknown** or **variable** like x. We join an unspecified number of chips together to form a bar called x:



Unknowns can be positive, negative, or zero. We use unknowns to represent quantities that will be known at a later time. Because unknowns are actually numbers, we treat them in the same manner as any other numbers.

Expressions

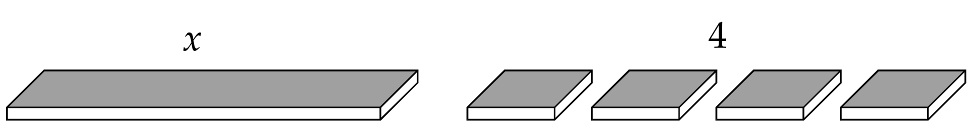
An **expression** is any quantity that stands for a number. Expressions may be as simple as one number or unknown, or they be lengthy statements including many numbers, unknowns, and operations:

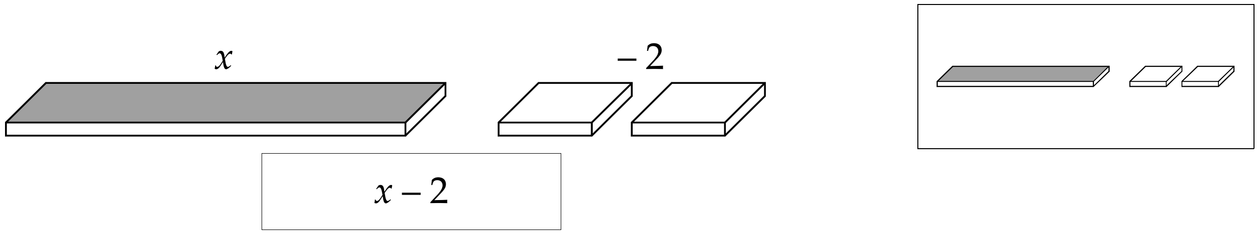
Examples of Expressions
$3x$
$3x + 1$
$^{-}17$
$3x + 2 + 6x - 2$

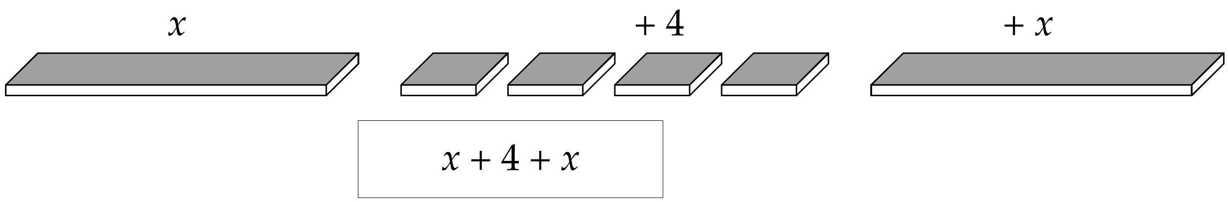
Simple Expressions

It is easy to visualize expressions that include only one or two symbols:

$x + 4$

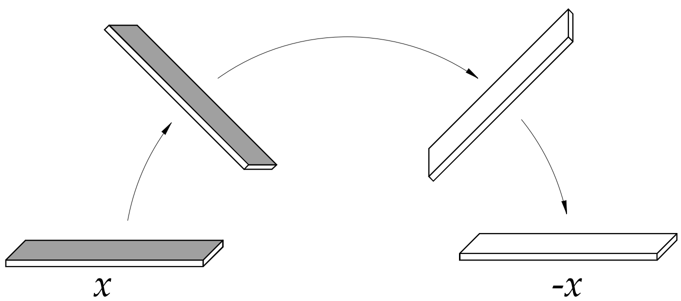






The Opposite of *x*

Just as we can find opposites of numbers by flipping the chips, the opposite of x can be shown as the x-bar flipped over:

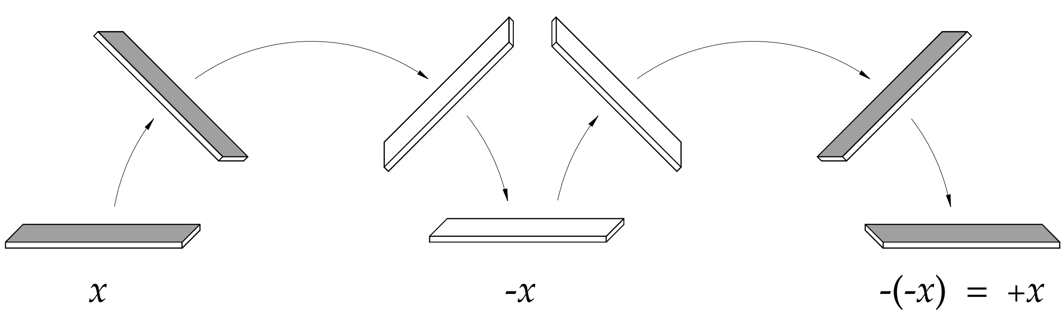


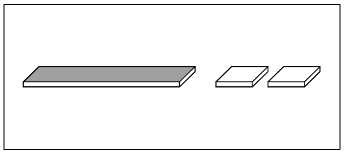
This $-x$ may be called **the opposite of x**, **the additive inverse of x**, or **negative x**. The last term—negative x—should be used with care. Because x may stand for either a positive or negative number, negative x stands for the opposite of x; it is not necessarily a negative number.

The opposite of the opposite is still the original amount:

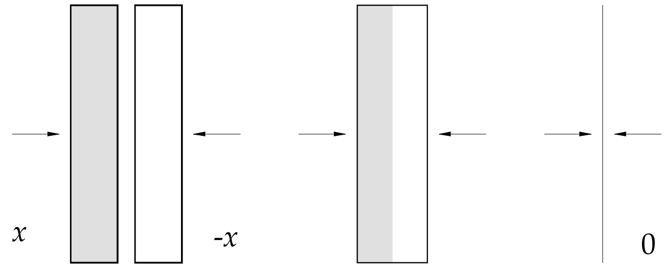
$$^{-}(^{-}5) = {}^{+}5$$

$$-(-x) = +x$$





Finally, *x* and -*x* are additive inverses. When added, they "cancel" to zero in the same way that $^{+}3$ and $^{-}3$ cancel:



Evaluating Expressions

An expression that includes unknowns has no exact value because we do not know the value of the unknown. If we choose a value for the unknown, we can then **evaluate the expression** to determine its value.

To evaluate an expression, simply substitute the value of the unknown into the expression and then carry out the indicated operations:

If $x = 7$, to evaluate $x + 5$:

$$x + 5$$

$$(7) + 5$$

$$12$$

If $x = {}^{-}5$, to evaluate $4 - x$:

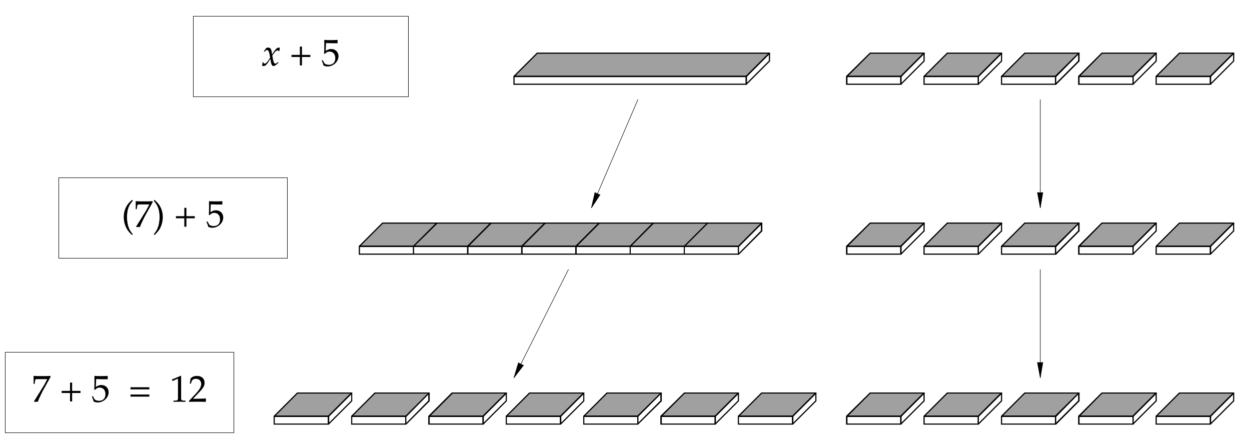
$$4 - x$$

$$4 - ({}^{-}5)$$

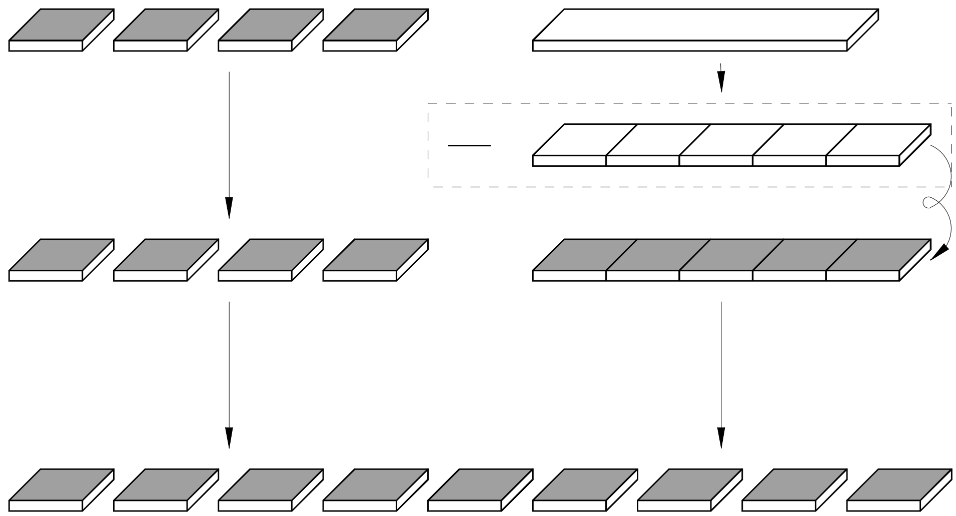
$$4 + 5$$

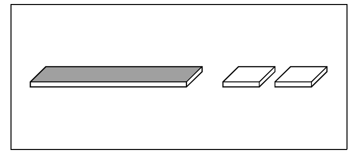
$$9$$

With the chips, we simply substitute the indicated number of units for the *x*-bar and then complete the count of unit chips. For $x + 5$, where $x = 7$:



For the second example, we start with a diagram of 4 – *x*, then we figure out the value of -*x* when *x* is $^{-}5$, and then we complete the count of unit chips:





The *x*-bar can stand for ***any*** *number—positive, negative, or zero.*

Exercises

Draw pictures of the following expressions. Evaluate each expression three different times—when *x* is 3, 0, and $^{-}2$:

1. $x + 6$

2. $2 + \text{-}x$

3. $x + 5 + x + x$

4. $^{-}5 + x + 5 + \text{-}x$

5. $3 + x + 5 + (\text{-}x) - x - 1$

6. $x + x + x - 5$

7. $x + x + x + x + x - 3$

8. $4 + x$

9. $x - x$

10. 5

11. $x + 3 + (^{-}3)$

12. $x - 3 - x - 2 - x - 1$

13. $3 - x$

14. $\text{-}x + 3$

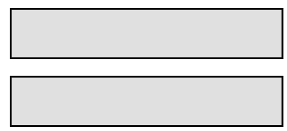
15. $x + (^{-}3)$

Section 2
Multiples of *x*

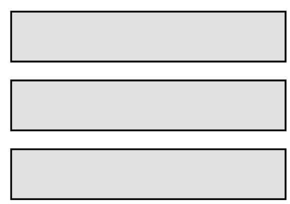
More than One *x*

If x is a certain quantity, the idea of several x's is a natural extension of our idea of one unknown:

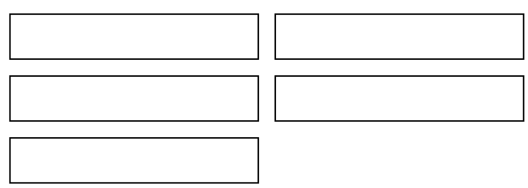
$2x$



$3x$



$^{-}5x$



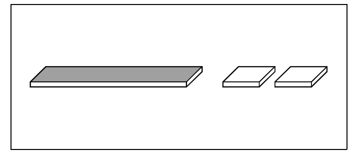
An expression like **2*x*** also can be thought of as a multiplication problem:

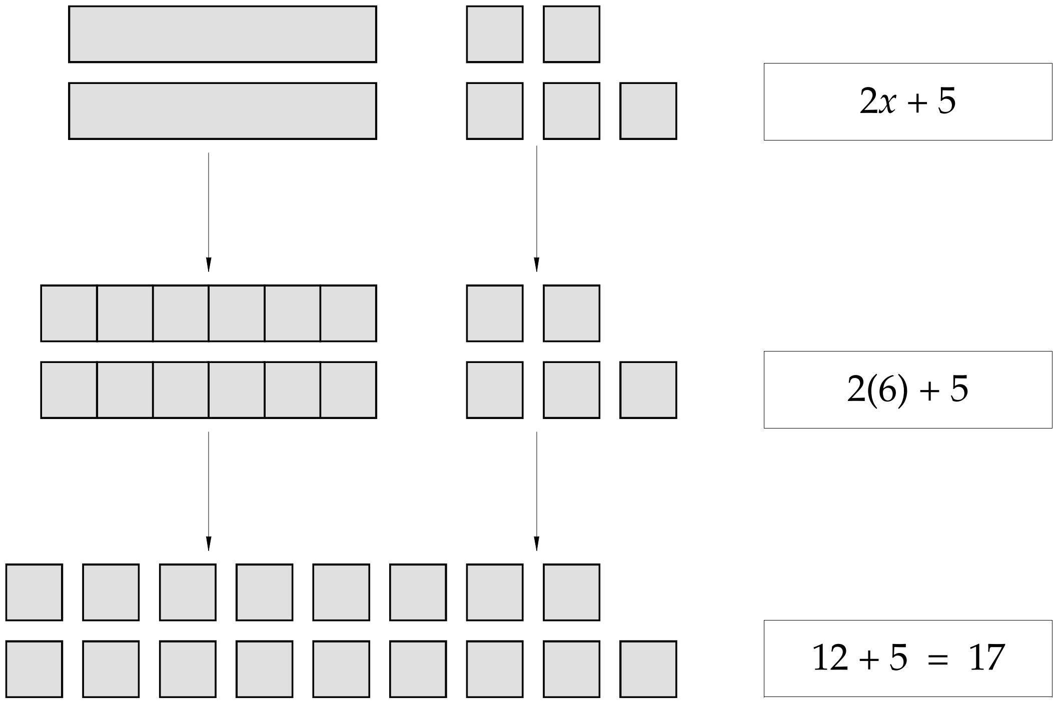
$$2x \ = \ 2 \cdot x \ = \ x + x$$

As we can see, we can call this expression "two x" or " two times x" and the meaning is still the same. Expressions such as $^{-}5x$ will be shown as 5 negative x-bars. In later chapters, we will see that the idea of $(^{-}5)\cdot(x)$ is also appropriate.

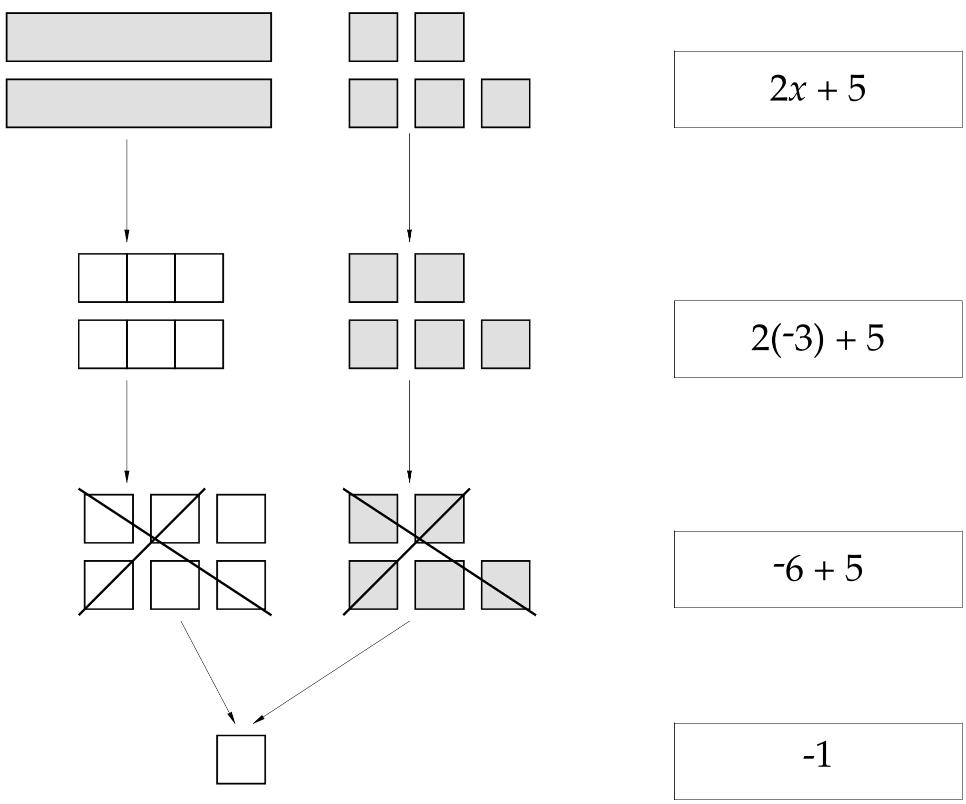
Evaluating Expressions with Multiple *x*'s

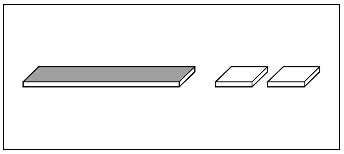
To evaluate an expression like those shown above, we still substitute a certain quantity for x, but we must be careful to carry out the indicated multiplications where a number is multiplied times x. For example, evaluate $2x + 5$, where $x = 6$:



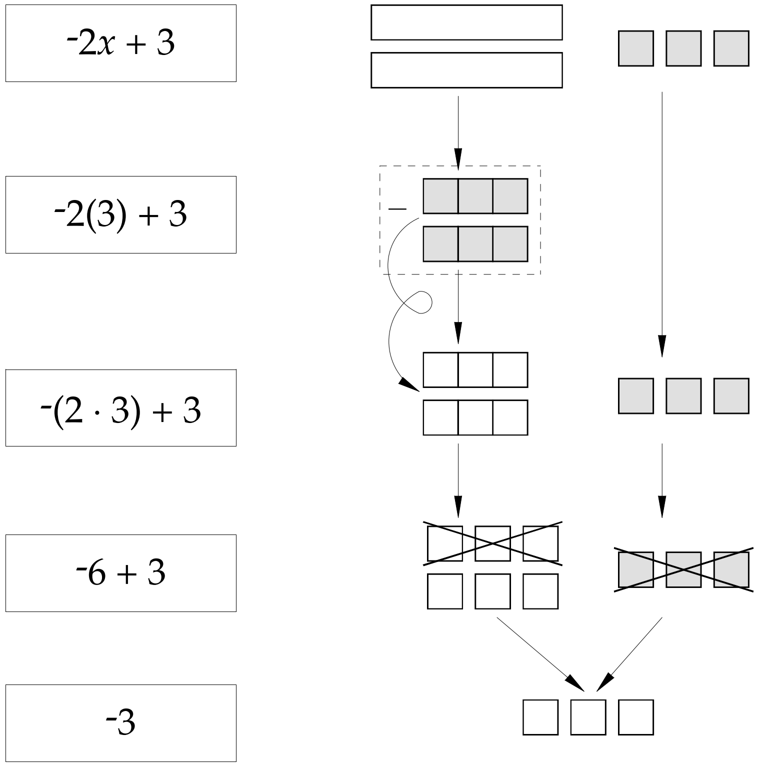


Repeating this example for a different value, where $x = {}^-3$:

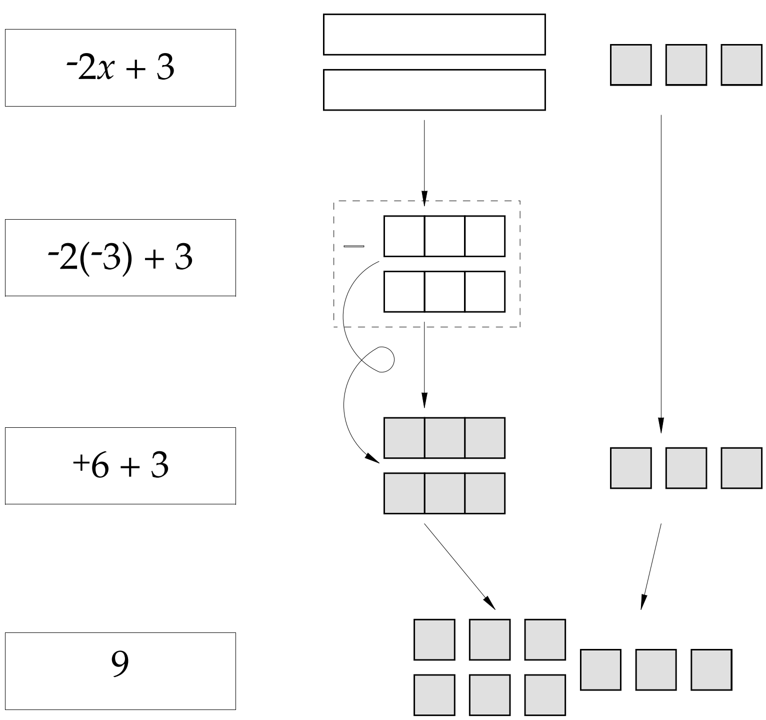




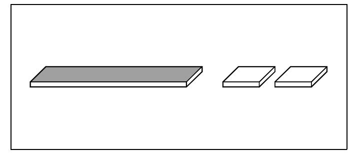
If the expression contains negative x's, we must first substitute the correct value for x; then we flip over the substituted chips to show the opposite of x. For example, to evaluate $^{-}2x + 3$, where x is 3:



Here is a second evaluation of $^{-}2x + 3$, where this time x is $^{-}3$:



Exercises



Use chips to represent the following expressions. Evaluate each expression four different times for $x = 1$, $x = 5$, $x = 0$, and $x = {}^{-}1$.

1. $3x - 15$

2. ${}^{-}3x - 12$

3. $2x + 5 + {}^{-}3x$

4. ${}^{-}3x + 2 + x + ({}^{-}3)$

5. $225x + 1 + {}^{-}225x$ (Do you need the chips?)

6. $5 - x$

7. $2 - 3x$

8. $0 - x + 16$

9. $3x - 2x + 6 - 2x$

10. $2 - ({}^{-}3x)$

11. $2x - 3 + x - 5$

12. ${}^{-}4 - x + 2 + 3x$

13. $5x - 2 + x$

14. $3 + 2x - 1 - x$

15. ${}^{-}({}^{-}2) + 3x + 5$

16. $x - 5 - 3x + 1$

17. ${}^{-}({}^{-}4x) + 3 + x$

18. ${}^{-}3 - 2x - ({}^{-}5)$

19. $x + x + 3x - 5$

20. $2 - 3x - 4x + 1$

Section 3
Combining Similar Terms

Combining Terms

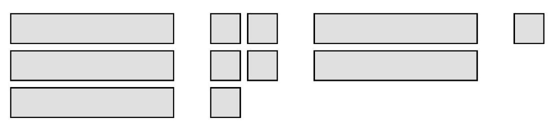
Each group of similar chips in an expression is called a **term**:

Expressions	Terms
$3x + 5 + 2x$	$3x$, 5, $2x$
$17 - 2x$	17, $^{-}2x$

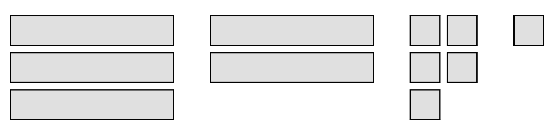
Notice that the **2** in **$2x$** is not a term.

Before we evaluate or use an expression, it is usually best to combine similar terms:

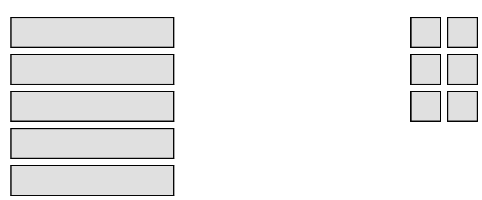
$3x + 5 + 2x + 1$



$3x + 2x + 5 + 1$

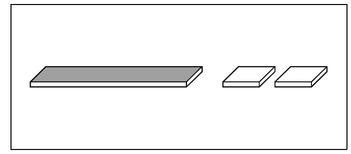


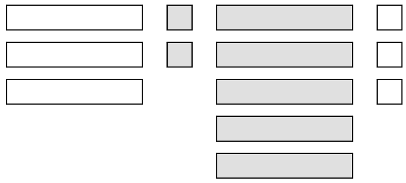
$5x + 6$



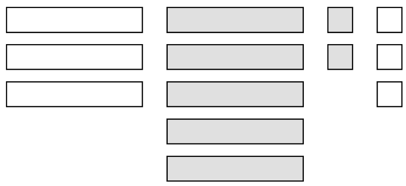
The process of combining similar terms will save time when evaluating expressions and will also be helpful in techniques presented in future chapters.

Positive x-bars and negative x-bars are combined in the same way as positive and negative chips:

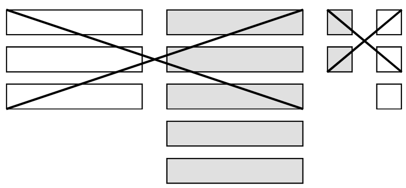


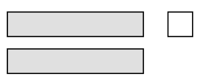


$^{-}3x + 2 + 5x - 3$



$^{-}3x + 5x + 2 - 3$



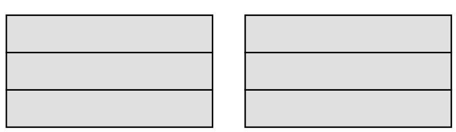


$2x - 1$

Multiplying Numbers and *x*'s

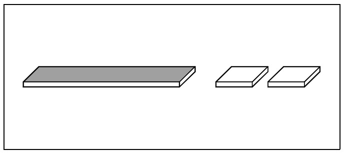
What is the meaning of:

$$2 \cdot 3x \ ?$$



Using symbols, we can write:

$$\begin{aligned} 2 \cdot 3x &= 2 \cdot (3 \cdot x) \\ &= 2 \cdot 3 \cdot x \\ &= (2 \cdot 3) \cdot x \\ &= 6 \cdot x \\ &= 6x \end{aligned}$$



If this seems too formal, think of the answer as 2 groups of $3x$. This is $3x + 3x$ or $6x$. Here are other examples:

$$\begin{aligned} 6x \cdot 5 &= (6 \cdot x) \cdot 5 \\ &= (x \cdot 6) \cdot 5 \\ &= x \cdot 6 \cdot 5 \\ &= x \cdot 30 \\ &= 30 \cdot x \\ &= 30x \end{aligned}$$

$$\begin{aligned} 6x \cdot (^-5) &= (6 \cdot x) \cdot (^-5) \\ &= (x \cdot 6) \cdot (^-5) \\ &= x \cdot 6 \cdot (^-5) \\ &= x \cdot [6 \cdot (^-5)] \\ &= x \cdot (^-30) \\ &= {}^-30 \cdot x \\ &= {}^-30x \end{aligned}$$

Common Errors

Most of the errors made by students can be avoided by paying attention to what the symbols *mean*. For example, consider the following errors made while combining similar terms:

$3x - x = 3$ (*Not* true)

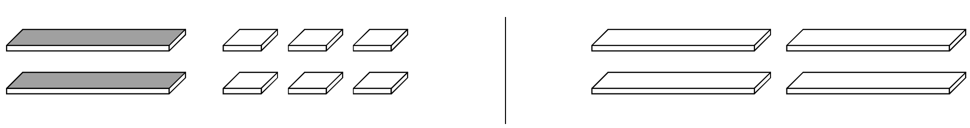
$3x + 6 = 9$ (*Not* true)

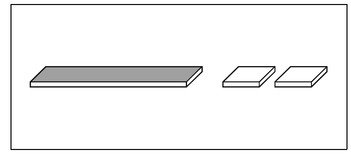
$3x + 6 = 9x$ (*Not* true)

$2x - 6 = {}^-4x$ (*Not* true)

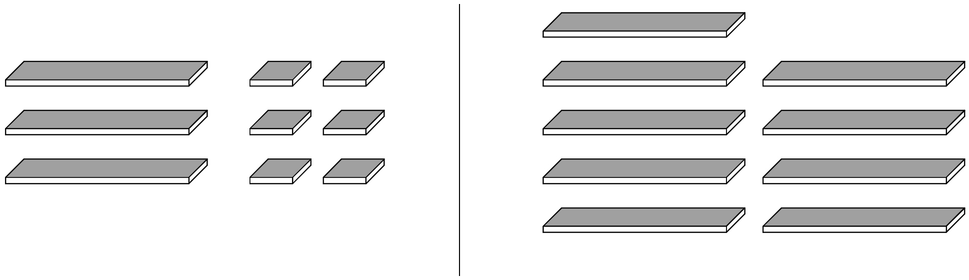
These errors usually occur when students are attempting to manipulate symbols by memorizing rules. When the chips are used, this type of mistake is much more obvious:

$2x - 6$ *is not* ${}^-4x$





$3x + 6$ *is not* $9x$



Exercises

Identify the terms in these expressions:

1. $3x - x + 5$

2. 0

3. $4x + 1 + 1$

4. $1 - 2 + x$

Simplify these expressions by combining similar terms:

5. $3x + (2x)(5) + 1$

6. $4x + 3 + (3)(2x) + 2$

7. $4x + 1 + 6x + 3 + x$

8. $^{-}4x + 3x + 1$

9. $2 + {}^{-}3x + 5x + {}^{-}6$

Evaluate these expressions *before* combining similar terms and *after* combining similar terms. Are the results the same? Do each problem with $x = 4$ and with $x = {}^{-}1$.

10. $3x + 2x + 1$

11. $4x + 3 + 2x + 2$

12. $4x + 1 + 6x + 3 + x$

13. $^{-}4x + 3x + 1$

14. $2 + {}^{-}3x + 5x + {}^{-}6$

Section 4
Expressions and Parentheses

Using Parentheses

Parentheses have the same meaning with unknowns as they do with exact numbers. You will remember that with exact numbers, parentheses indicated an operation or group of operations that were to be done first, before any other operations were done outside of the parentheses.

With unknowns, parentheses still indicate a group of terms that are together, but we cannot always complete the operations indicated because we do not know the value of the unknown term. Three examples are:

$$3(x + 5)$$

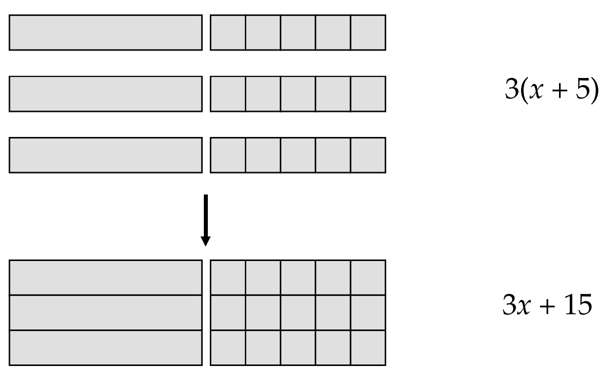
$$5 + 2(x - 3)$$

$$6 + 2(3x + 4) + x$$

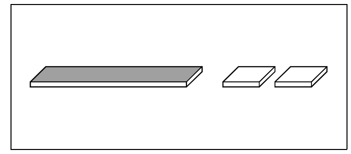
As we can see, the operations inside the parentheses cannot be immediately completed. *The symbols inside the parentheses still represent a group*. We can use the distributive property to finish the multiplication; then we combine similar terms. Here are the same examples worked out:

Example 1: $3(x + 5) = 3(x) + 3(5)$

$= 3x + 15$



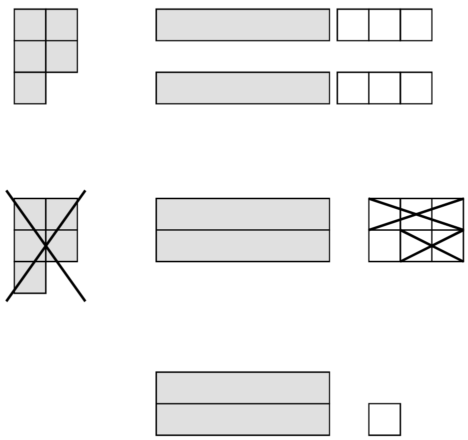
The distributive property shown in this diagram states that three groups of $(x + 5)$ is the same as $3x$ and 15. Three times the whole quantity is the same as three of each term.



Example 2: $5 + 2(x - 3) = 5 + 2(x + {}^{-}3)$

$= 5 + 2x + {}^{-}6$

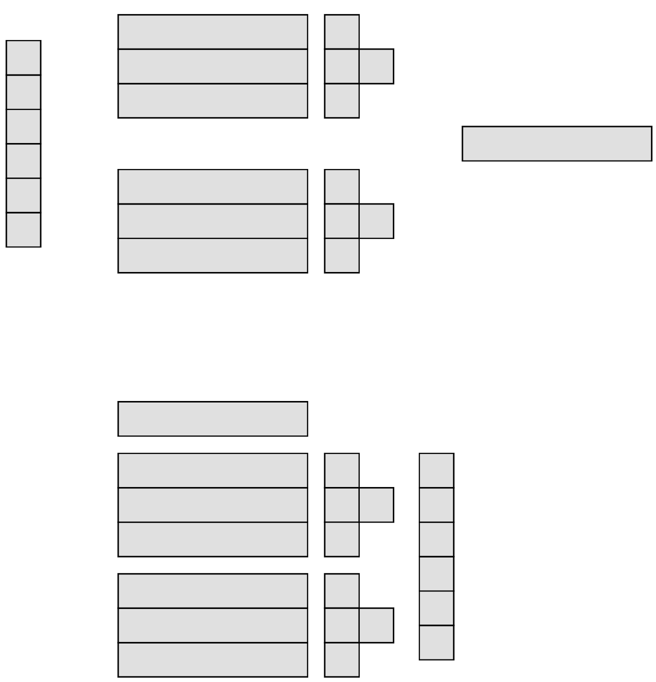
$= 2x - 1$

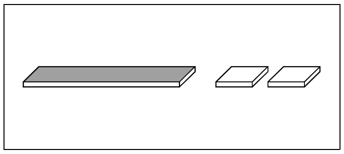


Example 3: $6 + 2(3x + 4) + x = 6 + 2(3x + 4) + x$

$= 6 + 6x + 8 + x$

$= 7x + 14$





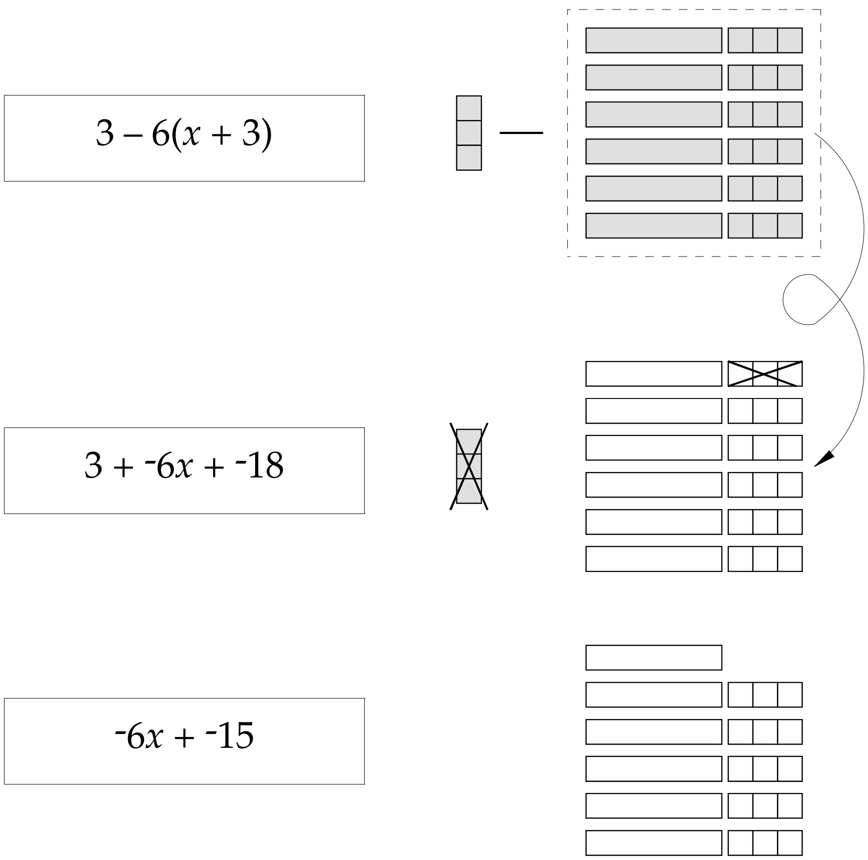
Negative Signs and Multiplication

Consider

$$3 - 6(x + 3)$$

The best way to work with this expression is to rewrite the subtraction as an addition:

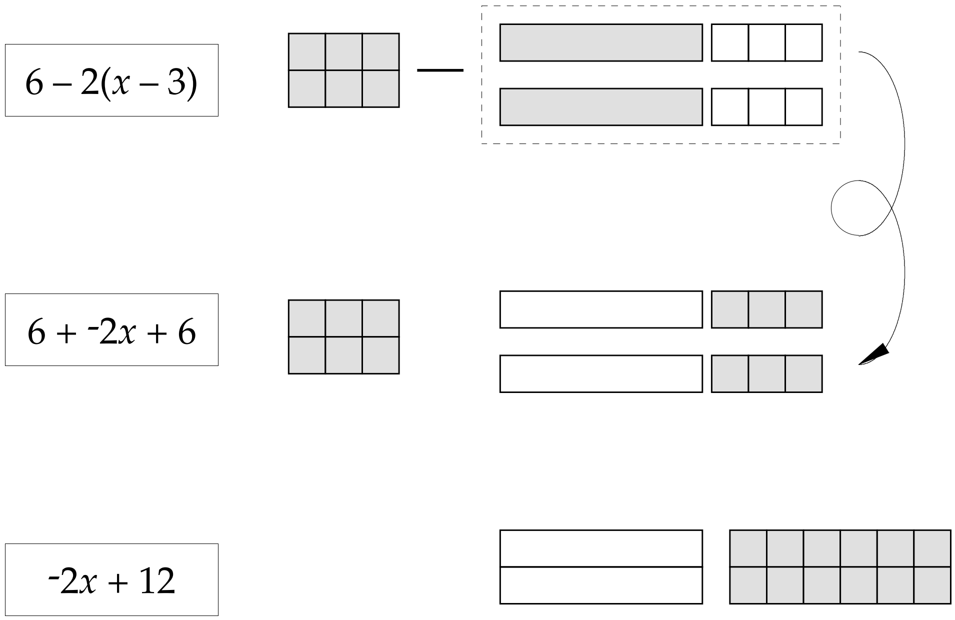
$$\begin{aligned} 3 - 6(x + 3) &= 3 + {}^-6(x + 3) \\ &= 3 + {}^-6x + {}^-18 \\ &= {}^-6x + {}^-15 \end{aligned}$$

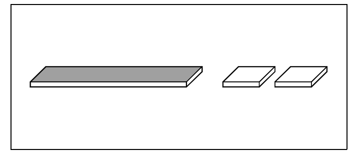


This technique is especially helpful where multiple negative signs are present:

$$\begin{aligned} 6 - 2(x - 3) &= 6 + ({}^-2)(x + {}^-3) \\ &= 6 + {}^-2x + 6 \\ &= {}^-2x + 12 \end{aligned}$$

Here is how this process is shown with the chips:





Summary

To simplify expressions containing parentheses:

- **Rewrite subtractions as additions.**
- **Carry out multiplications using the distributive property.**
- **Combine similar terms.**

Exercises

Simplify the following expressions:

1. $5(x - 3) + 2(3x + 1)$

2. $3 - (x + 5)$

3. $3 - 2(x - 5)$

4. $3 - 2(-x - 5)$

5. $-x - 3x + 4(5 + x)$

Simplify the expressions, then evaluate:

6.	$6x - 2(x + 4)$	$(x = 1)$
7.	$6 - (x - 5) - 3x$	$(x = {}^{-}1)$
8.	$x + 0(196x - 235)$	$(x = 256)$
9.	$2(3x + 2) - 5(3 + x) + 11$	$(x = {}^{-}17)$

Section 5
Expressions Containing Fractions

Fractions and Unknowns

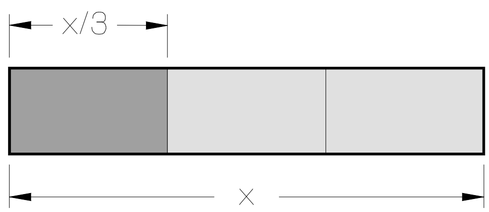
When we wish to represent part of an unknown, we use the same fractional notation that we use with everyday numbers:

$$\frac{1}{3} \text{ of } 6 \text{ means } \frac{1}{3} \cdot 6$$

$$\frac{1}{2} \text{ of } 7 \text{ means } \frac{1}{2} \cdot 7$$

$$\text{So } \frac{1}{2} \text{ of } x \text{ means } \frac{1}{2} \cdot x = \frac{1}{2} \cdot \frac{x}{1} = \frac{1 \cdot x}{2} = \frac{x}{2}$$

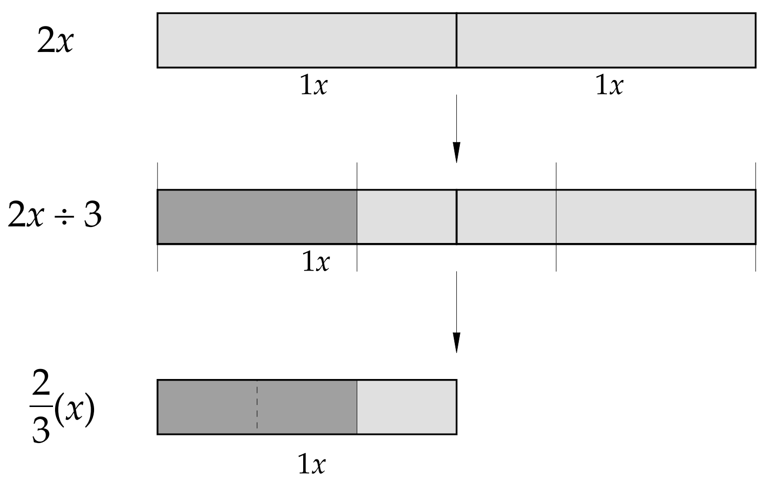
One-third of x would look like this:

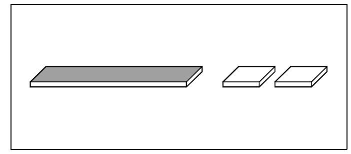


There are often several ways to show these fractional unknowns:

Meaning	Alternative Notations		
³⁄₅ of x	$\frac{3}{5} \cdot x$	$\frac{3}{5}x$	$\frac{3x}{5}$
²⁄₃ of x	$\frac{2}{3} \cdot x$	$\frac{2}{3}x$	$\frac{2x}{3}$
–²⁄₃ of x	$-\frac{2}{3} \cdot x$	$-\frac{2}{3}x$	$\frac{-2x}{3}$

If the alternatives seem to represent different quantities, here is a demonstration of the reasons why $(2x)/3$ is equal to $2/3(x)$:





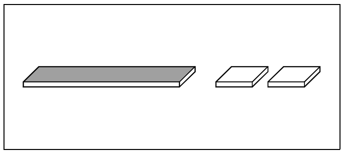
Simplifying Expressions with Fractional Unknowns

Fractional unknowns are simplified in the same way as numbers. When adding, find common denominators and then combine:

$$\begin{aligned}\frac{2x}{3}+\frac{3x}{4} &= \frac{2x}{3}\cdot\frac{4}{4}+\frac{3x}{4}\cdot\frac{3}{3}\\ &= \frac{2x\cdot 4}{3\cdot 4}+\frac{3x\cdot 3}{4\cdot 3}\\ &= \frac{8x}{12}+\frac{9x}{12}\\ &= \frac{8x+9x}{12}\\ &= \frac{17x}{12}\end{aligned}$$

When multiplying, use the same technique that we used with regular fractions:

$$\begin{aligned}\frac{2x}{3}\cdot\frac{3}{4} &= \frac{2x\cdot 3}{3\cdot 4}\\ &= \frac{6x}{12}\\ &= \frac{6}{12}\cdot x\\ &= \frac{1}{2}x\\ &= \frac{x}{2}\end{aligned}$$



Exercises

Simplify these expressions. Find common denominators as needed.

1. $\frac{x}{2}+\frac{x}{4}$
2. $\frac{x}{2}-\frac{3x}{4}$
3. $\frac{1}{2}\cdot\frac{x}{3}\cdot\frac{1}{4}$
4. $\frac{1}{2}\div\frac{3}{x}$
5. $\frac{1}{2}(2x+4)$
6. $\frac{1}{2}\left(2x+\frac{1}{2}\right)$
7. $\frac{1}{2}(6x)$
8. $\frac{2}{3}\cdot\frac{3x}{2}$
9. $\frac{3}{4}\cdot\frac{4x}{3}$
10. $12\left(\frac{x}{2}+\frac{x}{3}+\frac{5x}{6}\right)$
11. $\frac{2x}{3}+\frac{x}{5}$
12. $\frac{3x}{2}+\frac{2x}{3}$
13. $\frac{5x}{2}-x$
14. $\frac{5x}{2}-\frac{x}{3}$
15. $\frac{2}{3}\left(\frac{x}{6}\right)$
16. $\frac{3}{5}\left(\frac{2x}{9}\right)$
17. $\frac{1}{3}(6x-5)$
18. $\frac{3}{5}(10x+5)$

Section 6
Properties of Expressions

Properties of Numbers

All of the familiar associative, commutative, and distributive properties are true for expressions as well as for numbers. Because the unknowns represent numbers, there is usually no need to state separate properties for expressions.

Most of the following ideas are so intuitive that we have already used them without noticing anything new. It may be helpful, however, to restate some of the properties in a more formal manner.

Commutative and Associative Properties

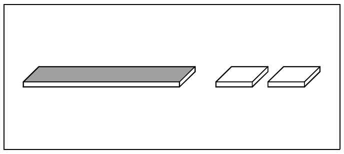
These properties concern the *order* and *grouping* of numbers or terms and are most useful in combining similar terms. For example, consider this illustration from a previous section:

$6x \cdot (^-5)$	$= (6 \cdot x) \cdot (^-5)$	
	$= (x \cdot 6) \cdot (^-5)$	Commutative Prop of Mult
	$= x \cdot 6 \cdot (^-5)$	
	$= x \cdot [6 \cdot (^-5)]$	Assoc. Prop. of Mult
	$= x \cdot (^-30)$	
	$= {^-30} \cdot x$	Comm. Prop. of Mult.
	$= {^-30}x$	

Because the parts of an expression that are being added or multiplied can be rearranged in different orders and groupings, we can more easily combine similar terms.

Commutative and Associative Properties

When *adding* or *multiplying* terms in an expression, the *order* or *grouping* of terms may be changed without affecting the value of the expression.



Distributive Properties

We have easily decided that

$$3x + 4x = 7x$$

More formally, this can be justified by the distributive property:

$$\begin{aligned} 3x + 4x &= 3(x) + 4(x) \\ &= (3 + 4)(x) \\ &= (7)(x) = 7x \end{aligned}$$

We have also been using this property in subtraction problems:

$$\begin{aligned} 3x - 4x &= (3 - 4)x \\ &= ({}^{-}1)x = -x \end{aligned}$$

We can use the idea in division problems and with fractions:

$$\begin{aligned} \frac{6x + 4}{2} &= \frac{6x}{2} + \frac{4}{2} \\ &= 3x + 2 \end{aligned}$$

$$\begin{aligned} \frac{6x - 4}{2} &= \frac{6x}{2} - \frac{4}{2} \\ &= 3x - 2 \end{aligned}$$

Expressions have the same properties as numbers because expressions *represent* numbers.

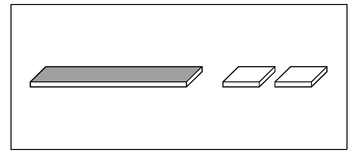
Distributive Properties

$$\mathbf{3x + 4x = (3 + 4)x = 7x}$$
$$\mathbf{3x - 4x = (3 - 4)x = {}^{-}1x}$$
$$\mathbf{\frac{6x - 4}{2} = \frac{6x}{2} - \frac{4}{2}}$$

or for any numbers a, b, and c, (c not zero)

$$\mathbf{ax + bx = (a + b)x}$$
$$\mathbf{ax - bx = (a - b)x}$$
$$\mathbf{\frac{ax - b}{c} = \frac{ax}{c} - \frac{b}{c}}$$

Properties of One and Negative One



For multiplication, one is the identity element. One times any number is the original number. With expressions, the property is of course the same:

$$1 \cdot 3x = 3x$$

$$1 \cdot ({}^{-}5x) = {}^{-}5x$$

$$(7x + 2) \cdot 1 = 7x + 2$$

$$1 \cdot x = 1x = x$$

We have also seen from the POSITIVE AND NEGATIVE NUMBERS chapter that multiplying $^{-}1$ times any number results in the opposite of that number:

$$\begin{aligned} {}^{-}1 \cdot 3x &= {}^{-}1 \cdot 3 \cdot x \\ &= ({}^{-}1 \cdot 3)x \\ &= ({}^{-}3)x \\ &= {}^{-}3x \end{aligned}$$

Finally, it is most important to understand that multiplying $^{-}1$ times x is $-x$:

$$^{-}1 \cdot x = -x$$

Properties of One

$$\mathbf{(1)(3x) = 3x}$$
$$\mathbf{(1)(x) = x}$$
$$\mathbf{({}^{-}1)(x) = -x}$$

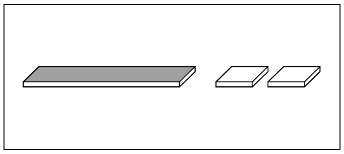
Properties of Zero

When zero is multiplied times an unknown quantity, we are taking an unknown number of zeros. The result is always zero:

$$0 \cdot 3x = 0$$

$$^{-}3x \cdot 0 = 0$$

$$(3x + 6) \cdot 0 = 0$$



Adding zero to any expression does not change the value of the expression:

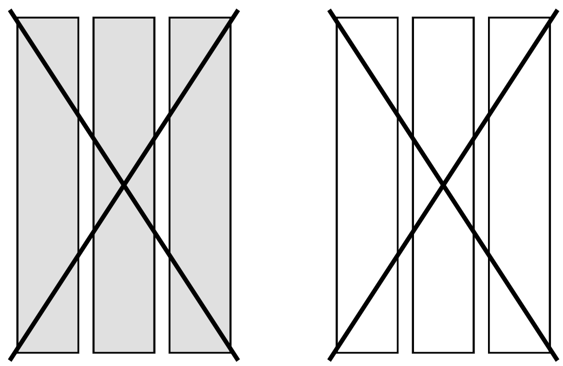
$$0 + 3x = 3x$$

$$^{-}3x + 0 = {}^{-}3x$$

$$0 + (4x - 73) = 4x - 73$$

If we add opposites together, they will cancel to zero:

$$3x + {}^{-}3x = 0$$



A more formal proof of this fact uses the distributive property:

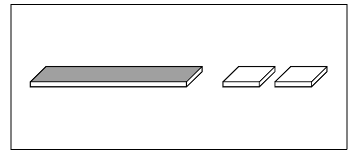
$$\begin{aligned} 3x + {}^{-}3x &= (3)x + ({}^{-}3)x \\ &= (3 + {}^{-}3)x \\ &= (0)x \\ &= 0 \end{aligned}$$

The opposite of any number of x-bars is the same number of -x bars.

Properties of Zero

$$\mathbf{0 + x = x}$$
$$\mathbf{(0)(x) = 0}$$
$$\mathbf{ax + {}^{-}ax = 0}$$

Order of Operations With Multiple Parentheses



When expressions have multiple levels of parentheses then, as before, we begin simplifying starting from the innermost parentheses and working our way out. For example, let's simplify

$$7x + 3[2x - 5(x - 3 - 2x)]$$

$7x + 3[2x - 5(-x - 3)]$ First we *combine like terms* inside the innermost

rentheses.

$7x + 3[2x + 5x + 15]$ Then we *multiply through* the innermost parentheses. Now the inner parentheses are gone.

$7x + 3(7x + 15)$ Next we combine like terms inside the remaining parentheses.

$7x + 21x + 45$ Then we multiply through the remaining parentheses.

$28x + 45$ When all parentheses are gone, combine like terms (if necessary) in the remaining expression.

As with other expressions having unknowns and numbers, the final result usually has two terms (one with a letter and one without) which cannot be combined.

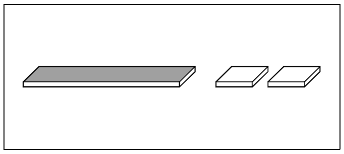
Exercises

Simplify the following expressions. Justify each step by referring to the appropriate property.

Example: $2(3x + 4) - 6x$

Solution:

$2(3x + 4) - 6x =$	$2(3x) + 2(4) - 6x$	Distributive
$=$	$6x + 8 - 6x$	
$=$	$8 + 6x - 6x$	Commutative
$=$	$8 + 0$	Inverses
$=$	8	



1. $5(x + 6) + x$

2. $2 - (3 + 2x)$

3. $-x(3) + 3x + 22 + x$

4. $9y - 3y + (6)(-y)$

5. $\frac{1}{3}x + \frac{2}{3}x$

6. $2 - 5(3 + x)$

7. $^{-}2 - 6(3 - x)$

8. $4x + 2x + {}^{-}2x + {}^{-}4x$

9. $\frac{6x + 2}{2} - 1$

10. $\frac{^{-}3x - 6}{^{-}3} + x$

11. $\frac{8x}{2} + \frac{9x}{3}$

12. $\frac{12x + 6}{6} + x + 1$

13. $(2 - 1)(3x)$

14. $-x + 4(x - 3)$

15. $\frac{4x + 4}{4}$

16. $\frac{6x - 6}{6}$

17. $\frac{6x + 1}{6}$

18. $(^{-}1)(5x)$

19. $(5)(7x)$

20. $(7x)(5)$

21. $(^{-}7x)(^{-}5)$

22. $(7x)(^{-}5)$

23. $(0)(3)(12)(6x)$

24. $24(3 - 3)(6x)$

25. $\frac{(6x)(0)}{6}$

26. $5[3x - 2(x + 7)]$

27. $4[3(2 - 3x) + 6x]$

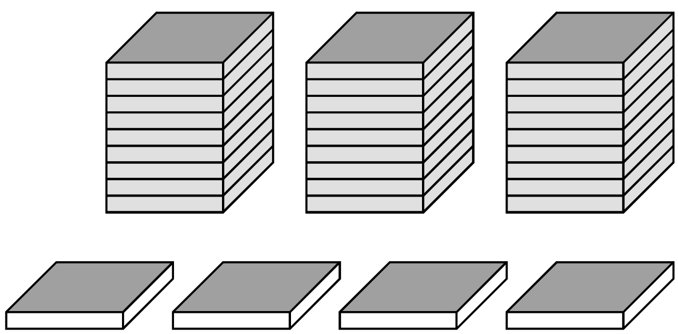
28. $7 - [2x + 5(6 - 3x)]$

29. $^{-}3x + 4[6 + 3(2x - 9)]$

30. $4x + 2[5 - 2(3x - 7)]$

Chapter 3

Equations



Section 1
Introduction to Equations

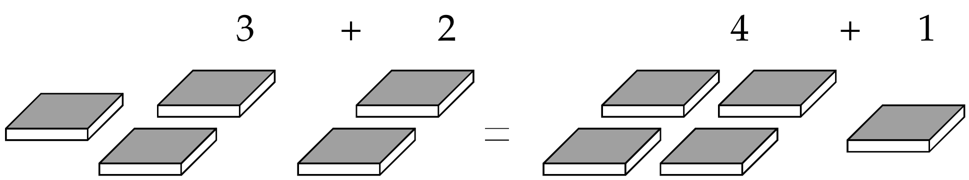
Equations

An **equation** is a number statement which says that two quantities are exactly the same. The symbol **=** (equals) is used between the quantities to show that the amount on the left is the same as the amount on the right. For example:

$$3 + 2 = 4 + 1.$$

Both the numbers on the left and the numbers on the right can be combined to give 5. So the equation really says

$$5 = 5 \quad \text{or} \quad \text{“Five equals Five”}$$



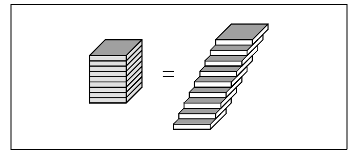
This is obvious if we know all of the numbers on both sides of the equation, but what if the equation has unknowns? When unknowns are included, we can use the fact that both sides are equal to find out the missing amount.

Equations versus Expressions

In the previous chapter, we worked with expressions that involved unknowns. *An expression is a quantity, while an equation is a statement that two quantities are equal.*

Expressions	Equations
$3x + 6$	$x + 3 = 5$
$^{-}17$	$2x - 3 = 15$
$3(5x - 4) + 17x$	$3(5x - 4) + 17x = 20$

An expression can stand for many different amounts, depending on what we choose for the unknown; in an equation, the unknown can only stand for numbers that make both sides of the equation equal. Here is a summary of the differences between equations and expressions:



	Expressions	**Equations**
Quantities	One quantity	Two amounts that are equal
Equals Sign	No	Yes
Meaning of Unknown	Many choices	Values that make the statement true

Exercises

Decide whether each item is an expression or an equation:

1. $x + 3$

2. $2(x - 5) + 7$

3. $0 = 0$

4. $2(x - 5) = 2$

5. $2(x - 5) = 2(x - 5)$

6. $\frac{1}{6}$

7. $\frac{3x}{5} = 12$

8. $3(x + 5) - 16x + 23$

9. $^{-}1$

10. 0

11. $0 = 0$

12. $3x + 2 = {}^{-}1$

13. $\frac{3x + 12}{x + 16}$

14. $x = y$

15. $y = 1$

Section 2
The Equation Game

Introduction

This game will help you to understand the meaning of equations and the methods by which they can be solved. As in many of the other sections of this book, you may find that you can easily discover the techniques of solving equations; in fact, you may already know a great deal about the subject.

The Rules of the Game

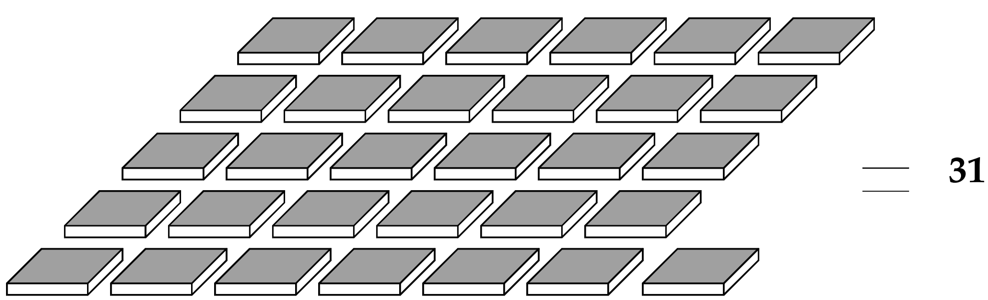
The game can be played alone or with a partner. You can pose equations and solve them yourself, or you can set up equations for your partner to solve.

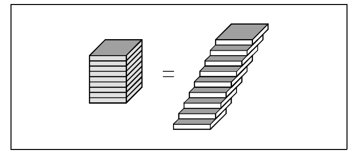
- **Begin by counting out any number of chips and writing the total in large numerals on one-half of a clean sheet of paper. A number between 10 and 50 chips works best.**

- **Divide up most (but usually not all) of the chips into a small number of stacks which are exactly equal in height.** *If you are playing alone, do not count the number of chips in a stack; you can tell if the stacks are equal by feeling the height of the chips.* **Place these stacks on the other half of the paper with the remaining chips arranged singly next to the stacks.**

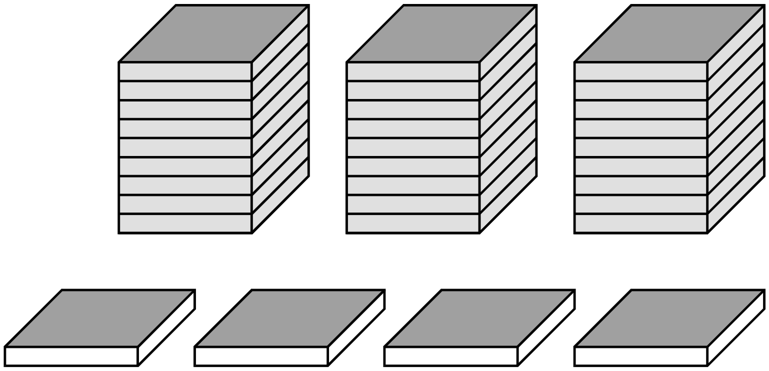
- **Without counting, determine how many chips are in a stack. We know that the total number of chips (stacks and single chips) equals the number written on the paper; use this information to discover how many chips are in a stack. Check the result by counting chips in the stack. If you are not correct, check that the stacks were the same height and that the total number of chips is correct.**

For example, count out 31 chips and write **31** on the paper.



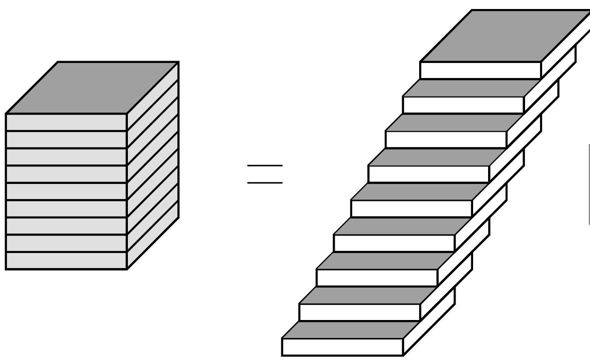


Lay out 4 chips singly and arrange the other chips into 3 equal stacks.



= 31

Calculate the number of chips in each stack. Thirty-one (31) chips minus the 4 extra gives 27 chips, and 27 divided into 3 equal stacks is 9. Check your answer by counting the chips in a stack.

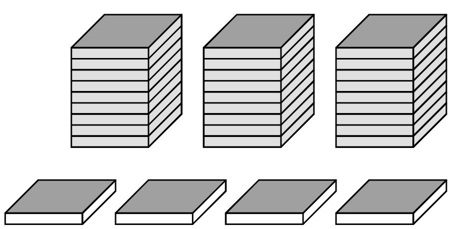


$x = 9$

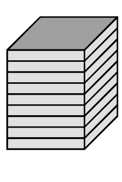
This process is called **solving an equation**. We write the equation as

$$3x + 4 = 31$$

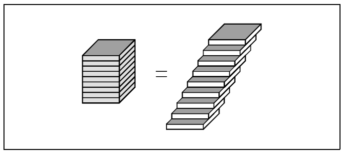
where ***x*** is a stack, **3*x*** is 3 stacks, and **4** is the 4 single chips.



= 31



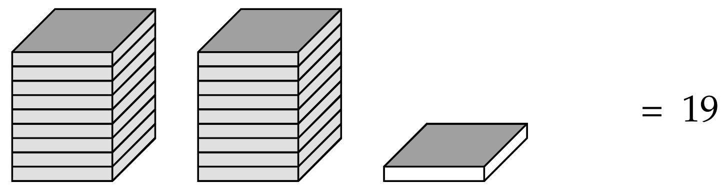
= 9



Exercises

Here are some sample equation games to play. Set up the chips, calculate the solution, and check your answer by counting the chips in a stack.

Example: $2x + 1 = 19$
Solution: $x = 9$



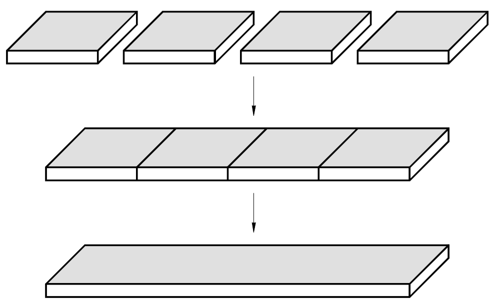
1. 23 chips: 4 stacks and 3 singles. ($4x + 3 = 23$)
2. 17 chips: 2 stacks and 1 single. ($2x + 1 = 17$)
3. 35 chips: 4 stacks and 3 singles.
4. 12 chips: 1 stack and 5 singles.
5. 29 chips:
   (You arrange stacks and singles. All stacks are the same height.)
6. 32 chips: 3 stacks and 11 singles.
7. 23 chips: 2 stacks and 5 singles.
8. 27 chips: 4 stacks and 7 singles.
9. 27 chips: 6 stacks and 3 singles.
10. 27 chips: 4 stacks and 3 singles.
11. 18 chips: 3 stacks and 3 singles.
12. 41 chips: 3 stacks and 5 singles.
13. 32 chips: 4 stacks and 4 singles.
14. 51 chips: 5 stacks and 6 singles.
15. 40 chips: 3 stacks and 1 single.
16. 47 chips: 2 stacks and 5 singles.
17. 38 chips: 5 stacks and 3 singles.
18. 50 chips: 6 stacks and 2 singles.
19. 35 chips: 4 stacks and 3 singles.
20. 29 chips: 3 stacks and 2 singles.

Section 3
Equations Using Unknowns

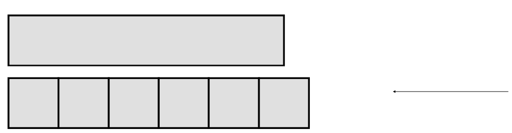
Using the Bar as *x*

As we learned in the last chapter, we can also represent an unknown amount with the long bar found in your packet.

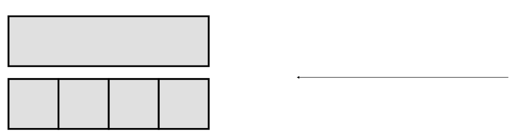
Instead of a stack of chips in a pile, the bar represents a group of chips in a line:



The bar *represents any unknown number of chips; you can imagine that it changes length in each example:*

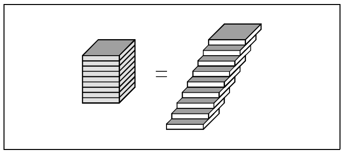


Actual Proportions: The bar is 5 ½ units long.



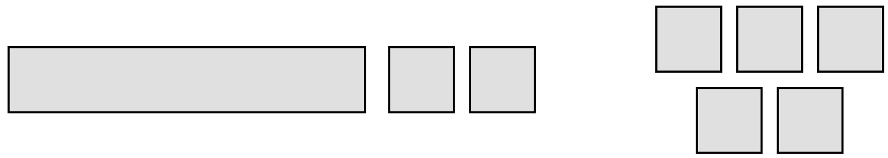
But it may stand for a quantity of 4 units.

Or it may stand for 2.



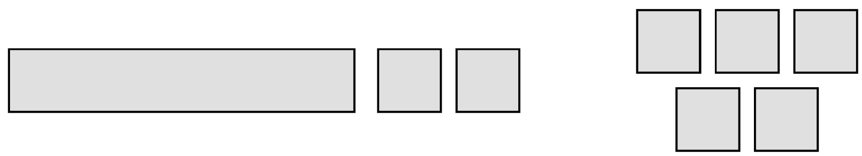
Equations Using Bars

In the following equation, what number does the bar represent?

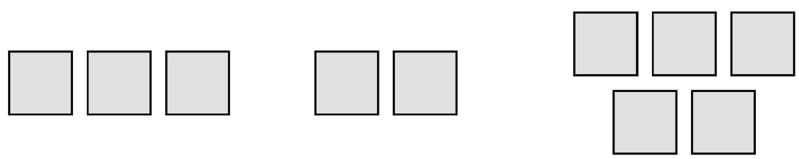


$$x + 2 = 5$$

We are trying to find out what number of chips are needed to replace the bar so that both sides of the equation are equal. The answer is 3, so the bar represents three chips:



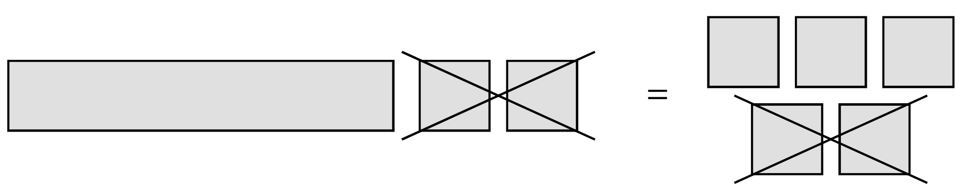
$$x + 2 = 5$$



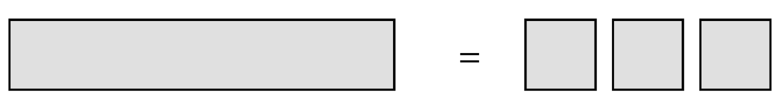
$$3 + 2 = 5$$

The best way to find the answer is to take chips away from each side until one side has only the bar left. In this case, we take 2 chips away and the bar must then be equal to 3 chips.

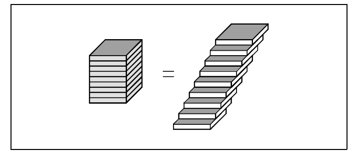
Take 2 chips away from each side.



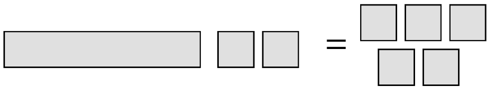
This leaves the bar equal to 3.



To check, replace the bar with 3 chips and make sure that there are equal numbers of chips on both sides.

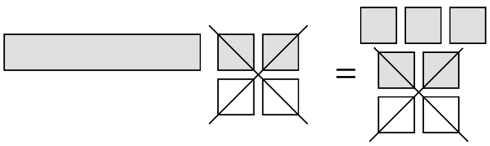


Another way to solve this equation is to add 2 negative chips to each side. Here is the process along with the algebra symbols that we will use:



$x + 2 = 5$

Represent the equation using a bar (unknown) and unit chips.



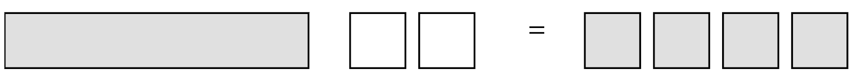
$x + 2 + {}^{-}2 = 5 + {}^{-}2$

Add ⁻2 to both sides.

$x = 3$

This leaves the solution: the value for the unknown.

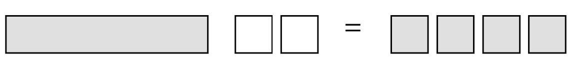
Here is a slightly different example:



$y + ({}^{-}2) = 4$

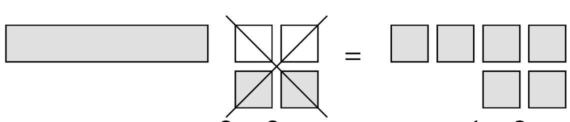
In order to find the value of the unknown (called ***y*** for variety) we look for a number that, when combined with ⁻2, becomes 4. The answer is ⁺6.

To solve this more easily, we can work to isolate the *y* bar by *adding* 2 positive chips to each side. This will cancel the negative chips and will help us discover the answer:



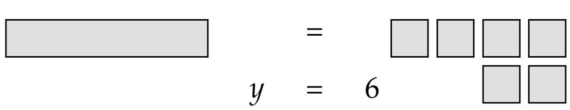
$y - 2 = 4$

Represent the equation using a bar (unknown) and unit chips.



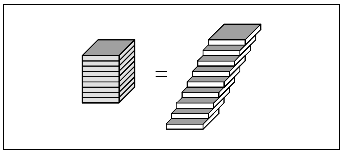
$y - 2 + 2 = 4 + 2$

Add ⁺2 to both sides.



$y = 6$

This leaves the solution: $y = 6$



Remember that

$$6 + {}^{-}2 = 6 - 2$$

You can use either form, but with chips it is often easier to represent the idea of adding ${}^{-}2$.

To find the solution:

- **"Isolate" the unknown by adding unit chips to both sides so that the chips other than the bar are cancelled out.**
- **Use positive chips to cancel negative chips, and negative chips to cancel positive chips.**
- **You are done when you have the bar alone, equal to a number of chips.**

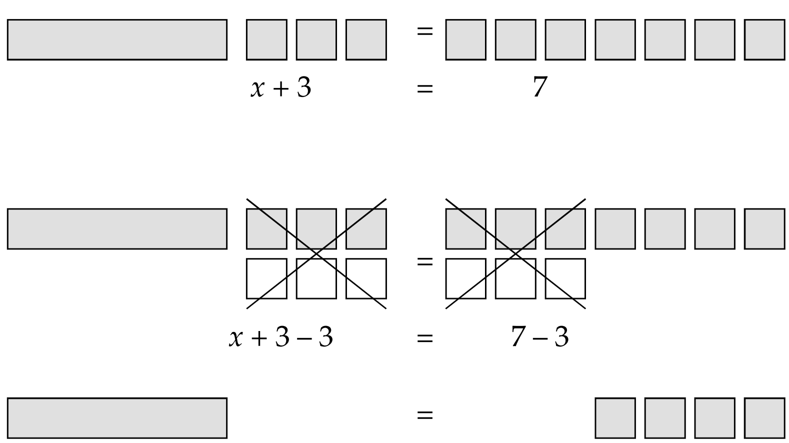
To cancel out units:

Add the Opposite

Exercises

Practice on these examples. Use the chips and also write out the algebra symbols for each problem.

Example: $x + 3 = 7$
Solution:



$x = 4$

1. $x + {}^{-}4 = 5$

2. $x + {}^{-}2 = {}^{-}3$

3. $y + 5 = 2$

4. $n - 4 = {}^{-}1$

5. $y + 2 = 2$

6. $x - 7 = 5$

7. $11 + x = 12$

8. $3 + y = {}^{-}13$

9. $1 + y = 0$

10. $x - 12 = 11$

11. $x + 12 = {}^{-}3$

12. $y - ({}^{-}3) = 5$

13. $y + ({}^{-}3) = 5$

14. ${}^{-}2 + x = 13$

15. $x + 0 = 0$

16. $x - 5 = 12$

17. $x + 5 = 12$

18. $y - 2 = {}^{-}3$

19. $y + 2 = {}^{-}3$

20. $5 + x = {}^{-}2$

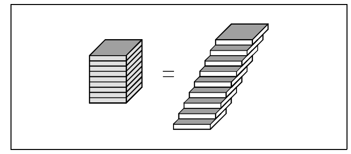
21. $7 + y = 4$

22. $n + 6 = 5$

23. $n - 3 = 5$

24. $x - 7 = {}^{-}7$

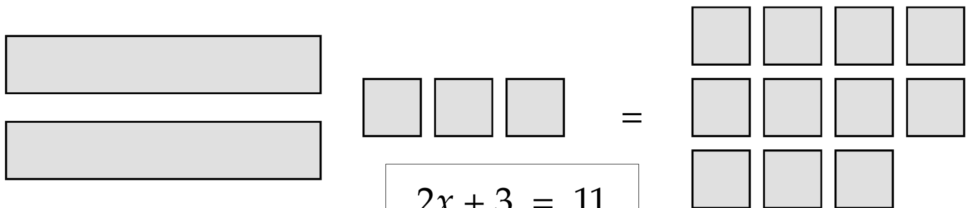
25. $x + 7 = {}^{-}7$



Section 4
Equations with Multiples of Unknowns

More than One Unknown

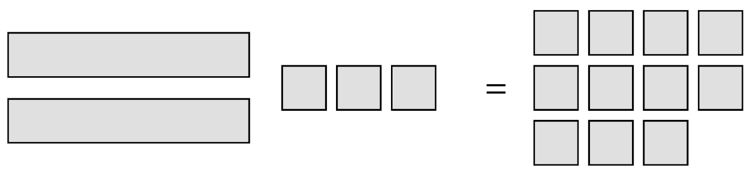
An equation may be more complicated than those that we have looked at thus far. For example, an equation may contain more than one x:



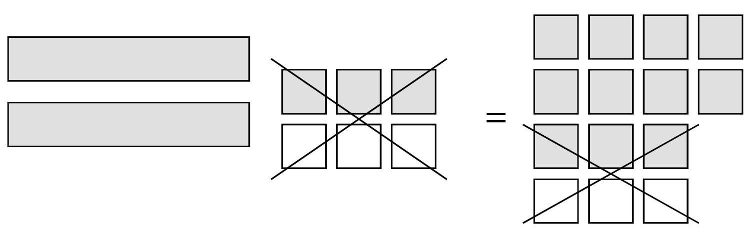
$$2x + 3 = 11$$

The first step in solving the equation is the same as for the simpler equations—we add $^-3$ to each side to isolate the unknowns. This gives:

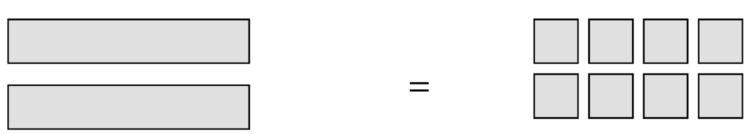
$$2x + 3 = 11$$



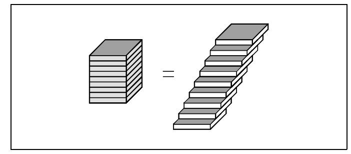
$$2x + 3 - 3 = 11 - 3$$

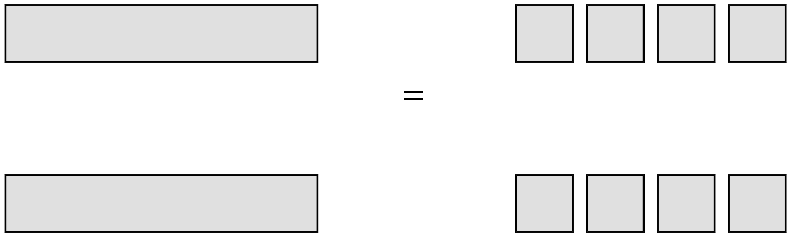


$$2x = 8$$

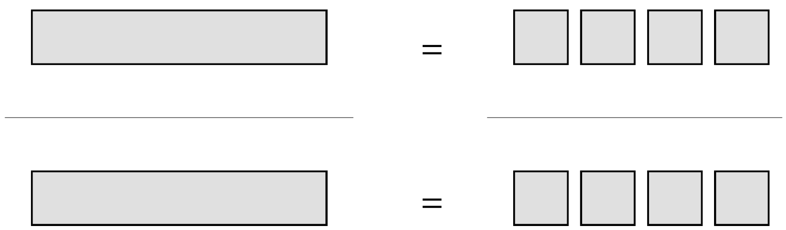


Although we now know what $2x$ is, we would like to know the value of x itself. Because the two sides are equal, we can split up the x bars into 2 groups and the unit chips into 2 groups.

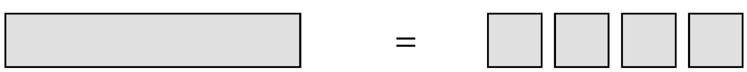




If the two sides are equal, then we can match up half of one side with half of the other side:



$$\frac{1}{2}(2x) = \frac{1}{2}(8)$$

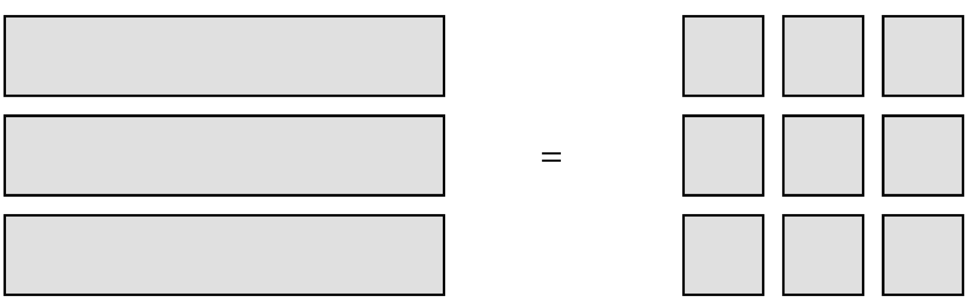


$$x = 4$$

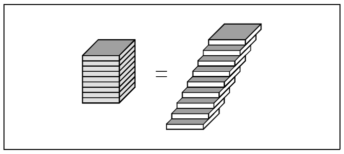
Our solution is 4.

Now we will do another example. Consider

$$3x = 9$$

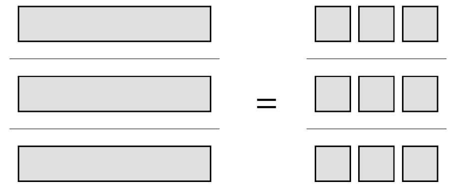


$3x \quad = \quad 9$



We divide each side into 3 groups and match up one group on each side giving x equal to 3:

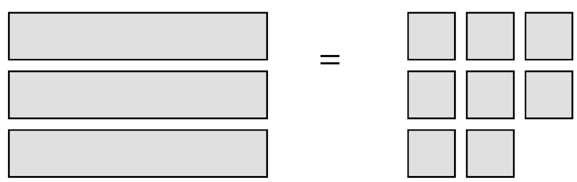
$$\frac{1}{3}(3x) = \frac{1}{3}(9)$$



$$x = 3$$

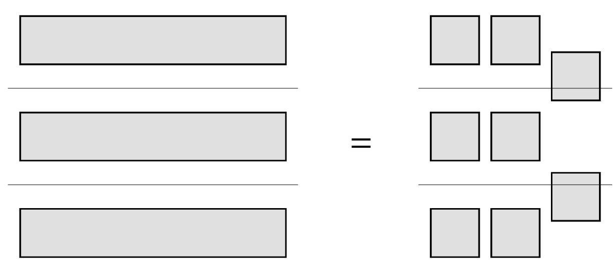
Equations that Result in Fractions

Sometimes an equation will result in a situation where the chips cannot be divided evenly into the desired number of groups. In these cases, the answer will contain a fraction. For example, $3x = 8$:

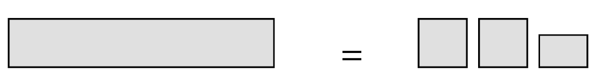


When we divide both sides into thirds, some chips on the right side must be cut to get 3 equal parts. The result is that $x = 2\,^2\!/_3$:

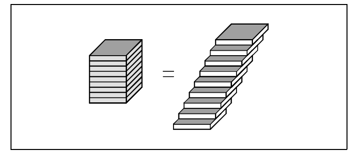
$$\frac{1}{3}(3x) = \frac{1}{3}(8)$$



$$x = 2\frac{2}{3} = \frac{8}{3}$$



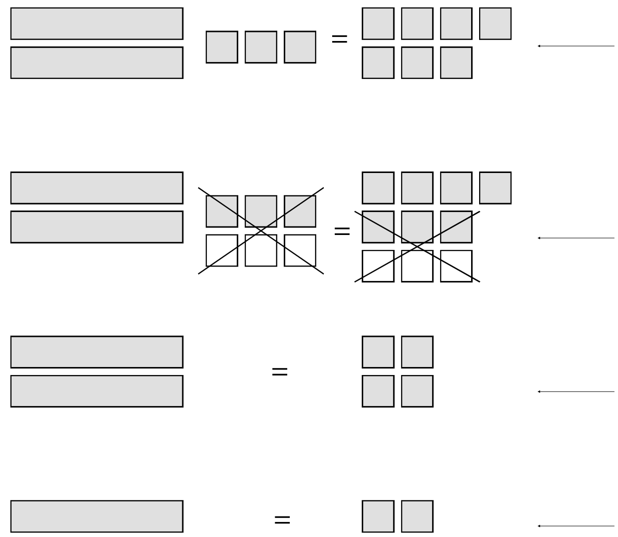
Exercises



Use chips to solve these equations. Write out the algebra steps for each problem:

Example: $2x + 3 = 7$

Solution:



$$2x + 3 = 7$$

$$2x + 3 - 3 = 7 - 3$$

$$2x = 4$$

$$x = 2$$

1. $3x + 5 = 17$

2. $3x + 4 = {}^{-}17$

3. $4x - 3 = 5$

4. $5x + 2 = 11$

5. $2y - 9 = {}^{-}5$

6. $6n - 2 = 3$

7. $2b + 5 = 5$

8. $5x + 1 = 11$

9. $2 + 3x = 35$

10. $3x - 2 = 8$

11. $6 + 3y = 21$

12. $0 + 2x = 0$

13. ${}^{-}3 + 5x = {}^{-}13$

14. ${}^{-}2 + 4x = {}^{-}10$

15. $3y - 12 = 12$

16. $7n + 7 = 8$

17. $3x + 1 = 5$

18. $5x = 3$

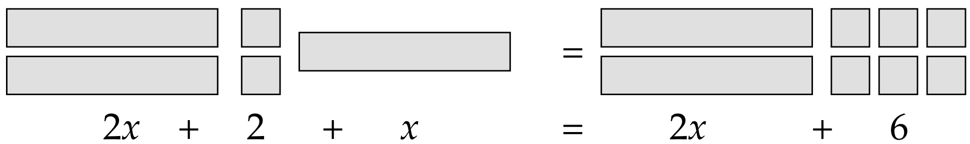
19. $2x - 1 = 0$

Section 5
Unknowns in More than One Term

Keeping the Equation Balanced

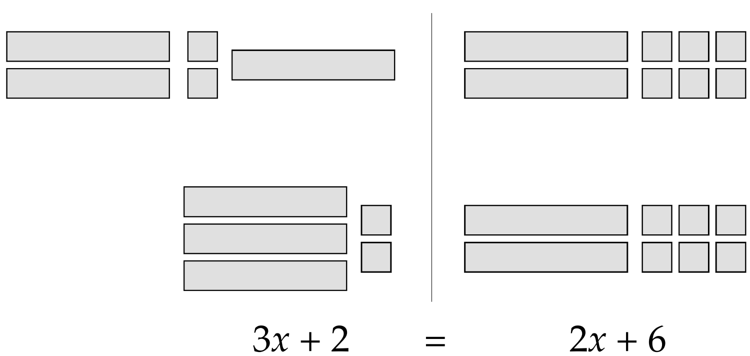
An equation may have unknowns in several places—on one side of the equation or on both sides. Consider the following equation:

$$2x + 2 + x = 2x + 6$$

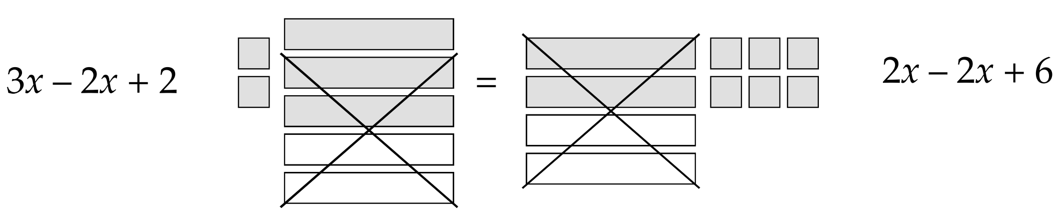


In an equation like this it is important to notice the position of the equals sign because it separates the left and right sides. The equation is like a balance and the amount on the left exactly balances the amount on the right.

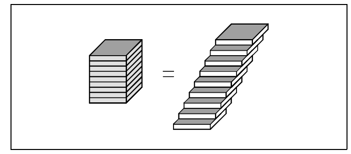
Our first step is to combine like terms on each side:

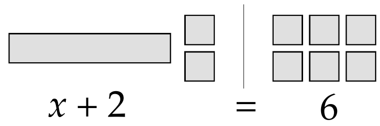


The next step is to add or remove unknowns from both sides. *We must add or remove equally on both sides or the equations will not remain balanced.* Since the right side has less unknowns, we can cancel out these by adding two negative bars to both sides:

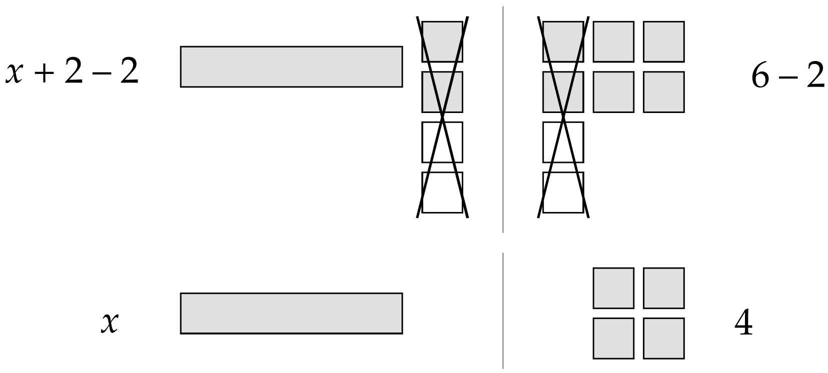


This leaves us with unknowns on one side only. When we combine similar terms, we are left with:



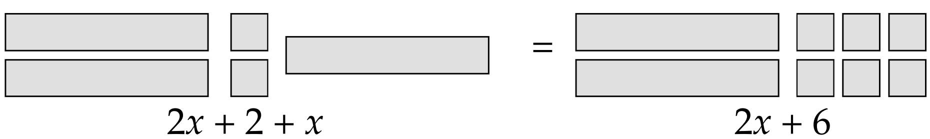


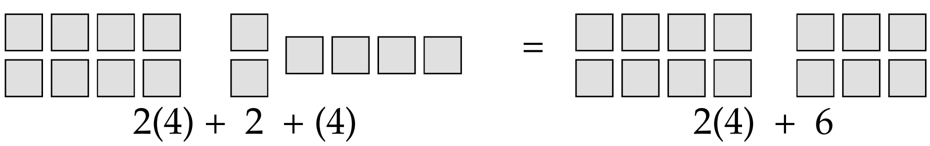
From this point, we can solve the equation exactly as before:



$x = 4$

Our solution is that $x = 4$. To check our answer, we replace each x bar on both sides of the equation with 4 chips and then confirm that both sides have a balanced (equal) number of chips:

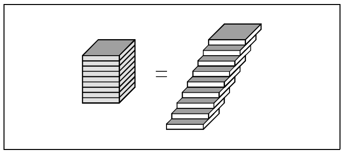




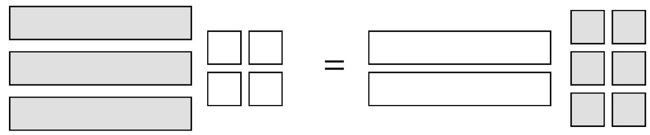
$14 = 14$

To summarize these steps:

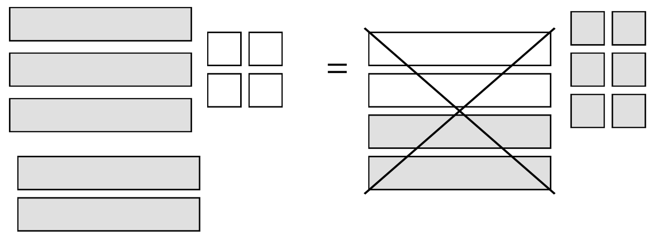
- **Combine similar terms on each side of the equation.**
- **Eliminate the unknowns from one side by adding the opposite type of bars. Add negatives to eliminate positives, and add positives to eliminate negatives.**
- **Add positive or negative chips to cancel out the units and to "isolate" the unknown.**
- **Multiply both sides by ½, ⅓, etc. to match up a single unknown with the correct number of chips.**



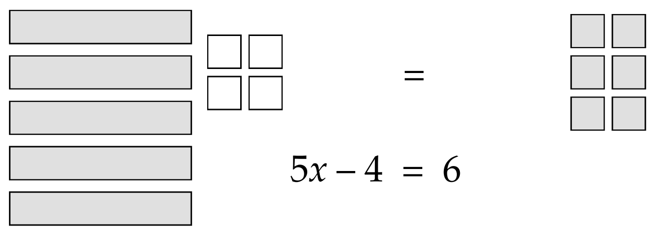
Here is another example:



$$3x - 4 = {}^{-}2x + 6$$

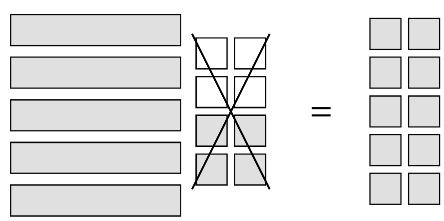


$$3x + 2x - 4 = {}^{-}2x + 2x + 6$$

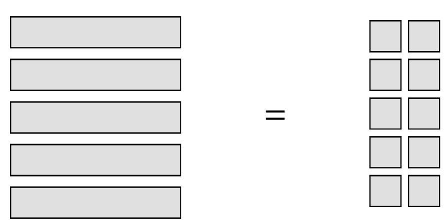


$$5x - 4 = 6$$

We can now solve as before:

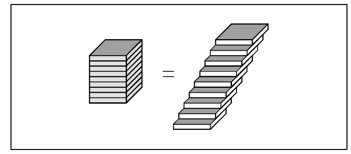


$$5x - 4 + 4 = 6 + 4$$



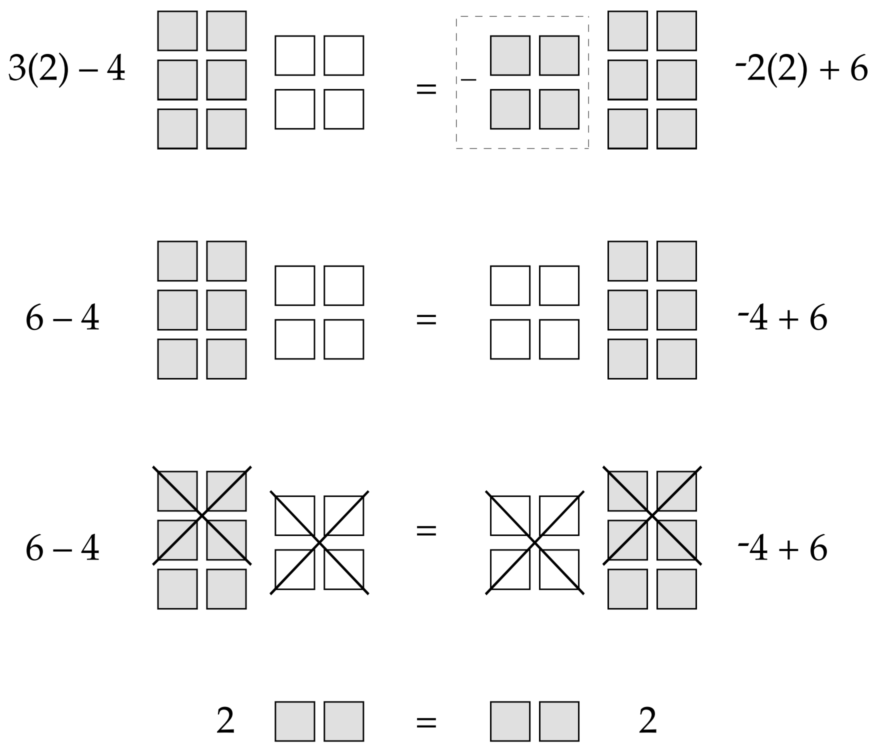
$$5x = 10$$

$$x = 2$$



To check our result of $x = 2$, we replace the x's with 2 chips and the $-x$'s with $^{-}2$ chips:

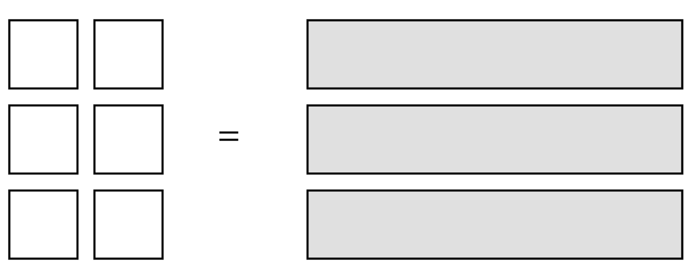
$$\begin{aligned} 3x - 4 &= {}^{-}2x + 6 \\ 3(2) - 4 &= {}^{-}2(2) + 6 \\ 6 - 4 &= {}^{-}4 + 6 \\ 2 &= 2 \end{aligned}$$



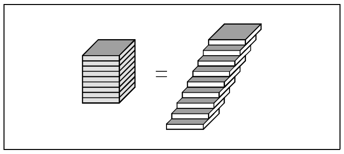
$$2 = 2$$

Unknowns on the Right Side of the Equation

When we isolate the unknowns, the unknowns may be on the right side of the equation instead of on the left.



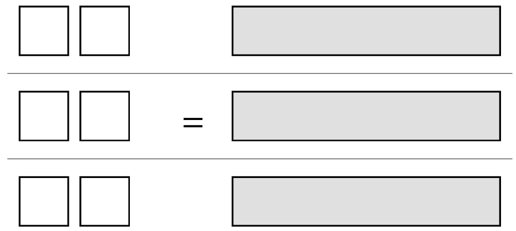
$$^{-}6 = 3x$$



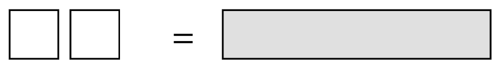
Because

$$^{-}6 \;=\; 3x \quad \text{has the same meaning as} \quad 3x \;=\; ^{-}6$$

we do not have to swap the sides of the equation. Instead, we continue solving in the usual way:



$$\frac{1}{3}(^{-}6) \;=\; \frac{1}{3}(3x)$$

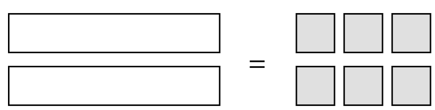


$$^{-}2 \;=\; x$$

Negative Unknowns

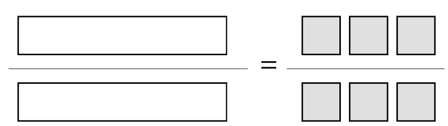
In the final step of solving an equation, we may be left with negative unknowns:

$^{-}2x \;=\; 6$

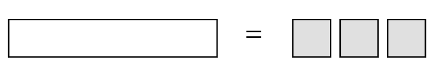


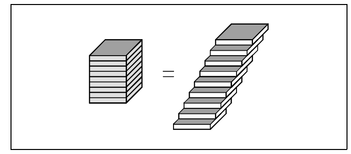
We can use our usual method to isolate the negative *x* by multiplying both sides by one-half:

$\frac{1}{2}(^{-}2x) \;=\; \frac{1}{2}(6)$

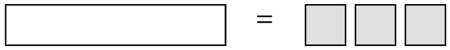


$-x \;=\; 3$





But now we have the value of the *opposite* of x instead of the value of x itself. It is clear that if the opposite of x is 3, then x is ${}^{-}3$. We can show this physically by flipping all of the chips on both sides:



$-x = 3$

$x = {}^{-}3$

This keeps our equation balanced, because *if two quantities are equal, their opposites are also equal.* With symbols, it is often written as:

$$-x = 3$$
$${}^{-}1(-x) = {}^{-}1(3)$$
$$x = {}^{-}3$$

Multiplying both sides by negative one can be shown as flipping the chips on both sides.

Exercises

Do these problems using the chips. Write out the steps.

1. $3x + 5 - x = x - 6$

2. $2x - 4 + x = -x + 8$

3. ${}^{-}2y - 2 + y = 2y + 7$

4. $6 - 3n = n + 5 - 3$

5. $4y - 3 = {}^{-}3 - 6y$

6. $1x + 2x + 3x = 4x + 4$

7. $4 - z - 2z - 3z = {}^{-}20$

8. $6x = 2x - 12$

9. $7x - 5x + x = 14 + x$

10. $10 + x = {}^{-}12x + 5x - 6$

11. $9y + 2 = 6y - 4$

12. $-x = 5$

13. $-x = {}^{-}3$

14. ${}^{-}7x = {}^{-}3x$

15. ${}^{-}2x + 6 = x - 9$

16. $-x + 6 = -2x + 2$

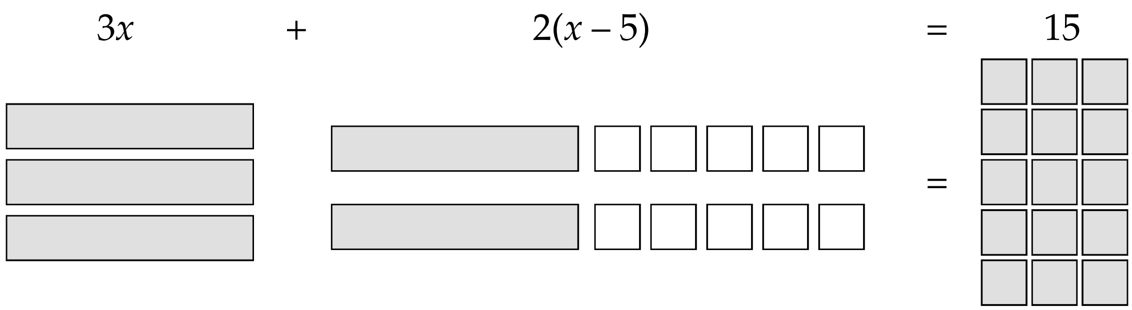
17. $-x + 1 = 5x + 1$

Section 6
Equations with Parentheses

Using the Distributive Property

Some equations may contain complicated expressions including parentheses. For example:

$$3x + 2(x - 5) = 15$$



Notice that the 2 is not a term. It cannot be eliminated by adding $^{-}2$ to both sides. In most cases, it is necessary to use the distributive property to multiply out the expression **2(*x* – 5)** so that we can combine terms and proceed with the solution of the equation:

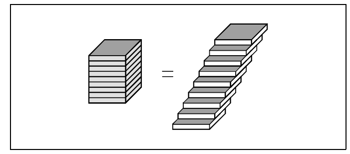
$$\begin{aligned} 3x + (2 \cdot x) - (2 \cdot 5) &= 15 \\ 3x + 2x - 10 &= 15 \\ 5x - 10 &= 15 \\ 5x &= 25 \\ x &= 5 \end{aligned}$$

Consider the equation:

$$5 + 3(x + 2) = 2(x + 1) + 12$$

Again, we cannot eliminate any of the parts of $3(x + 2)$ or $2(x + 1)$ until we multiply out these expressions using the distributive property:

$$\begin{aligned} 5 + 3(x + 2) &= 2(x + 1) + 12 \\ 5 + 3x + (3 \cdot 2) &= (2 \cdot x) + (2 \cdot 1) + 12 \\ 5 + 3x + 6 &= 2x + 2 + 12 \end{aligned}$$



From here on, the procedure is the same as in the previous section:

$$\begin{aligned} 5 + 3x + 6 &= 2x + 2 + 12 \\ 11 + 3x &= 2x + 14 \\ 11 + 3x - 11 &= 2x + 14 - 11 \\ 3x &= 2x + 3 \\ 3x - 2x &= 2x + 3 - 2x \\ x &= 3 \end{aligned}$$

Subtraction and the Distributive Property

When the product of two amounts is *subtracted* in an equation, we must be careful to use signed numbers correctly. Consider the equation:

$$5 - 2(x + 1) = 1$$

This is not the same as:

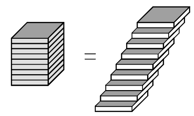
$$5 - 2x + 1 = 1 \quad \text{(Why not?)}$$

Instead, rewrite the subtraction as addition:

$$\begin{aligned} 5 - 2(x + 1) &= 1 \\ 5 + {}^{-}2(x + 1) &= 1 \\ 5 + ({}^{-}2 \cdot x) + ({}^{-}2 \cdot 1) &= 1 \\ 5 + {}^{-}2x + {}^{-}2 &= 1 \\ 3 + {}^{-}2x &= 1 \\ {}^{-}2x &= {}^{-}2 \\ x &= 1 \end{aligned}$$

If there are two subtractions, you may want to rewrite both as addition:

$$\begin{aligned} 10 - 3(2x - 5) &= 19 \\ 10 + {}^{-}3(2x + {}^{-}5) &= 19 \\ 10 + ({}^{-}3 \cdot 2x) + ({}^{-}3 \cdot {}^{-}5) &= 19 \\ 10 + {}^{-}6x + 15 &= 19 \\ {}^{-}6x + 25 &= 19 \\ {}^{-}6x &= {}^{-}6 \\ x &= 1 \end{aligned}$$



Summary

We can now add an initial step to our plan from the previous section:

- ***Use the distributive property to complete any multiplications of expressions in parentheses.***

- **Combine similar terms on each side of the equation.**

- **Eliminate the unknowns from one side by adding the opposite type of bars. Add negatives to eliminate positives, and positives to eliminate negatives.**

- **Add positive or negative chips to cancel out the units and to "isolate" the unknown.**

- **Multiply both sides by ½, ⅓, etc. to match up a single unknown with the correct number of chips.**

Exercises

Solve these equations using chips and algebra symbols:

1. $4(x + 1) - 3 = 3(x - 2) + 13$

2. $5 - 2(x - 2) = 5(x + 1) + 4$

3. $3(2x + 1) - 2(x - 1) = 21$

4. $6(1 - x) + 3 = {}^{-}3x - 3$

5. $3(3 + 2x - 1) = 2x - 1(3 - 2x) + 9$

6. $5(x + 1) - 4x = 2$

7. $2x + 3(x - 2) = 9x + 2$

8. $7y - 6(y - 1) + 3 = 9$

9. $3 - y(1 + 2) = {}^{-}2y - 5$

10. $2(3x + 1) - 5(x + 2) = 1 - 10$

11. $3(7 - 2x) = 14 - 8(x - 1)$

12. $1 - 3(x + 4) = {}^{-}5x - 5$

13. $1 - 1(x - 1) = x$

14. $5x + 6x + 3(x - 2) = {}^{-}6 + x$

15. $3x + 2(3x - 1) = 6x$

16. $5(x - 2) - 3x = 6$

17. $2x - 9 = 3(2 - x)$

18. $5 - 2(1 - x) = 3(x - 4)$

19. $2(4x + 3) - 3(x - 2) = 3x + 8$

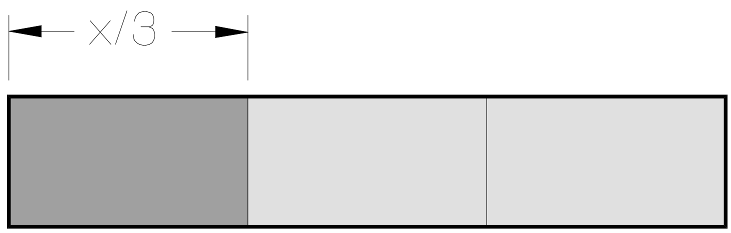
20. $8x - 3(2x + 5) = x - 4$

Section 7
Equations with Fractions or Decimals

Simplifying Equations with Fractions

If fractions occur in an equation, there is an easy technique for creating an equivalent equation without fractions. For example:

$$\frac{x}{3} + 4 = x$$



First, think of $x/3$ as x divided into 3 pieces, or one-third of x.
Although equations of this type can be solved in the usual way by subtracting parts of x from both sides, it is usually easier (with symbols or chips) to multiply both sides of the equation by a number so that the resulting equation has no fractions. *Multiplying both sides of an equation by the same number creates an equivalent equation with the same solution as the original equation.*

$$\frac{x}{3} + 4 = x$$

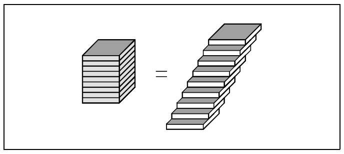
$$3\left(\frac{x}{3} + 4\right) = 3(x)$$

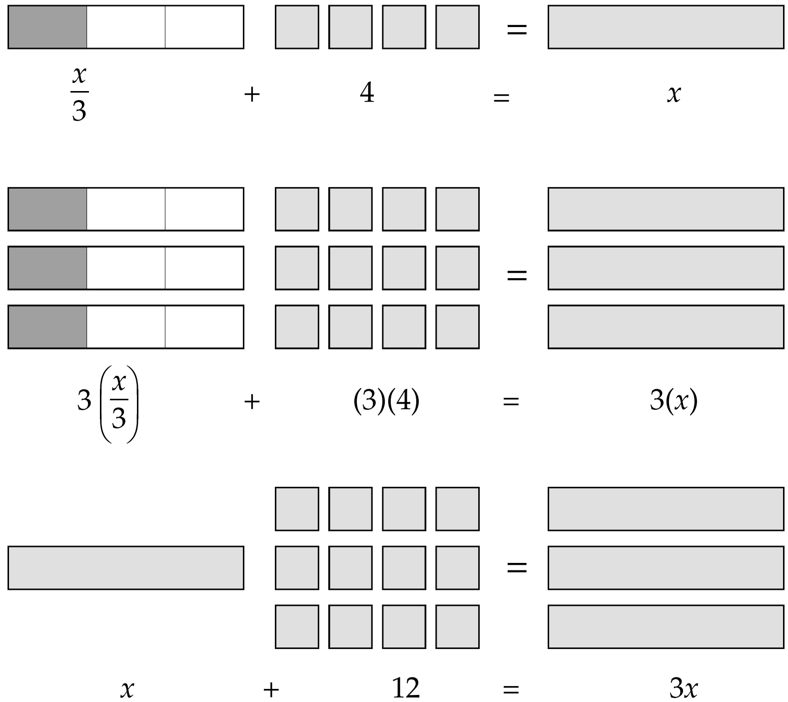
$$3\left(\frac{x}{3}\right) + 3(4) = 3x$$

$$\frac{3x}{3} + 12 = 3x$$

$$x + 12 = 3x$$

Notice that you must multiply 3 times each term on both sides.





After this point, the steps are the same as in previous sections:

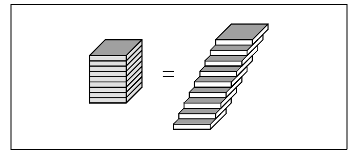
$$
\begin{aligned}
x + 12 &= 3x \\
x + 12 - x &= 3x - x \\
12 &= 2x \\
\frac{1}{2}(12) &= \frac{1}{2}(2x) \\
\frac{12}{2} &= \frac{2x}{2} \\
6 &= x
\end{aligned}
$$

How do you choose the number that you will use to multiply? If x is divided by 3, multiply by 3. If x is divided by 4, multiply by 4. If we triple $x/3$ (one-third of x), we will get one x.

If the equation has more complicated fractions, we still multiply by a number which will cancel the denominators. For example:

$$\frac{2}{3}x + 2 = \text{x}$$

It is still useful to multiply both sides by a number, and we use the same number as in the previous example:



$$\begin{aligned}
\frac{2}{3}x + 2 &= x \\
3\left(\frac{2}{3}x + 2\right) &= 3(x) \\
3\left(\frac{2}{3}x\right) + 3(2) &= 3(x) \\
\left(\frac{3}{1}\cdot\frac{2}{3}\right)x + 6 &= 3x \\
\frac{6}{3}x + 6 &= 3x \\
2x + 6 &= 3x \\
2x + 6 - 2x &= 3x - 2x \\
6 &= x
\end{aligned}$$

Equations with Several Fractions

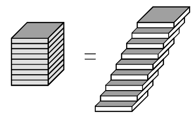
An equation may contain fractions with *different* denominators. We will still multiply both sides by a number, but this time we will use a number that will eliminate all of the fractions.

The number we want will have to be divisible by all of the denominators, or it will not "cancel" when it is multiplied times each term. The lowest number that is divisible by a group of numbers is called the **least common multiple** or **least common denominator**. Here is an example of the steps:

$$\frac{x}{3} + \frac{x}{4} = \frac{x}{2} + 1$$

The least common denominator of 3, 4, and 2 is 12.

$$\begin{aligned}
12\left(\frac{x}{3} + \frac{x}{4}\right) &= 12\left(\frac{x}{2} + 1\right) \\
12\left(\frac{x}{3}\right) + 12\left(\frac{x}{4}\right) &= 12\left(\frac{x}{2}\right) + 12(1) \\
\frac{12x}{3} + \frac{12x}{4} &= \frac{12x}{2} + 12 \\
4x + 3x &= 6x + 12 \\
7x &= 6x + 12 \\
7x - 6x &= 6x - 6x + 12 \\
x &= 12
\end{aligned}$$



Equations with Decimals

Decimal numbers such as .1 and 3.034 are often called decimal fractions because they represent fractions with denominators of 10, 100, 1000, etc. Because decimals are really fractions, we solve equations with decimals in the same way that we solve equations with fractions.

Consider this equation:

$$.3x + .2 = 1.7$$

We find a number (the least common multiple) that we can use to multiply times both sides to eliminate the decimals. The correct choice is to multiply by 10, because the equation could be written as:

$$\frac{3}{10}x + \frac{2}{10} = \frac{17}{10}$$

Multiplying by 10 will eliminate the decimals and will result in a new equation that is easier to solve:

$$\begin{aligned} 10(.3x + .2) &= 10(1.7) \\ (10 \cdot .3x) + (10 \cdot .2) &= 17 \\ \mathbf{3x + 2} &= \mathbf{17} \end{aligned}$$

We can now solve the equation in the usual way:

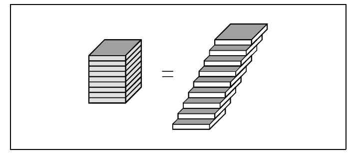
$$\begin{aligned} 3x + 2 &= 17 \\ 3x + 2 - 2 &= 17 - 2 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

Equations may also contain decimals with different numbers of decimal places. Again, we multiply both sides by the power of 10 (10, 100, 1000, etc.) that will eliminate decimals from all of the numbers. Consider this equation:

$$.03x + .7 = x - 3.18$$

There are three numbers with decimal points. Two of them (.03 and 3.18) have two decimal places and one (.7) has one decimal place. We need to multiply by 100 to eliminate all of the decimal places:

$$\begin{aligned} 100(.03x + .7) &= 100(x - 3.18) \\ 3x + 70 &= 100x - 318 \\ 3x + 388 &= 100x \\ 388 &= 97x \\ 4 &= x \end{aligned}$$



To check our answer:

$$.03x + .7 = x - 3.18$$
$$.03(4) + .7 = (4) - 3.18$$
$$.12 + .7 = .82$$
$$.82 = .82$$

To review, we chose 100 as our number to multiply because it was the least common multiple. We could have rewritten the original equation to show why this is true:

$$.03x + .7 = x - 3.18$$

is also:

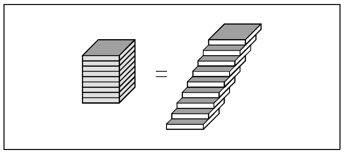
$$\frac{3}{100}x + \frac{7}{10} = x - \frac{318}{100}$$

The common denominator for 10 and 100 is clearly 100.

Fractions and Decimals: How to Multiply

With both fractions and decimals, we multiply both sides of the equation by the least common multiple. With fractions, we look at the denominators and choose the least common denominator. With decimals, we look at the number of decimal places and we multiply by the appropriate power of 10 (10, 100, 1000 …).

For this equation:	Multiply by:	Reason:
$\frac{x}{6} + \frac{x}{8} = 7$	24	Common denominator of 6 and 8
$\frac{x}{3} + \frac{3x}{4} = \frac{x}{6} + 11$	12	Common denominator of 3, 4, and 6
$.02 + .13x = .15$	100	Maximum of 2 decimal places
$3 + .001x = 3.1$	1000	Maximum of 3 decimal places



It is important to understand that you do not have to multiply these equations, but it is usually easier to do so. If you do not multiply both sides, you can solve the equations by subtracting and dividing with the fractions or decimals:

$$
\begin{aligned}
.03x + .7 &= x - 3.18 \\
.03x + .7 + 3.18 &= x - 3.18 + 3.18 \\
.03x + 3.88 &= x \\
.03x + 3.88 - .03\mathrm{x} &= x - .03x \\
3.88 &= .97\mathrm{x} \\
\frac{3.88}{.97} &= \frac{.97}{.97}x \\
4 &= x
\end{aligned}
$$

Summary

Now we can add one more step to our list:

- **Use the distributive property to complete any multiplications of expressions in parentheses.**
- ***If fractions or decimals are present, multiply both sides of the equation by the least common multiple (least common denominator).***
- **Combine similar terms on each side of the equation.**
- **Eliminate the unknowns from one side by adding the opposite type of bars. Add negatives to eliminate positives, and positives to eliminate negatives.**
- **Add positive or negative chips to cancel out the units and to "isolate" the unknown.**
- **Multiply both sides by ½, ⅓, etc. to match up a single unknown with the correct number of chips.**

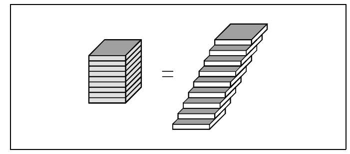
Exercises

Solve for x.

1. $\frac{x}{12} + 1 = x - 21$

2. $\frac{x}{2} = x + 4$

3. $\frac{x}{2} = {}^{-}1$



4. $\frac{x}{2} + \frac{x}{3} = \frac{5}{6}$

5. $\frac{x}{3} + x = 7$

6. $x - \frac{x}{4} = 9$

7. $\frac{x}{3} + \frac{x}{2} = \frac{x}{4} + \frac{7}{2}$

8. $\frac{2x}{3} + 2 = \frac{2}{3}$

9. $\frac{1}{5}x + 3 = 6$

10. $\frac{3}{4}x + 1 = x - 2$

11. $3 - \frac{2}{3}x = 4x - 5$

12. $6.9 + x = 3.3$

13. $.3 + 2x = 3.9$

14. $.2x + 3.1 = 3.9$

15. $.02 + .13x = .15$

16. $3 + .001x = 3.1$

17. $.2x + .8 = x - 4$

18. $3x + 4x = 6.8 + .2x$

19. $.002x = 0$

20. $\frac{3}{5}x + .2 = .1x + 10.2$

(What is the common denominator for 5 and 10?)

Section 8
Special Solutions

When the Variable Disappears

Some equations may contain the same number of *x*'s on both sides. This may be obvious:

$$3x + 7 = 3x + 7$$

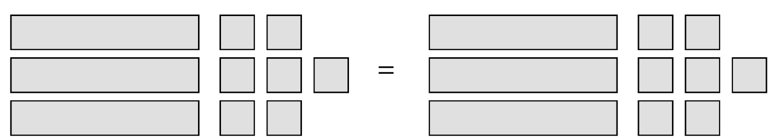
or it may occur after you have combined similar terms:

$$\begin{aligned} 3x + 2(x + 1) &= 5(x + 1) \\ 3x + 2x + 2 &= 5x + 5 \\ 5x + 2 &= 5x + 5 \end{aligned}$$

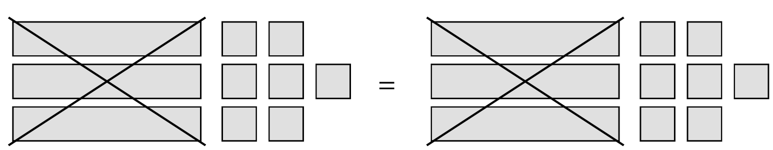
If you proceed in the usual way by subtracting *x*'s, you will get strange results. In the first case:

$$\begin{aligned} 3x + 7 &= 3x + 7 \\ 3x + 7 - 3x &= 3x + 7 - 3x \\ 7 &= 7 \end{aligned}$$

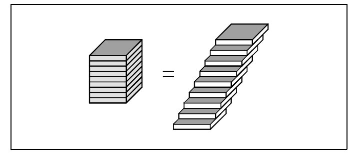
$3x + 7 = 3x + 7$



$3x + 7 - 3x = 3x + 7 - 3x$

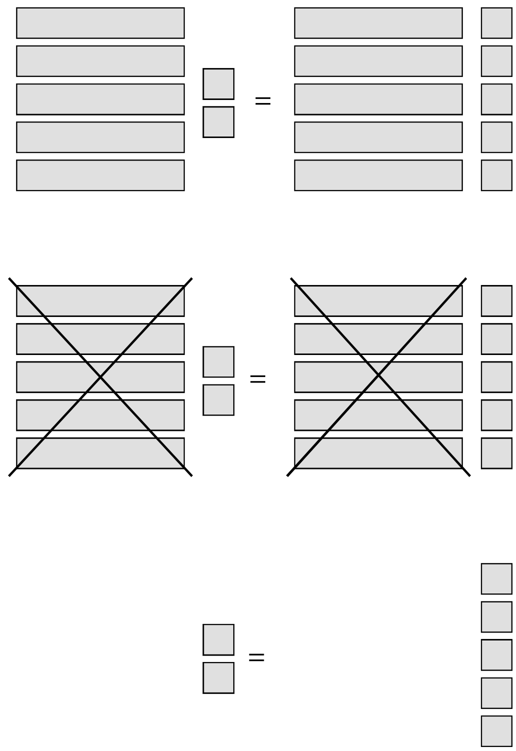


$7 = 7$



In the second case:

$$5x + 2 = 5x + 5$$
$$5x + 2 - 5x = 5x + 5 - 5x$$
$$2 = 5$$



The two cases above have a different meaning:

- **7 = 7 Since this is a true statement, the equation will be true for any x. You can choose any value for x, and the equation is still true. The equation is true for all x.**

- **2 = 5 Since this statement is false, the equation is false for any x. We substitute any value for x, but the equation will always turn out to be false. There is no solution.**

Exercises

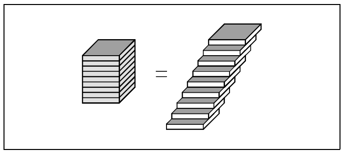
Solve for x. Determine if there is a solution, if there is no solution, or if the equation is true for all values.

1. $2x + 3x = 5x$

2. $4x + 3x = 7x + 1$

3. $2(x + 1) = 8$

4. $2(x + 1) - 2 = 3(x + 4) - (x + 12) + 1$



5. $3(3 + 2x - 1) = 4x - 1(3 - 2x) + 9$

6. $2(x + 3) - 2x = 5$

7. $2x + 3 = 3$

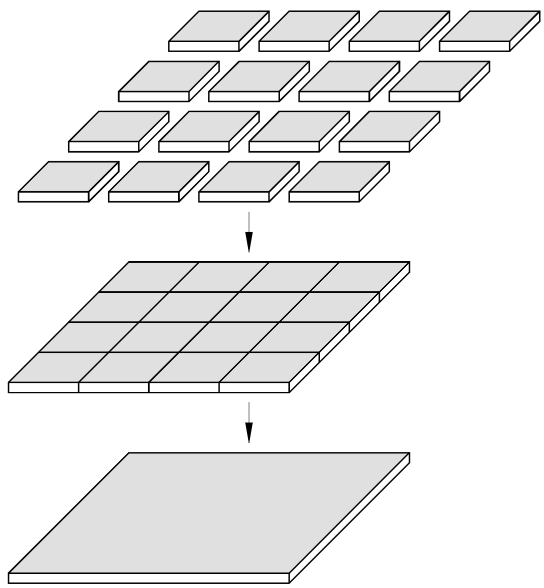
8. $^{-}3(x - 1) = x + 4(2 - x) - 5$

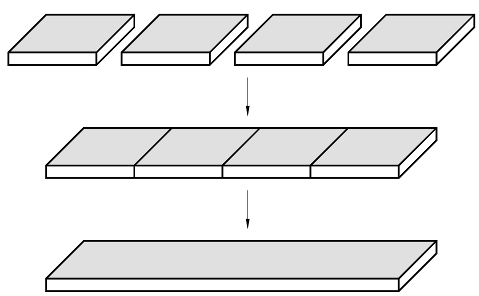
9. $1 + x - (x - 1) = 2$

10. $3(x + 1) = 3x + 1$

Chapter 4

Polynomials

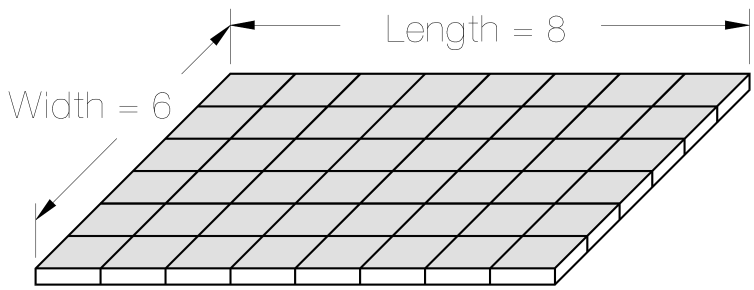




Section 1
Using Unknowns: 1, x, x^2

The Meaning of Multiplication

To do multiplication with unknowns we must remember how we do multiplication with integers. When we multiply integers we are making rectangles, and the product (the answer to the multiplication problem) is the area of the rectangle:

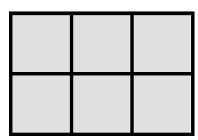


Length × Width = $8 \cdot 6 = 48$ units

To get the sign of the answer (product), we start with the colored side up, and then flip the chips once for each negative (-) sign in the problem.

$(+3) \cdot (+2) = 6$

(No Flips)



$(+3) \cdot (^-2) = {}^-6$

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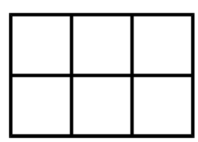
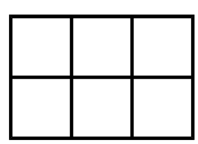


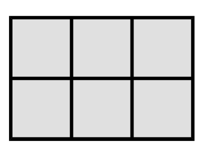
$(^-3) \cdot (+2) = {}^-6$

(One Flip)

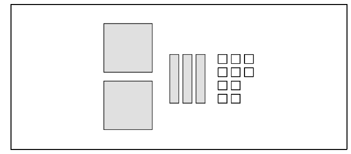


$(^-3) \cdot (^-2) = +6$

(Two Flips)

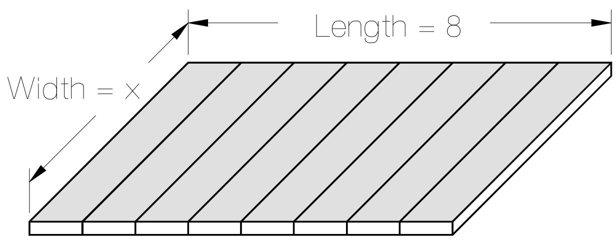


We should note here that the two numbers we are multiplying become the **dimensions** of the resulting rectangle. If just one of these dimensions is negative, then the rectangle ends up white side up (negative). If both dimensions are negative, then the product (the rectangle) will end up positive, with colored side up, just as it does when neither side is negative.



Multiplying With Unknowns

So far, we have been multiplying lengths and widths that are numbers. Can we make areas that have lengths or widths of x? Multiplication will still have the same meaning, but the sides may have dimensions involving x.



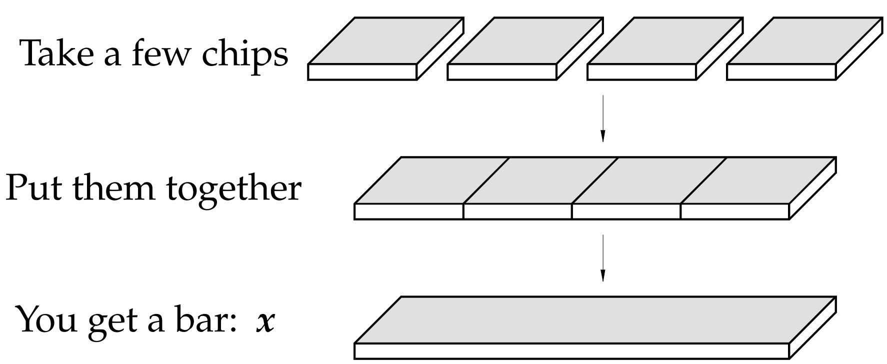
Length times Width $= 8 \cdot x = 8x$

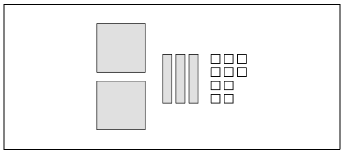
.
As we begin making rectangles using both integers and unknowns, the process for determining the sign of the rectangle will remain the same. If just one dimension (side) of a rectangle is negative, the white side is up and the result is negative; if both or neither sides are negative, then the colored side is up and the answer is positive.

When we're using unknowns, we can still think of making rectangles, but now our rectangles will have bars as well as units.

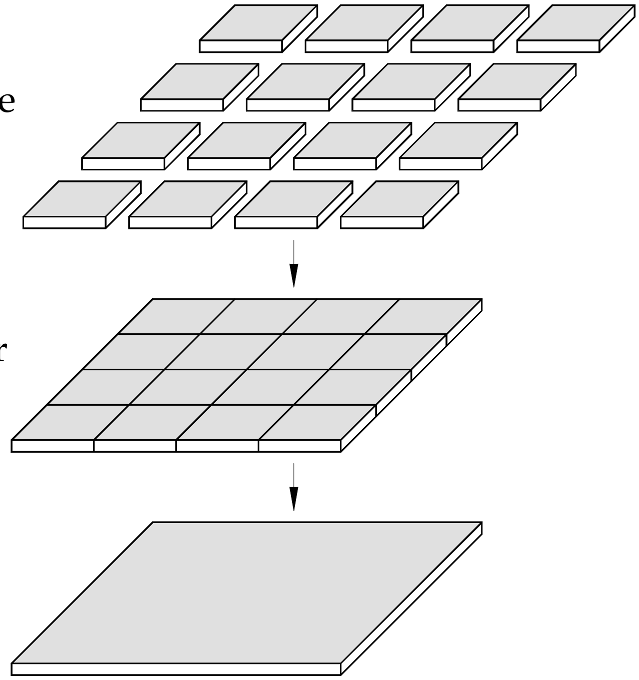
Definition of x and x^2

First, let's define x. Take a few chips and line them up in a row. Imagine that they are joined together in a bar, but then erase the boundaries so that all we see is a bar of unknown length. **This is x.**





Next, we will take a few chips and form a square. If we put the chips together and imagine that we cannot see exactly how many chips there are, we have built a square that is an unknown width and length. This is $\boldsymbol{x \cdot x}$ or $\boldsymbol{x^2}$:

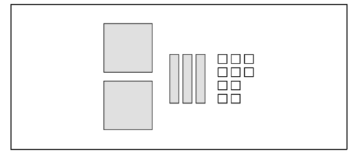


More on x and x^2

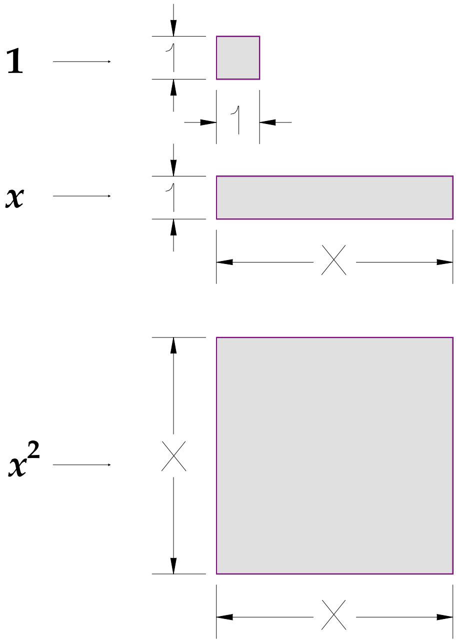
The following table shows the names and sizes of our new pieces. Note that the value of each piece is equal to its area.

Piece	Value	Length ×	Width =	Area
Chip or Little Square	**1** (Unit)	1	1	1
Bar	x	x	1	x
Big Square	x^2	x	x	x^2

Some of the pieces have sides in common. The unit and the x both have a side of one. The $\boldsymbol{x}$ and the $\boldsymbol{x^2}$ both have a side of x. The x does not represent a specific number of chips; it represents *any* unknown number of chips. If you try to match up unit chips along the long (x) side of the $\boldsymbol{x}$ bar, you will find that neither 5 nor 6 nor any number of chips fits exactly.



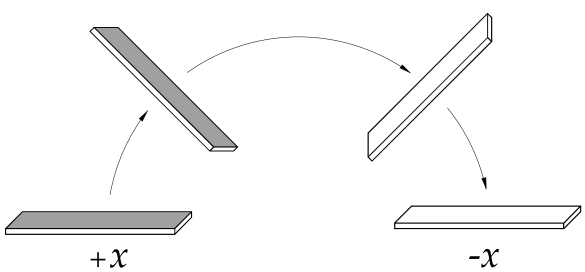
Our set of chips now looks like this:



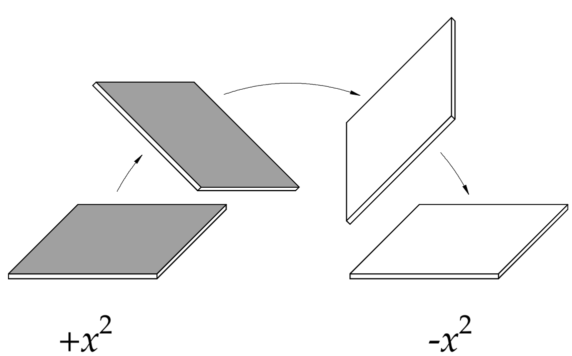
The Opposites of *x* and x^2

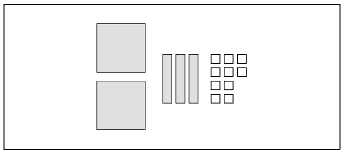
Unknowns can also have opposites. We have already been introduced to the idea that flipping a unit chip to the white side represents $^{-}1$; now we will put together the opposites of *x* and x^2.

First, let's review the idea of the opposite of the *x* bar. This will be written as **-*x*** **a**nd will be called **negative *x*** or **the opposite of *x***.



In the same way, we can construct **$-x^2$**:

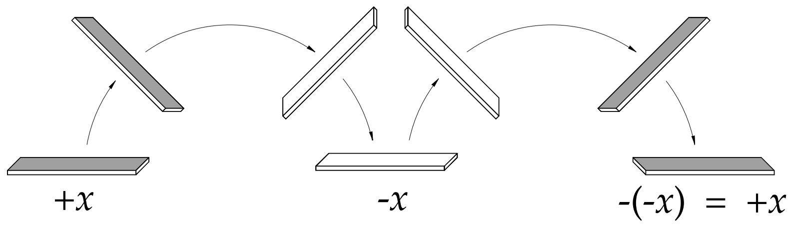




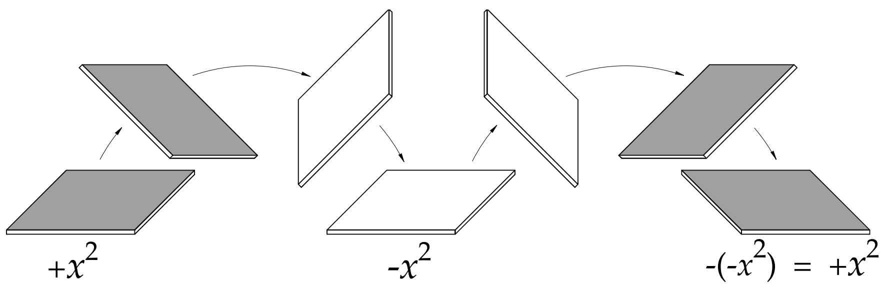
These new pieces behave in the same way as single chips—*Flipping an* x *or* x^2 *changes the sign.*

$$-(x) = -x$$
$$-(-x) = +x$$
$$-(x^2) = -x^2$$
$$-(-x^2) = +x^2$$

For x:



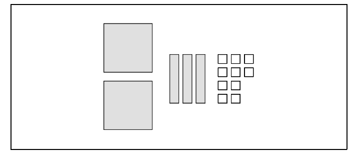
For x^2:

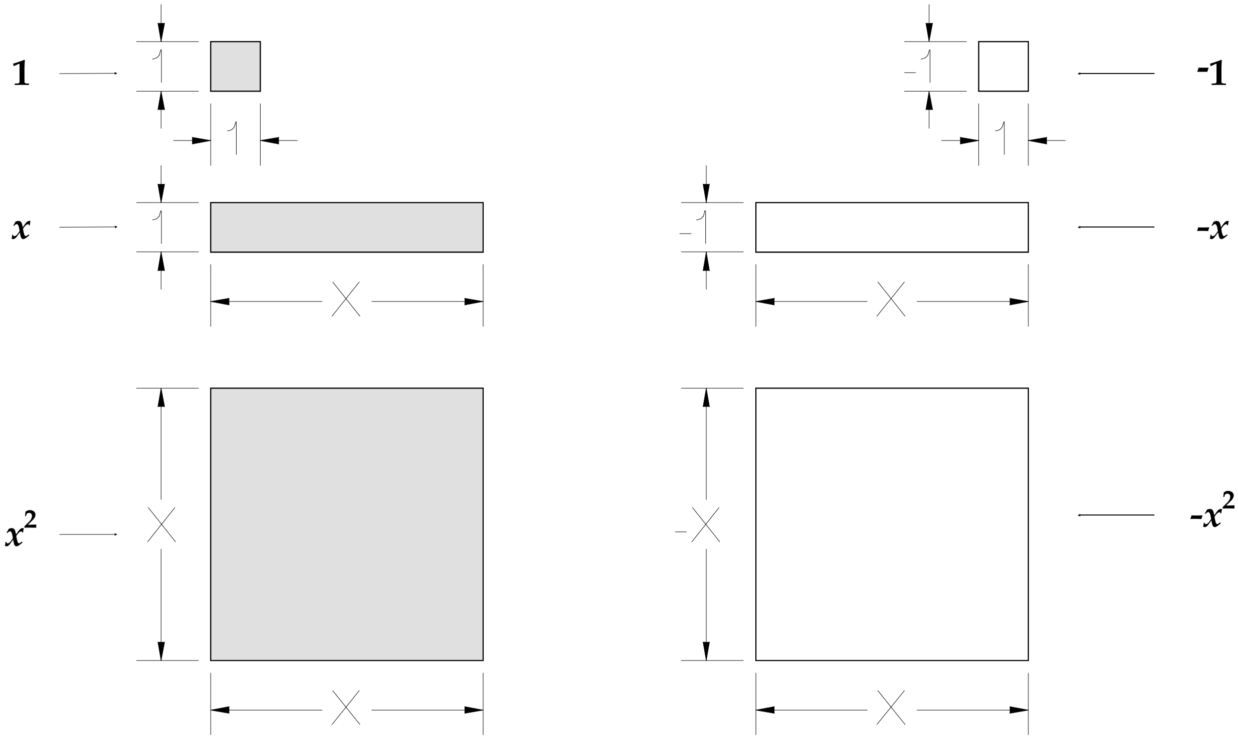


It is best to think of $-x$ and $-x^2$ as the opposites of x and x^2 ($-x$ *is not necessarily a negative number*!). Here is an expanded table of our pieces:

Piece	**Value**	**Length** ×	**Width** =	**Area**
Chip or Little Square	**1** (Unit)	1	1	1
Negative Chip	**-1** (Unit)	1	-1	-1
Bar	x	x	1	x
Opposite Bar	$-x$	x	-1	$-x$
Big Square	x^2	x	x	x^2
Opposite Big Square	$-x^2$	x	$-x$	$-x^2$

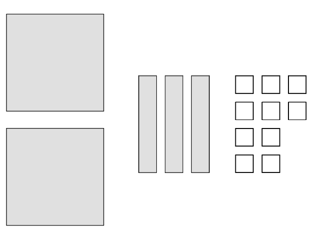
Finally, here are all of the new pieces:



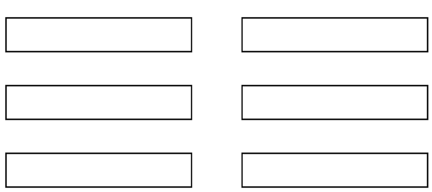


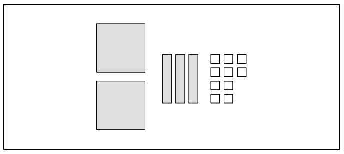
Polynomials

When we have an assortment of pieces such as units, x's, and x^2 chips, we call this a **polynomial**. A polynomial can have many types of pieces

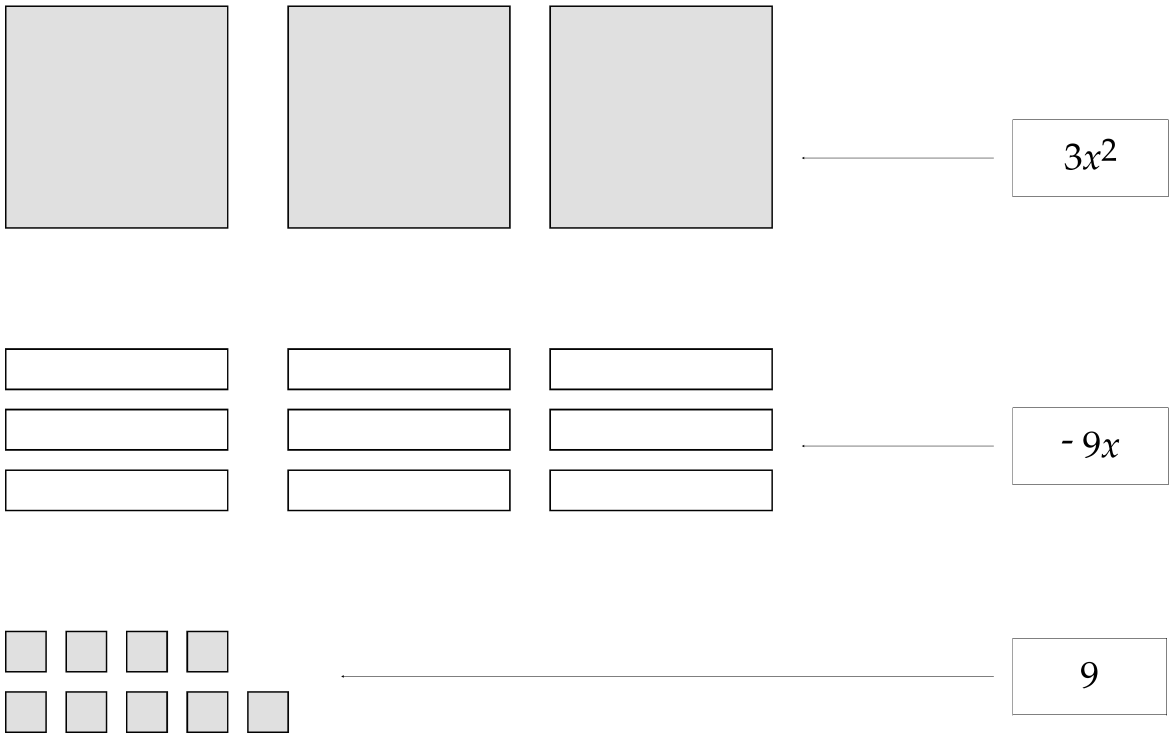


or just one kind of piece.





Each group of like shapes is called a **term**. When we have two bars we have two x's, or **2*x***. Three x^2 pieces can be written as a **3*x*²** term. Here are some examples of terms:



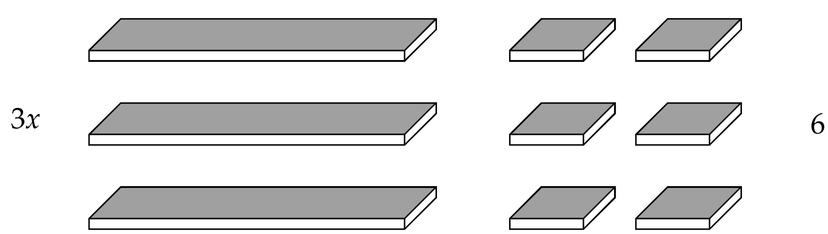
An expresion with *two* terms is called a **binomial**. An expression with *three* terms is called a **trinomial**.

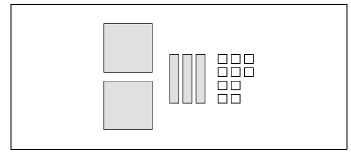
Exercises

Set up the following expressions with chips and identify individual terms:

Example: $3x + 6$

Solution: Terms are **3*x*** and **6.**





1. $7x$

2. $7x - 2$

3. $4x^2$

4. $3x^2 - 6$

5. $6 - 2x^2$

6. $2x^2 - 3x + 12$

7. ${}^-2x^2 - 5x - 1$

8. ${}^-0x^2$

9. $5 - 3x^2$

10. $2x + 3$

11. $x^2 - 5x + 6$

12. $2x - x^2 + 4$

13. $4x + 3x^2$

14. $2x^2 - 7$

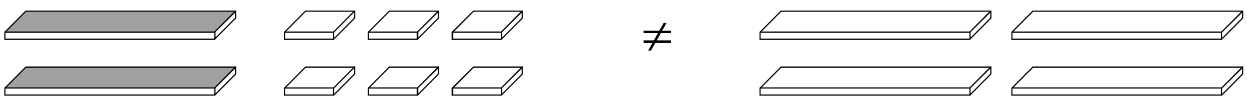
15. $3x^2 - 5x + 2$

Section 2
Adding and Subtracting Polynomials

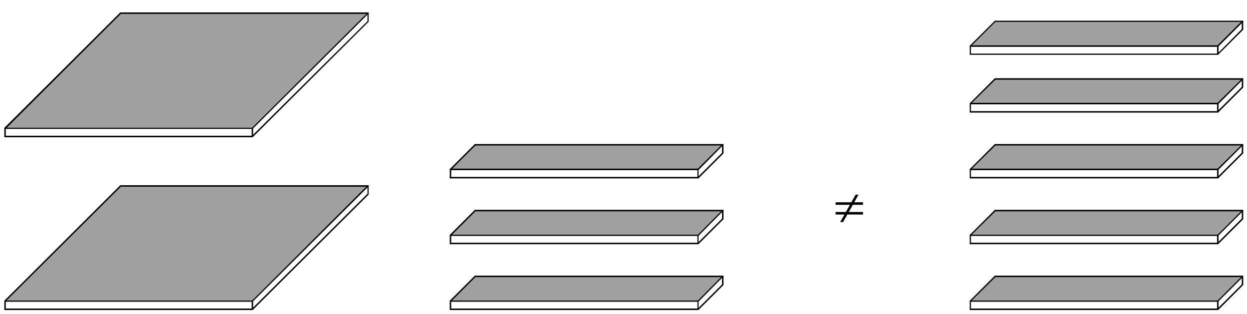
Combining Like Terms

If a polynomial has two separate kinds of pieces (bars and chips), they are not the same size and shape. This means that, in the symbolic language of algebra, we must also have ***two separate terms***; one with ***x***'s (bars) and the other with units (chips). These two terms are made up of ***different pieces*** and therefore they *cannot be combined*.

$2x - 6$ is not $^{-}4x$



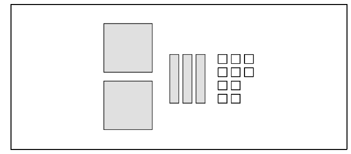
$2x^2 + 3x$ is not $5x$



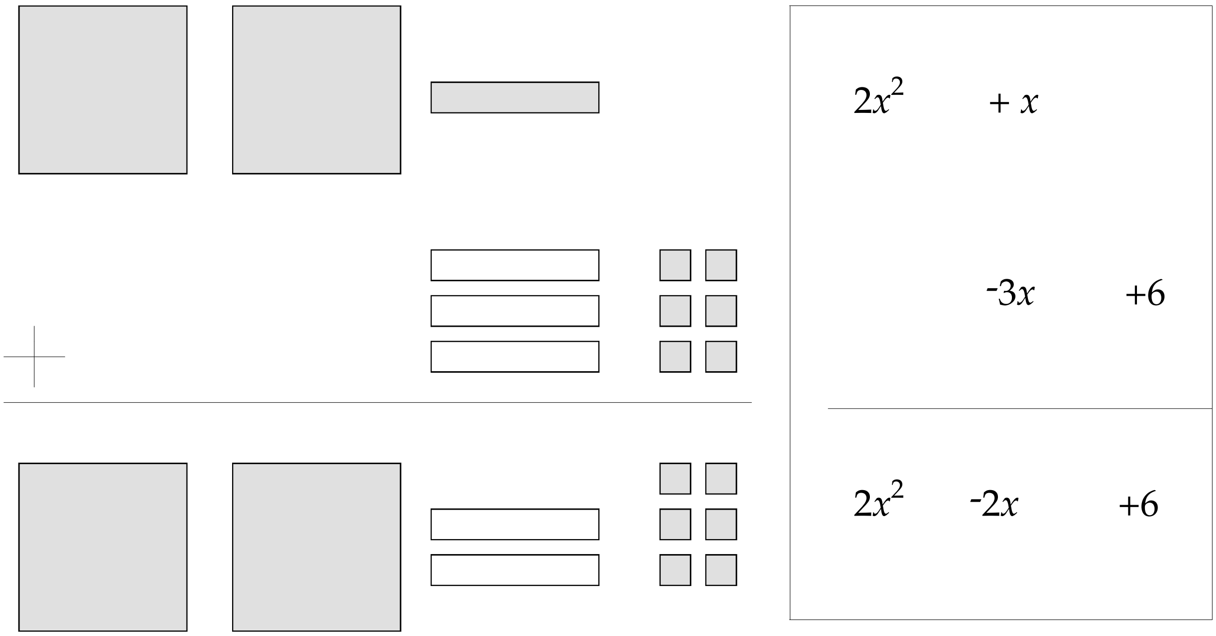
We cannot combine these terms because different shapes cannot be treated as if they are the same; they must be kept separate, ***x*'s** in one term and units in another.

If you use the chips and think of polynomials as groups of shapes, it will be easy to work with them without needing to memorize any rules. Just combine similar shapes.

Adding Polynomials



Adding two polynomials is done in the same way that we add units—we take similar pieces from each polynomial and combine like terms.



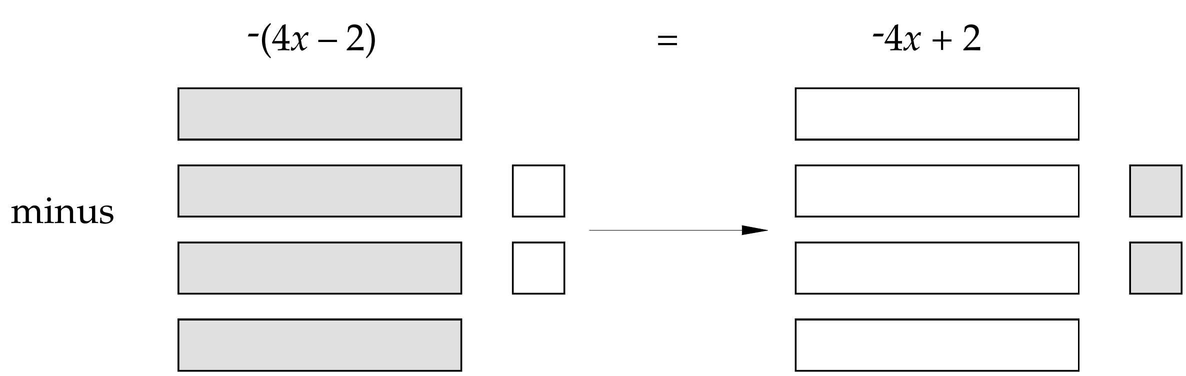
The algebra symbols show like *terms* being combined; the chips show like *pieces* being combined.

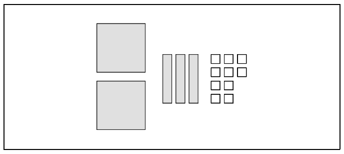
Subtracting Polynomials

To subtract a polynomial, we think of adding the opposite:

$$(3x - 5) - (4x - 2) = (3x - 5) + {}^-(4x - 2)$$

Just as with signed numbers, the negative sign means take the opposite, or *flip the chips*.



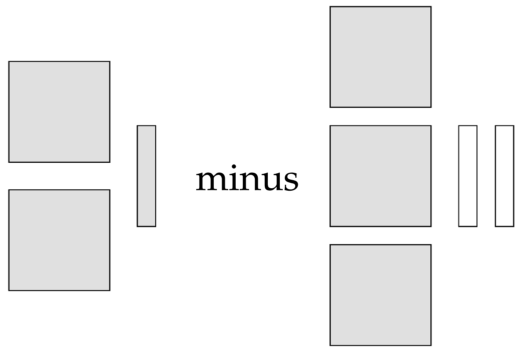


So for each subtraction, write the problem as an addition (flip the subtracted chips) and proceed as usual by combining like terms.

- **identify the two polynomials**
- **subtraction becomes adding the opposite**
- **find the opposite**
- **add**

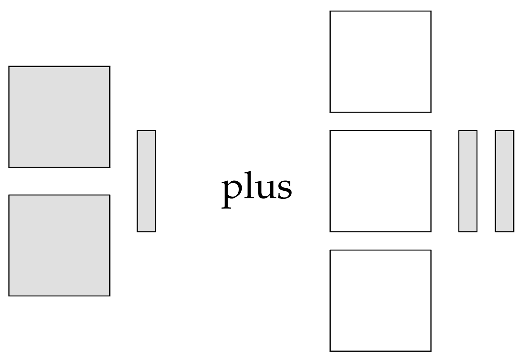
Here is an example of this process:

$(2x^2 + x) - (3x^2 - 2x)$

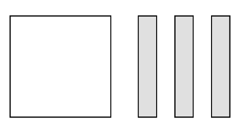


$(2x^2 + x) + {}^{-}(3x^2 - 2x)$

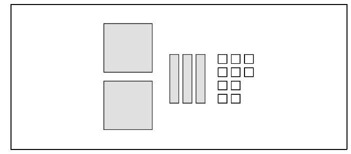
$(2x^2 + x) + {}^{-}3x^2 + 2x$



$-x^2 + 3x$



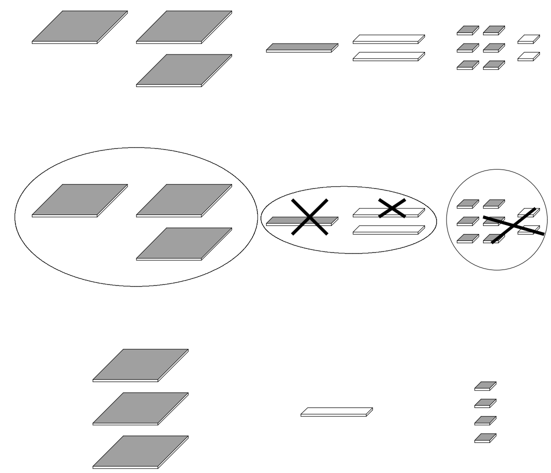
Exercises



Use your chips to set up the following problems. Combine similar shapes (terms).

Example: $x^2 + 2x^2 + x - 2x + 6 - 2$

Solution: $3x^2 - x + 4$



1. $3x - 2x$

2. $^{-}3x + 5 - 6x$

3. $5 - 4x^2 + x$

4. $2x^2 + x + x^2 + 3x$

5. $^{-}3x^2 - x^2$

6. $5x + 5 + (^{-}6x) + (^{-}3) + 2x^2$

7. $2x + x^2 - 5x - 3$

8. $^{-}3x^2 + 5 - x - 7$

9. $2x - 5 + x^2 + 3x$

10. $5 - (^{-}3x) + x + 7$

Use chips to complete these addition problems and write the algebra symbols as well:

11. $(3x - 2) + (5x - 6)$

12. $(x^2 + 3x + 3) + (2x^2 - x)$

13. $(^{-}2x^2 - x - 1) + (2x^2 + x - 1)$

14. $(2x - 5) + (x^2 + 3x + 2)$

15. $(x^2 - 3x + 1) + (x^2 - 7)$

16. $(^{-}5x + 3) + (2x^2 - 3x)$

17. $(x^2 + 3x - 2) + (3x^2 - x - 5)$

18. $(^{-}3x + 5) + (4x^2 - 5)$

Perform the following subtractions:

19. $(3x - 2) - (5x - 6)$

20. $(x^2 + 3x + 3) - (2x^2 - x)$

21. $(^{-}2x^2 - x - 1) - (2x^2 + x - 1)$

22. $(6x - 2) - (3 - 2x)$

23. $(x^2 + 3x - 1) - (x^2 - 2x + 5)$

24. $(2x + 3) - (x^2 - 5x)$

25. $(2x^2 - 5)$ - $(x^2 + 5x - 6)$

26. $(3x^2 - 5x + 1) - (x^2 - 3x - 2)$

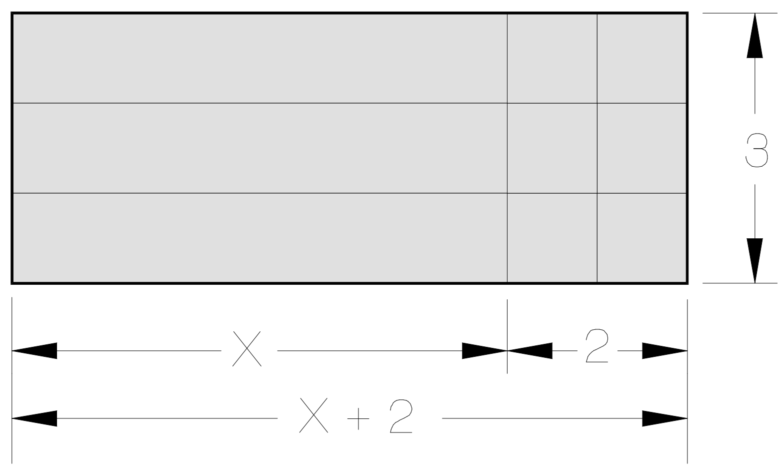
Section 3
Multiplying Polynomials

Multiplying with One Unknown

If we have a product (multiplication) like

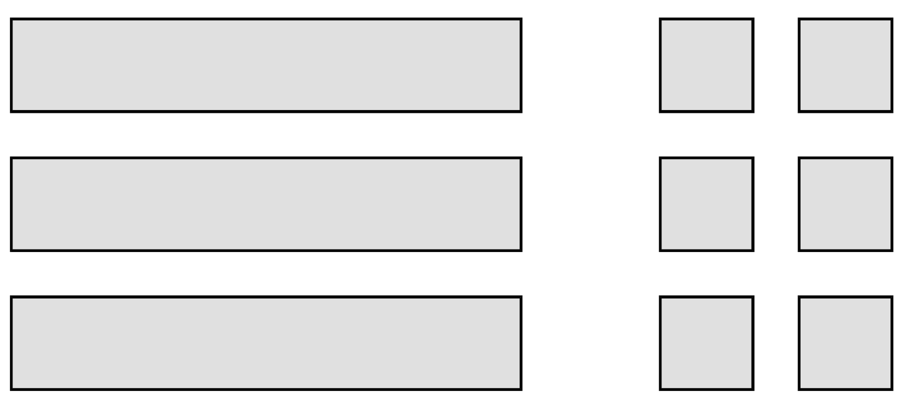
$$(3)\cdot(x + 2)$$

we make a rectangle with dimensions (sides) of **3** and ***x* + 2**, like this:

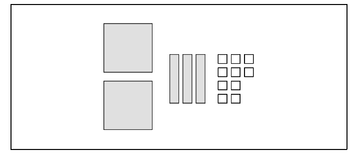


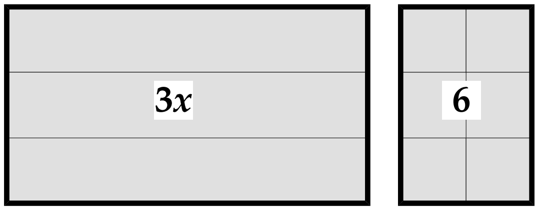
The product, or area, will just be the sum of the pieces we use, which is three bars (**3*x***) and six little squares (**6**):

$$3(x + 2) = 3x + 6$$

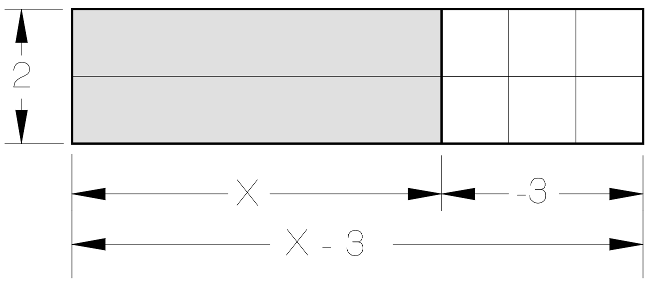


The product, or area, is **$3x + 6$.** We can think of this as being two smaller rectangles added together: one rectangle **3** by x, and the other rectangle **3** by **2**. In this case both rectangles are positive.





If one piece of our product is negative, the product will look like this:



This time one of the smaller rectangles is positive ($2 \cdot x = 2x$) while the other smaller rectangle is negative ($2 \cdot {}^{-}3 = {}^{-}6$). Thus the product is.

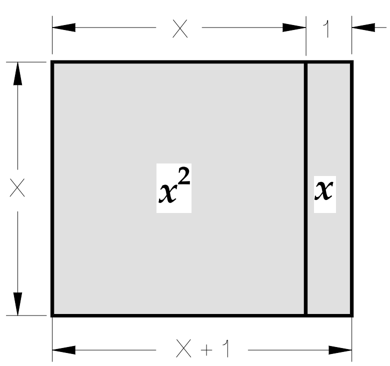
$$2(x - 3) = 2x - 6$$

Using Unknowns in Both Dimensions

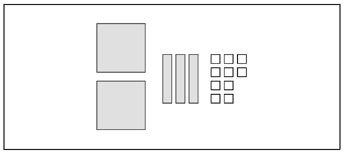
If we wish to find the product

$$x(x + 1)$$

we build a rectangle x wide and $x + 1$ long. This is a rectangle made up of two smaller rectangles. One is x by x or x^2, the other is x by **1** or x**:**



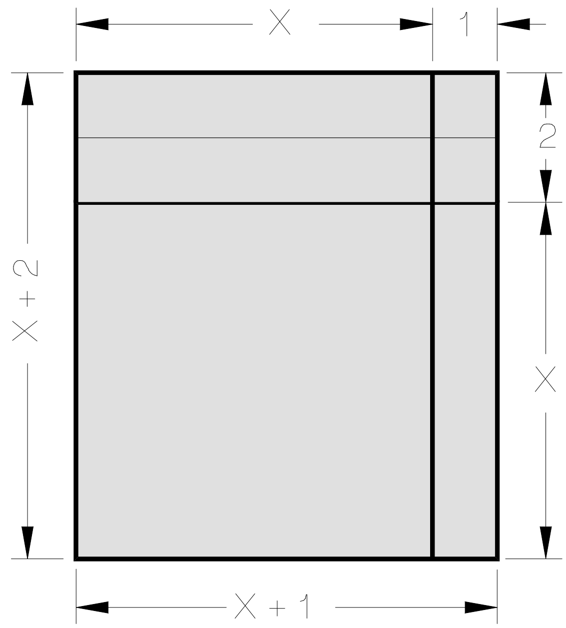
$$(x)(x + 1) = x^2 + x$$



If we wish to find the product

$$(x + 2)(x + 1)$$

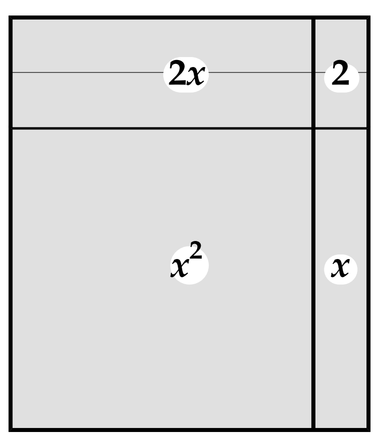
we must build a rectangle having each factor (**x + 2** and **x + 1**) as one dimension of length or width.

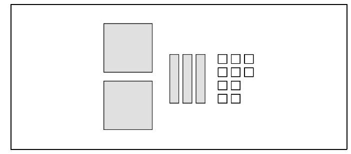


As can be seen in this illustration, the result is a large rectangle which can be subdivided into four smaller rectangular areas. In the upper right are two small chips, a rectangle 1 by 2 units. At the top left are two bars, defining a rectangle 2 by x. At the lower right is one bar, in a rectangle **1** by **x**. Finally, the large square forming the lower left corner is the rectangle with sides **x** by **x** and area **x^2**.

The area of the larger rectangle is the sum of these 4 areas:

$$(x + 2)(x + 1) = x^2 + x + 2x + 2$$





This time, two of the terms **are** made of the same size pieces; the x and the $2x$ are both made up of bars, so they can be combined giving

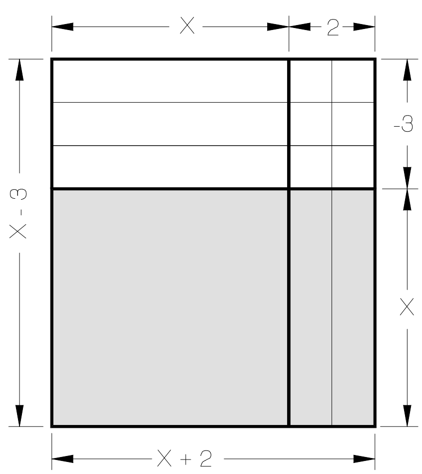
$$(x + 2)(x + 1) = x^2 + 3x + 2$$

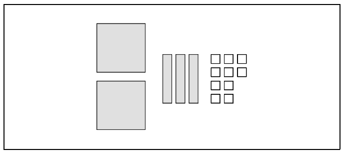
Each of the four smaller rectangles inside the large rectangle represents a piece of the product we are seeking. When using symbols, we can find these four smaller areas by using a technique called the **FOIL method**. This is defined as shown below:

F	**F**irst times **F**irst	$(x+2)(x+1)$	$x \cdot x = x^2$
O	**O**utside times **O**utside	$(x+2)(x+1)$	$x \cdot 1 = x$
I	**I**nside times **I**nside	$(x+2)(x+1)$	$2 \cdot x = 2x$
L	**L**ast times **L**ast	$(x+2)(x+1)$	$2 \cdot 1 = 2$

Each piece of the product, the x^2, the $1x$, the $2x$, and the 2, is one of the smaller rectangles in our figure.

If one or both sides of any of these smaller rectangles is negative, then we use our rules for signs to determine the sign of that particular rectangle. For example, let's illustrate the product of $(x - 3)(x + 2)$:



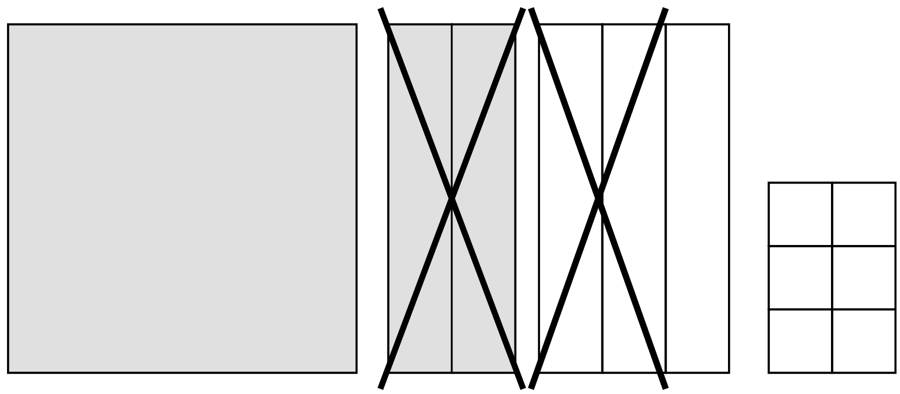


In this example, of the four rectangles within the figure, two are positive and two are negative (white side up).

$$(x-3)(x+2) = x^2 + 2x - 3x - 6$$

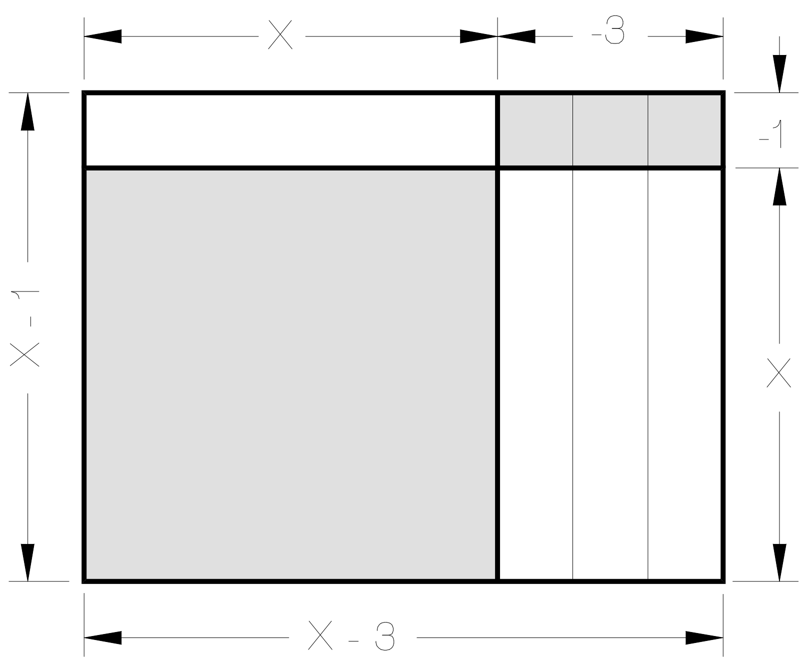
If we combine like terms, the x's (positive bars) will be cancelled out by the negative bars leaving:

$$(x-3)(x+2) = x^2 - x - 6$$



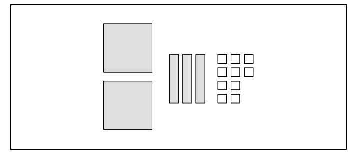
Again, each piece of the rectangle comes from one piece of the product when using the FOIL method. When some parts of our area are positive and other parts are negative, we can think of the product, or area of the figure, as being the difference of the areas, or the area left over when the white is taken away from the colored area.

Here's an example having two negative terms:

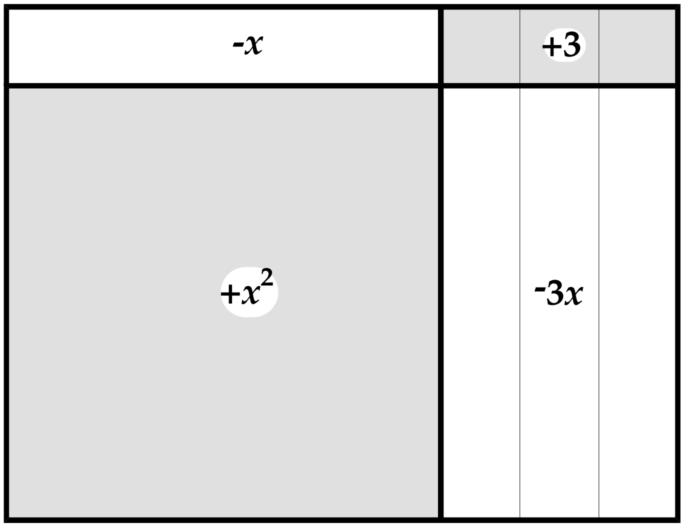


This is

$$(x - 1)(x - 3)$$



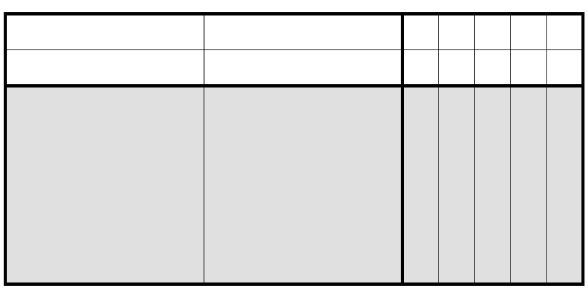
Can you explain why the ***x*-bars** are turned white side up, and the three chips are turned colored side up?



Here is one final example. Find the product

$$(x - 2)(2x + 5)$$

This requires forming a rectangle of dimensions **(*x* – 2)** by **(2*x* + 5):**



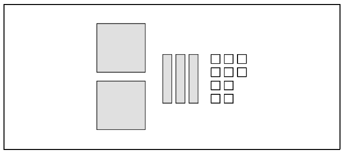
From this we see

$$(x - 2)(2x + 5) \;=\; 2x^2 + 5x - 4x - 10$$

Combining like terms gives

$$(x - 2)(2x + 5) \;=\; 2x^2 + x - 10$$

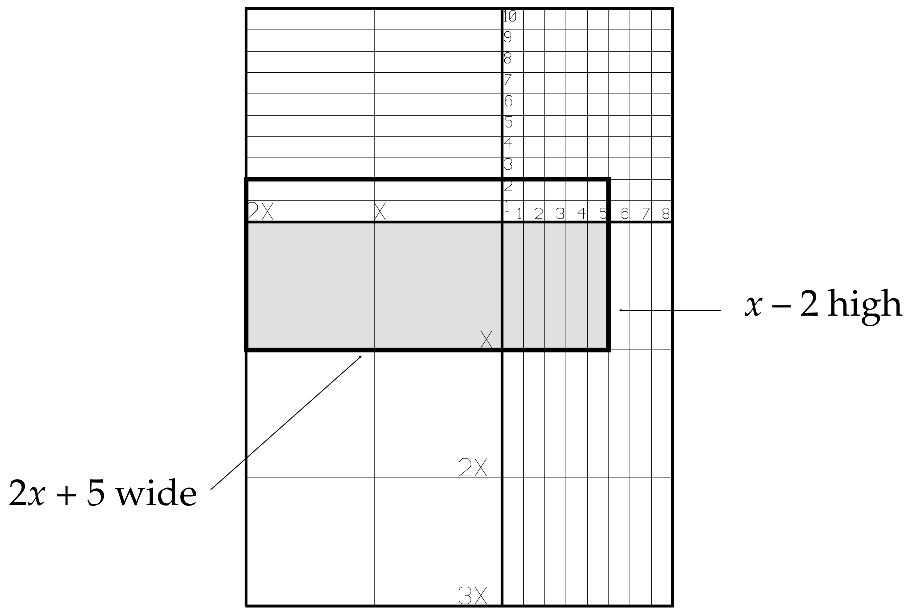
Do you understand where the signs on each term came from?



A plastic grid is included with this book. You can use the grid instead of the chips to plot multiplication of polynomials. Use a water-based marker to outline the rectangles or chips you want to use. You can also mark areas as positive or negative.

The grid is ruled in units of x's and **ones.** The darker lines across the grid (one horizontal and one vertical) are the lines which separate the four smaller rectangles within the larger figure. Remember that each of these smaller rectangles has its own sign and represents one term of the product.

Here is the example above, using the grid:

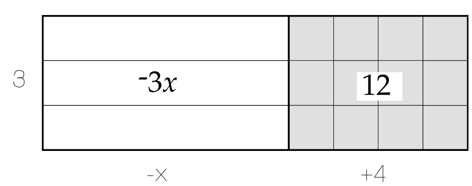


Exercises

Look at the example products and then use your chips to do the following multiplications.

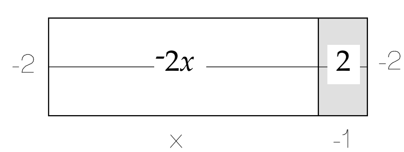
Example: $3(-x + 4)$

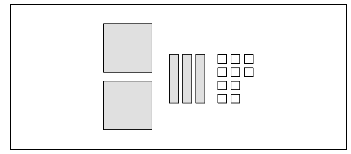
Solution: $^{-}3x + 12$



Example: $^{-}2(x - 1)$

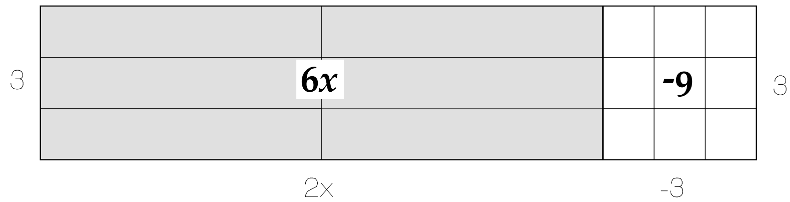
Solution: $^{-}2x + 2$





Example: $3(2x - 3)$

Solution: $6x - 9$



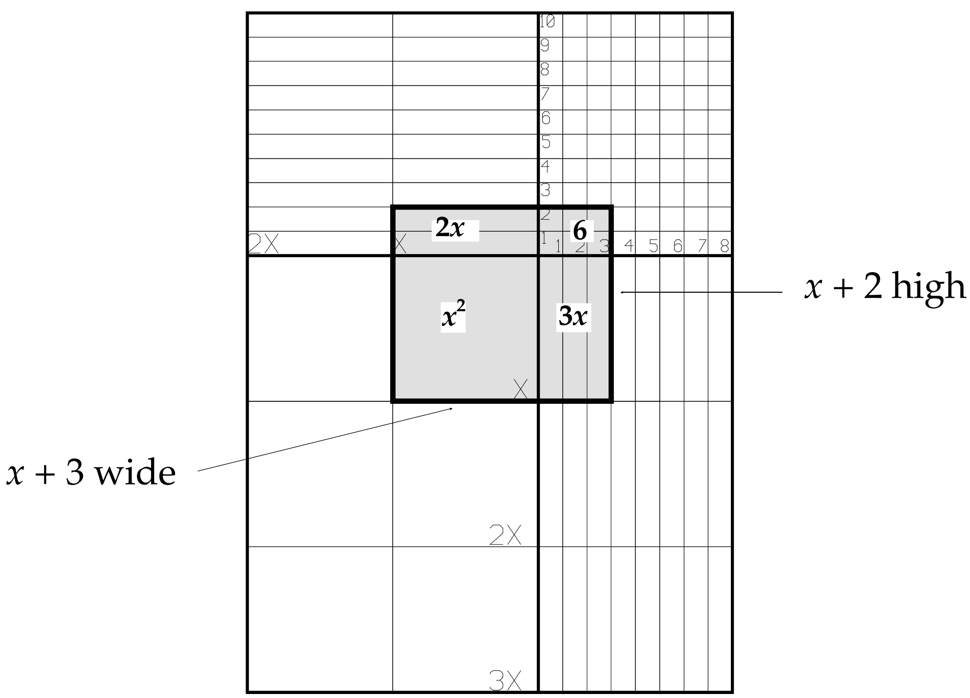
Multiply:

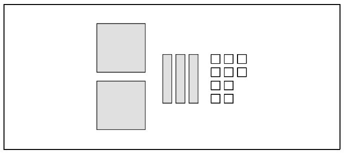
1. $2(x - 4)$
2. $3(2x + 1)$
3. $3(-x + 1)$
4. $^-2(x - 3)$
5. $^-2(-x - 1)$
6. $2(3x - 1)$
7. $^-3(-x + 3)$
8. $^-2(2x - 5)$

Try these problems using chips or the grid:

Example: $(x + 3)(x + 2)$

Solution: $x^2 + 3x + 2x + 6 \;=\; x^2 + 5x + 6$





9. $5(2x - 3)$

10. $-3(x - 5)$

11. $2(-2x + 1)$

12. $-5(-2x - 3)$

13. $-5(3x - 2)$

14. $2(5 - 3x)$

15. $-4(3 - x)$

16. $(x + 4)(x + 1)$

17. $(x - 3)(x + 4)$

18. $(x - 1)(x - 5)$

19. $(x + 5)(x - 3)$

20. $x(x - 6)$

21. $(2x + 1)(x - 4)$

22. $-x(3x - 2)$

23. $(2x - 3)(x - 2)$

24. $(x + 3)(x - 5)$

25. $(x - 2)(x - 6)$

26. $(x + 3)(2x - 1)$

27. $(2x - 3)(x + 2)$

28. $-x(3 - 2x)$

29. $(x - 2)(2x + 1)$

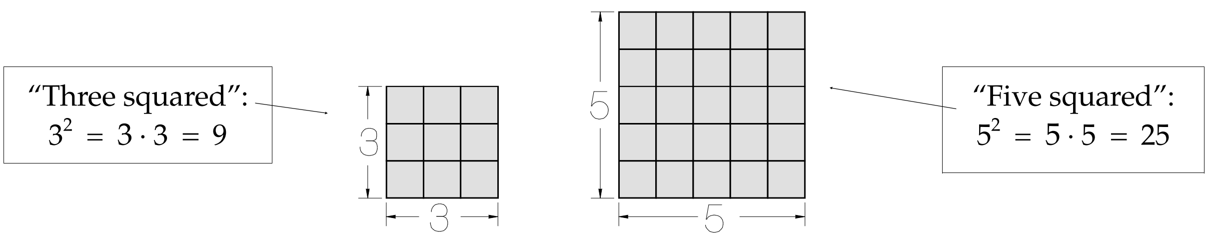
30. $(2x - 1)(2x + 3)$

Section 4
Special Products

Perfect squares

Two types of polynomials are considered special. These special polynomials are called **perfect squares** and **the difference of two perfect squares**.

Any time we make a rectangle where the length and width are the same, we get a square. This is obviously true if the sides of the square are just numbers.



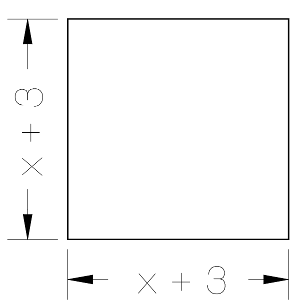
In fact, numbers which can be made into a square in this way are called **perfect square** numbers. The first six perfect square numbers are

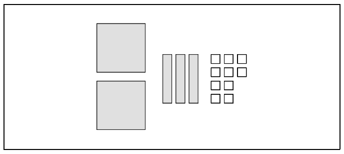
$$1,\ 4,\ 9,\ 16,\ 25,\ 36$$

Can you name the next six perfect square numbers in the series? If you take a number of chips from this list you will be able to arrange them into a perfect square, just as the name suggests.

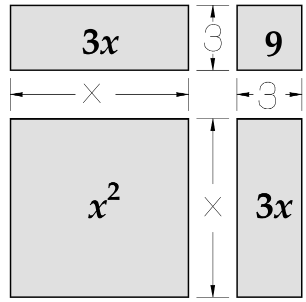
In the same way, if we multiply a polynomial having two terms (a **binomial**) times itself, we get a rectangle which has the same length and width: a *perfect square*.

Using chips, if we multiply the quantity $(x + 3)$ times itself, giving $(x + 3)^2$ or x plus three, quantity squared, we will be making a rectangle having the same length and width: a square.





Breaking this square into its four smaller areas we find that two of them, the units and the x^2's, are smaller squares.



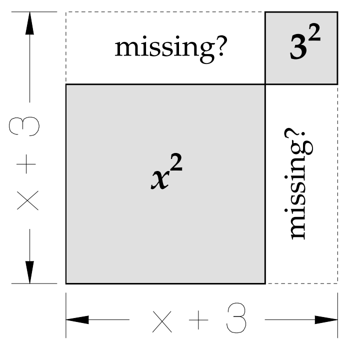
The remaining rectangles, the x's, are both the product of one side of each of these smaller squares $(x \cdot 3)$.

Although this example seems obvious when working with chips, it is important to remember when using the symbolic language of algebra, that

$$(x+3)^2 \quad \textbf{\textit{is not}} \quad x^2+3^2$$

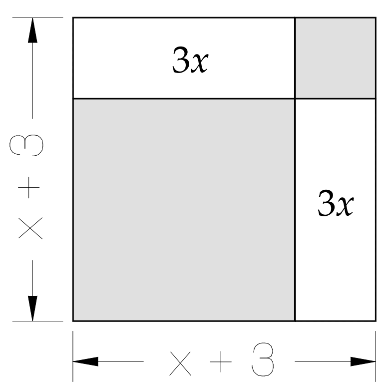
With the chips, we can see that these two expressions cannot be equal:

$(x+3)^2$ is not just x^2 plus 3^2



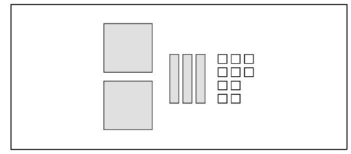
We must include the two rectangles which each have area $3x$. Remember the FOIL method:

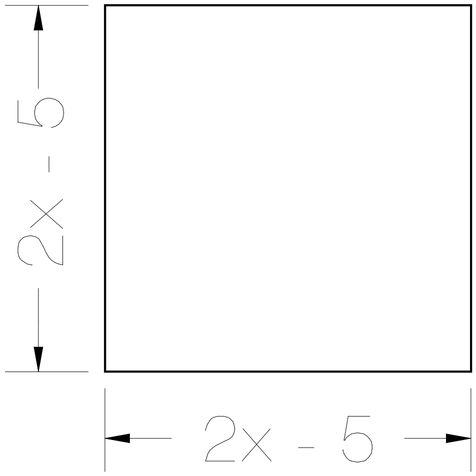
$$(x+3)^2 = (x+3)(x+3)$$
$$= x^2 + \mathbf{3x} + \mathbf{3x} + 9$$
$$= \mathbf{x^2 + 6x + 9}$$



When we include all four of the areas shown we get the correct result.

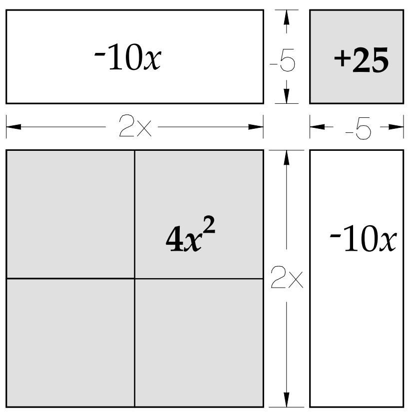
Now consider a second example of multiplying a binomial by itself to form a perfect square: $(2x - 5)^2$, or two x minus 5, quantity squared:





$$(2x - 5)^2 = (2x - 5)(2x - 5)$$

Filling in the four smaller rectangles within this diagram we again find two squares and two rectangles.



The two squares are both positive (colored side up),

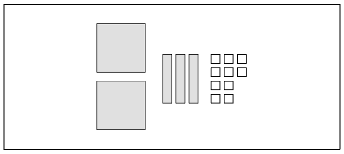
$$(2x)(2x) = 4x^2$$

$$({}^-5)({}^-5) = {}^+25$$

but this time the x-bars are negative (white side up) since

$$({}^-5)(2x) = {}^-10x$$

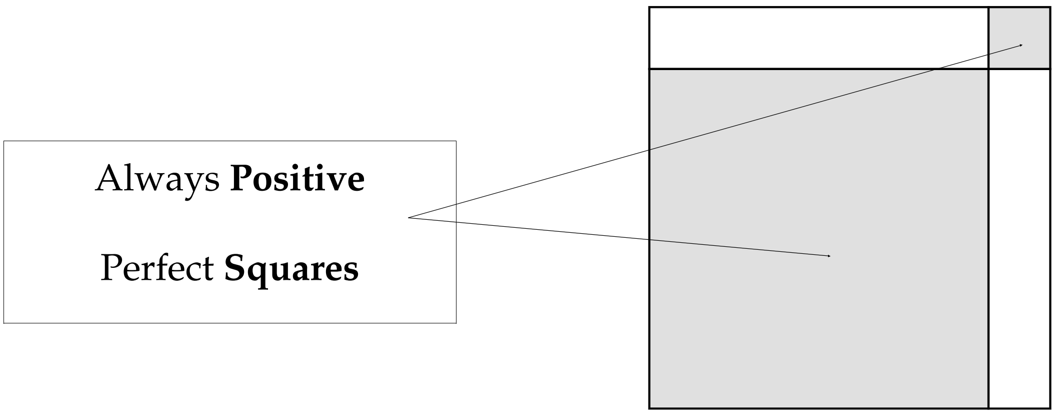
$$(2x)({}^-5) = {}^-10x$$



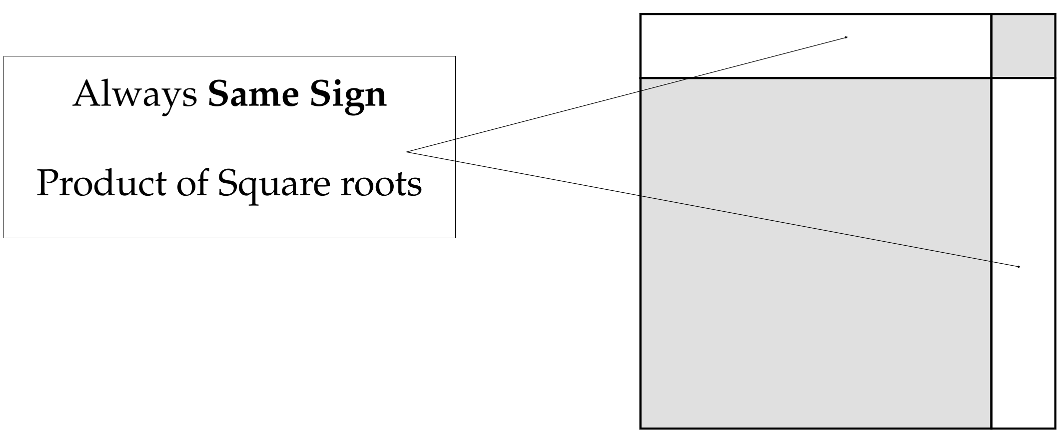
Again, using the FOIL method of symbol multiplication we get all four of the included areas and the correct result:

$$\begin{aligned}(2x-5)^2 &= (2x-5)(2x-5)\\ &= (2x)(2x)+(2x)(-5)+(-5)(2x)+(-5)(-5)\\ &= 4x^2-10x-10x+25\\ &= \mathbf{4x^2-20x+25}\end{aligned}$$

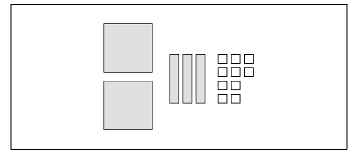
The smaller squares (x^2 pieces and units) within a perfect square are always positive in value (colored side up). This is because we get both of them by multiplying a number times itself, which always gives a positive result.



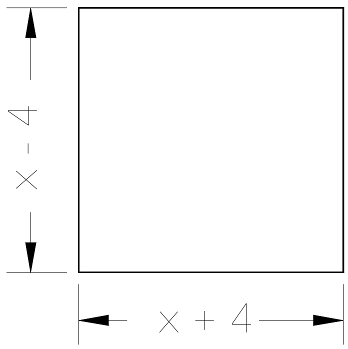
As the two examples demonstrate, the x-bars in a perfect square trinomial can sometimes be positive and sometimes be negative,. But in any one perfect square, all of the x-bars must be the *same sign*, either all plus or all minus. The number of x-bars will always equal the product of the square roots of the units square and the x^2 square, times two (because there are two groups of x-bars).



The Difference of Two Perfect Squares

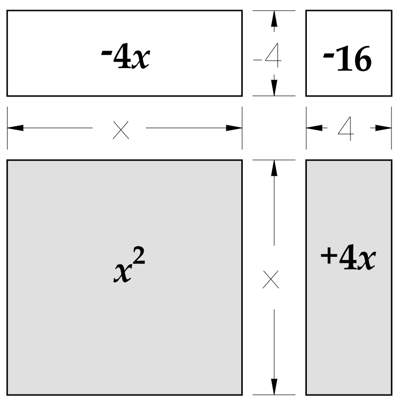


The two binomials $(x + 4)$ and $(x - 4)$ look very similar to each other; their only difference is the sign on the second term. If we multiply these two binomials together we get an interesting result.

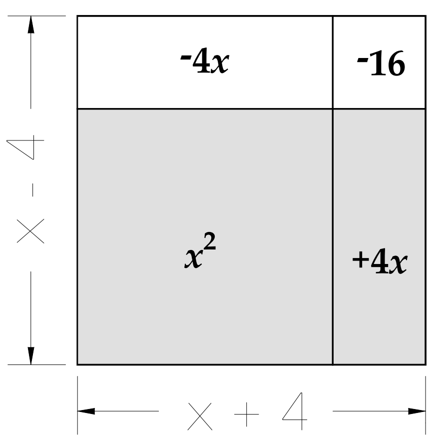


$$(x + 4)(x - 4)$$

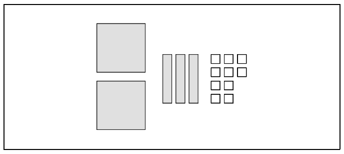
We again have a figure which appears square, but this time the two sides will have some pieces of different colors.



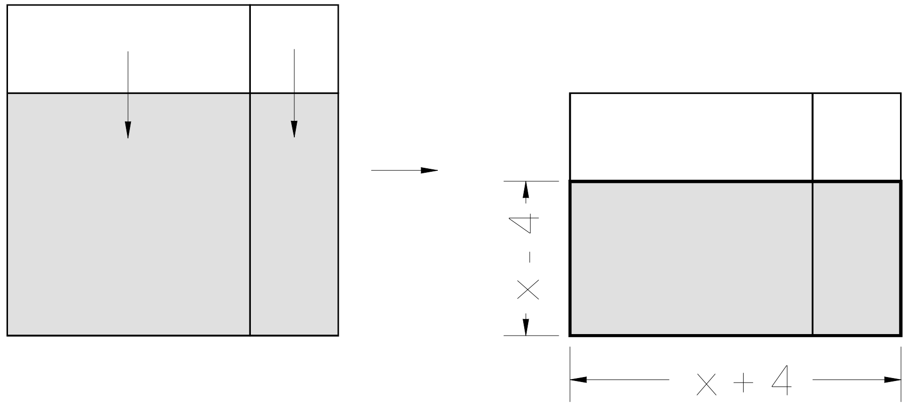
Shown as one rectangle, our example now looks like:



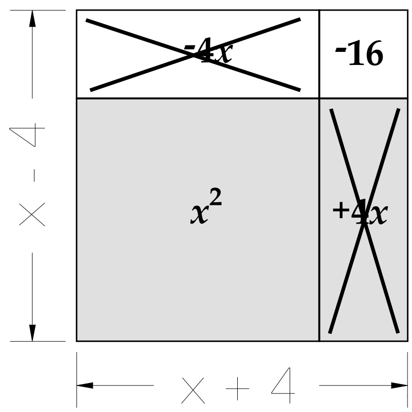
$$(x + 4)(x + 4) \;=\; x^2 - 4x + 4x - 16$$



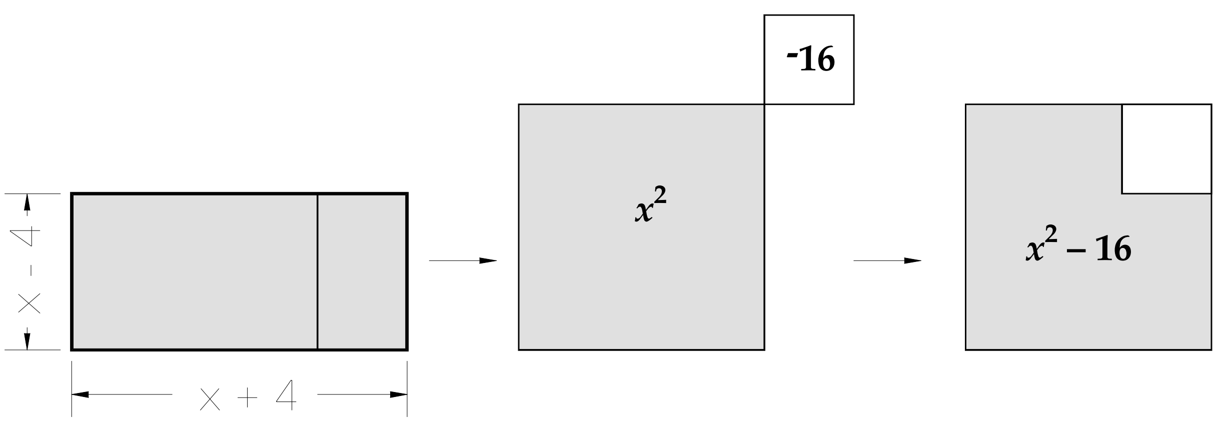
If we overlay the pieces and subtract the negative areas from the positive areas, we see that the resulting area is not really a square, but a rectangle having dimensions $(x + 4)$ and $(x - 4)$.

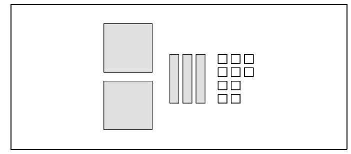


If we let the positive and negative pieces cancel in a different way we get an equivalent and surprising result.



The $^{+}4x$ and the $^{-}4x$ cancel each other out, leaving only x^2 and $^{-}16$. So we see





Now we can see why such products, products of binomials which differ only in the sign on the second term, are called *the difference of two perfect squares*; they are one square subtracted from another. We can use the FOIL method of multiplying symbols to verify this result.

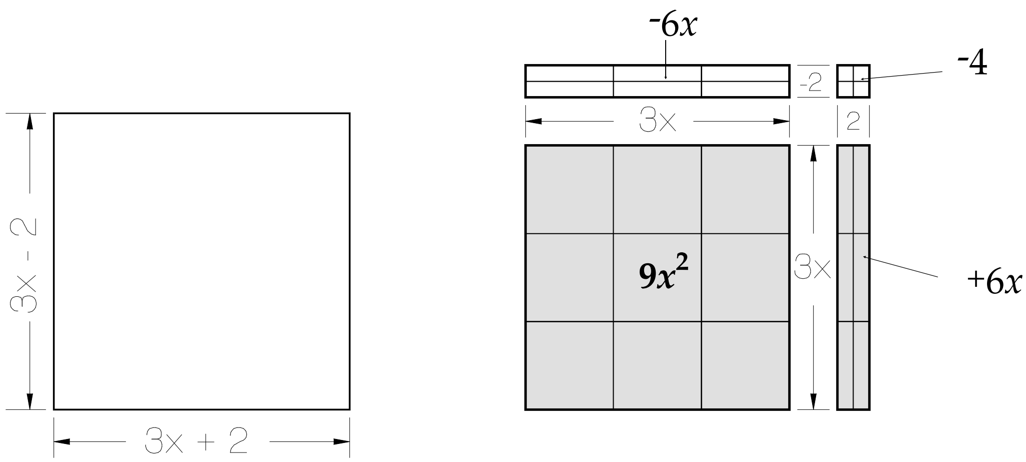
$$\begin{aligned}(x+4)(x-4) &= (x)(x)+(x)(-4)+(4)(x)+(4)(-4)\\ &= x^2-4x+4x-16\\ &= x^2+0x-16\\ &= x^2-16\end{aligned}$$

In the result, each of the two terms is a perfect square and the negative sign means to take the *difference*, or subtract, one perfect square from the other.

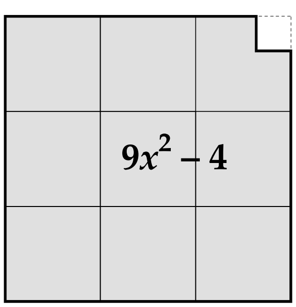
Here is a second example of a product which will generate the difference of two perfect squares:

$$(3x+2)(3x-2)$$

Again the two binomials in the product differ only in the sign on the second term.

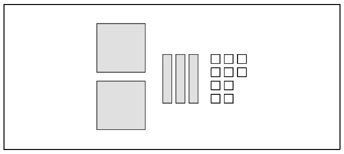


We can use both a diagram and the FOIL method to obtain the results of the product.



$$\begin{aligned}(3x+2)(3x-2) &= (3x)(3x)+(3x)(-2)+(2)(3x)+(2)(-2)\\ &= 9x^2-6x+6x-4\\ &= 9x^2-4\end{aligned}$$

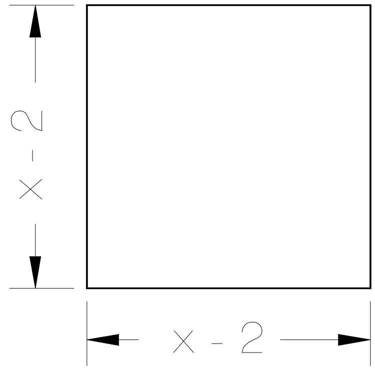
In both results we are subtracting one perfect square from another; *the difference of two perfect squares*.



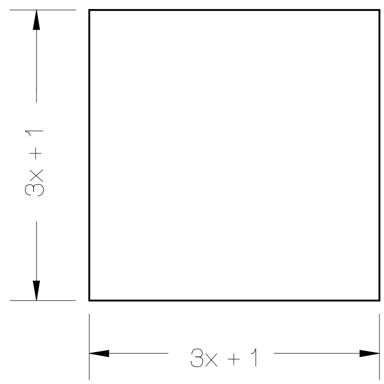
Exercises

Fill in the four smaller rectangles included in these perfect squares and then use the FOIL method to get the same results using algebraic symbols.

1. $(x-2)^2 =$



2. $(3x+1)^2 =$



Multiply out the following perfect squares; verify using a sketch.

3. $(7)^2$

4. $(2x-7)^2$

5. $(3x)^2$

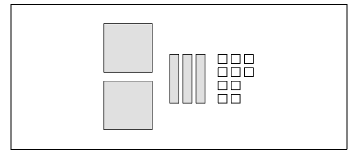
6. $(3x+2)^2$

7. $(x+4)^2$

8. $(2x-1)^2$

9. $(x-9)^2$

10. $(5x+3)^2$



Choose only the products which will generate the difference of two perfect squares, and work out only those products both in a diagram and using the FOIL method to verify your results.

11. $(2x + 5)(2x - 3)$

12. $(x + 2)(x - 2)$

13. $(3x + 4)(3x + 4)$

14. $(3x + 5)(5x - 3)$

15. $(3x + 5)(3x - 5)$

16. $(x - 7)(x + 7)$

17. $(2x - 1)(2x + 3)$

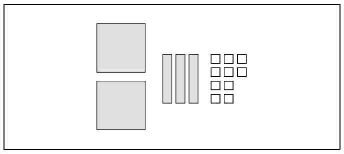
18. $(2x - 1)(x + 1)$

19. $(2x + 1)(2x - 1)$

20. $(3x - 2)(2x + 3)$

21. $(5x - 6)(5x + 6)$

22. $(7x - 1)(7x + 1)$



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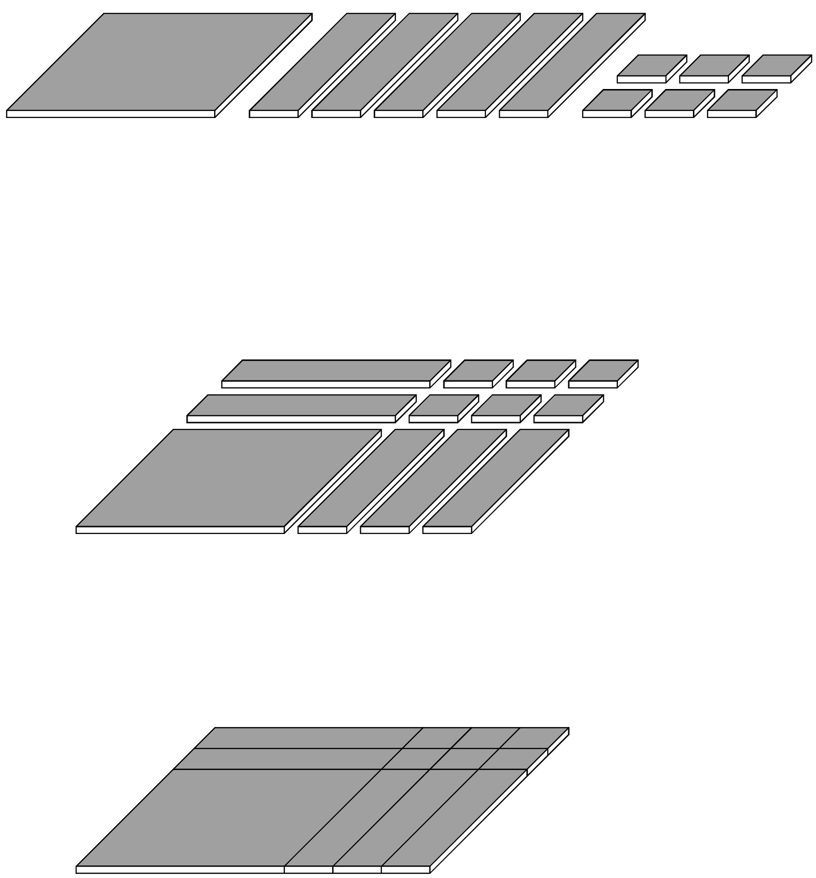
Chapter 5

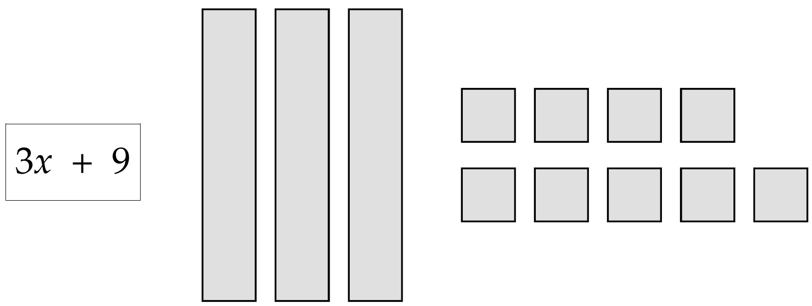
Factoring Polynomials



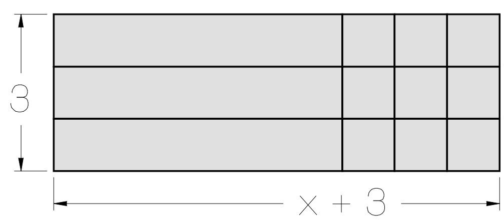
Section 1
Introduction: Rectangles and Factoring

The Meaning of Factoring

Factoring means taking an amount and rewriting the amount as a multiplication problem. Using chips, factoring is the process of taking a group of pieces and arranging them to form a rectangle. The factors are the dimensions (the length and width) of the rectangle. Start with $3x + 9$:

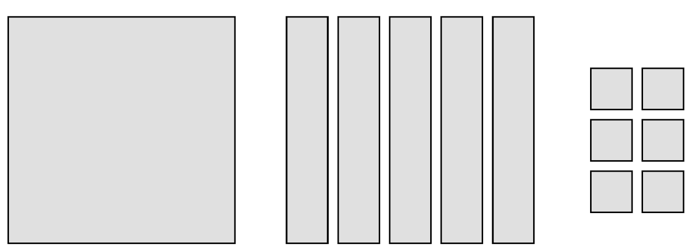


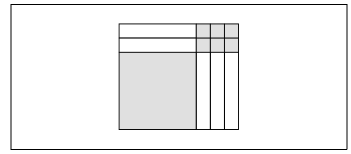
The polynomial $3x + 9$ makes a rectangle that is **3** by $\boldsymbol{x + 3}$:



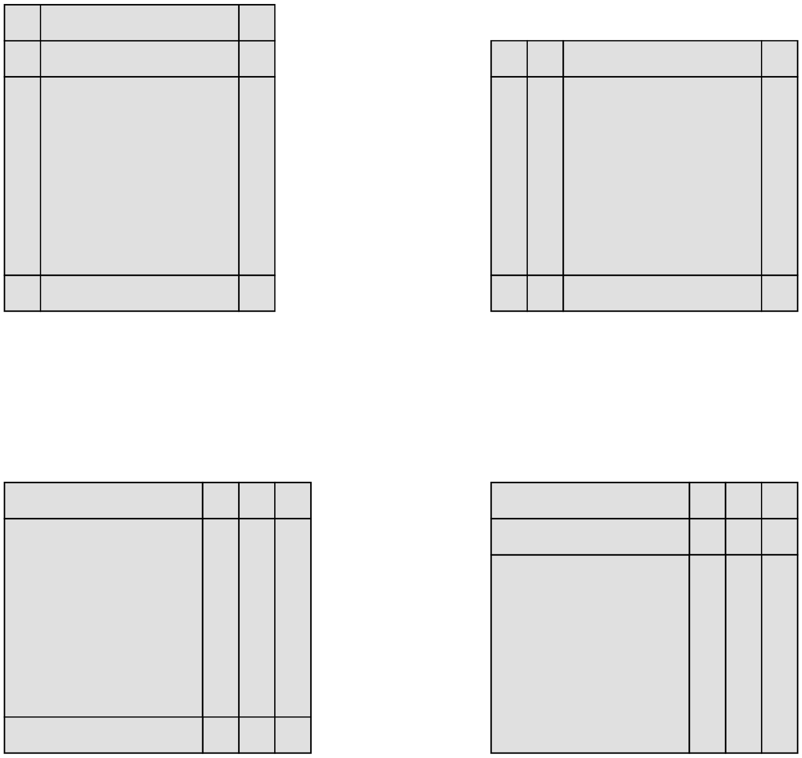
The **factors** of $3x + 9$ are **3** and $\boldsymbol{x + 3}$. This is the same as saying that the product of 3 and $x + 3$ is $3x + 9$. In both forms, *the rectangle means multiplication.*

Sometimes there are several ways to make a rectangle from a group of pieces. Start with the following chips:





From these pieces we could make the following rectangles:

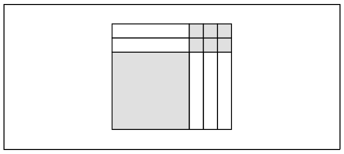


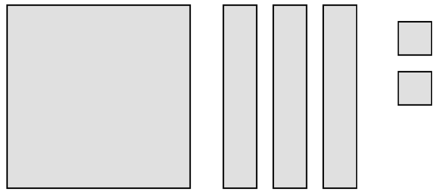
If we check the side lengths of each of these rectangles we find that they all have one direction which is a bar and three unit squares long ($x + 3$), and another which is a bar and two unit squares long ($x + 2$):

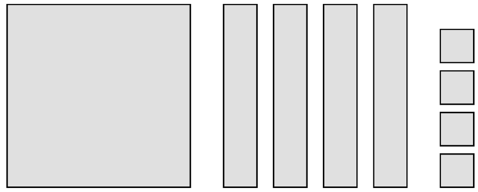
large square		five bars		six units	Factors into a rectangle	1 bar and 3 units	by	1 bar and 2 units
x^2	$+$	$5x$	$+$	6	$=$	$(x + 3)$	$\cdot$	$(x + 2)$

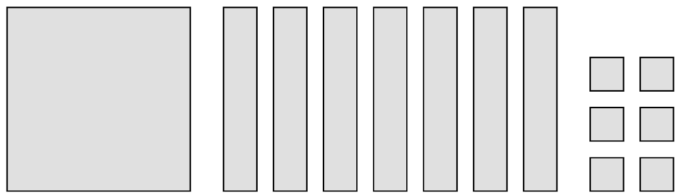
Exercises

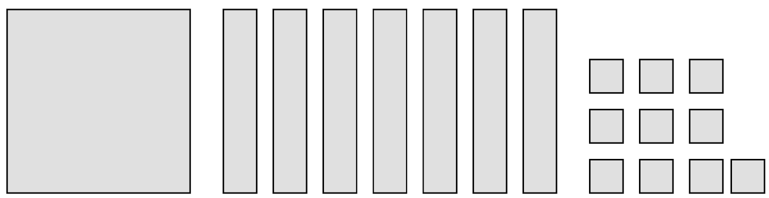
Make rectangles from the following groups of pieces. Remember that there must be no pieces left over and no holes or bumps in the rectangle.



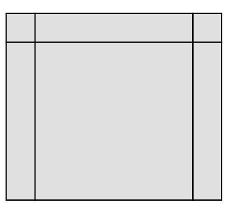
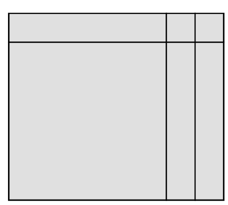
1. 

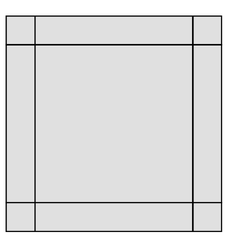
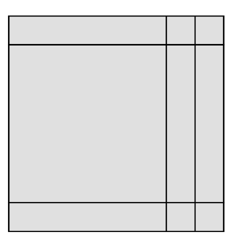
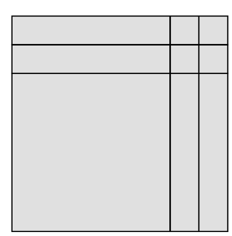
2. 

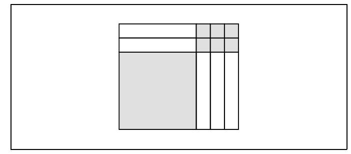
3. 

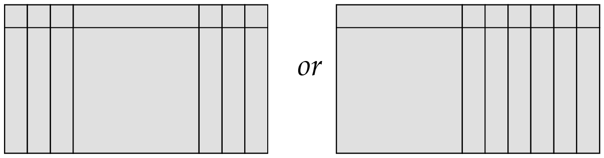
4. 

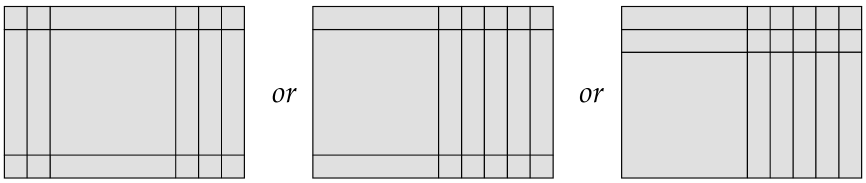
Some possible solutions look like this:

1.  *or* 

2.  *or*  *or* 

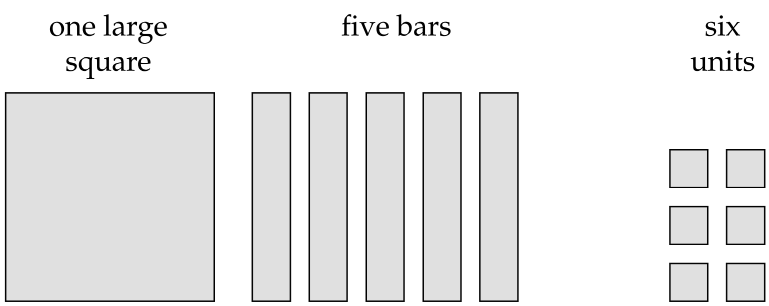


3. 

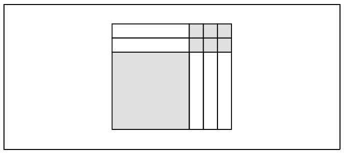
4. 

A Clue is in the Units

Look at these two similar examples (both shown before):

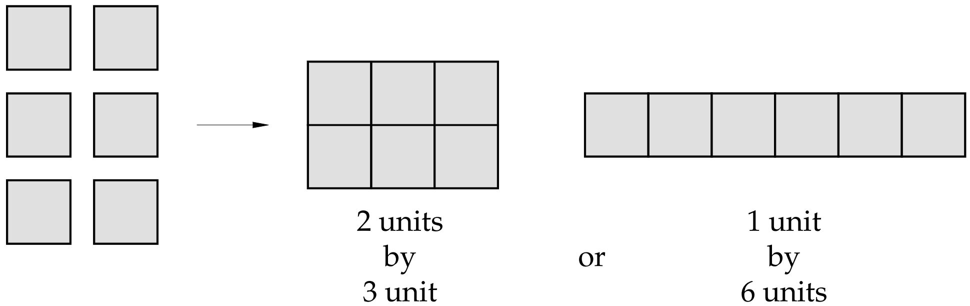


These two groups of pieces differ only in the number of bars. Obviously, since the two groups shown have different numbers of bars, the rectangles they make must have different dimensions.

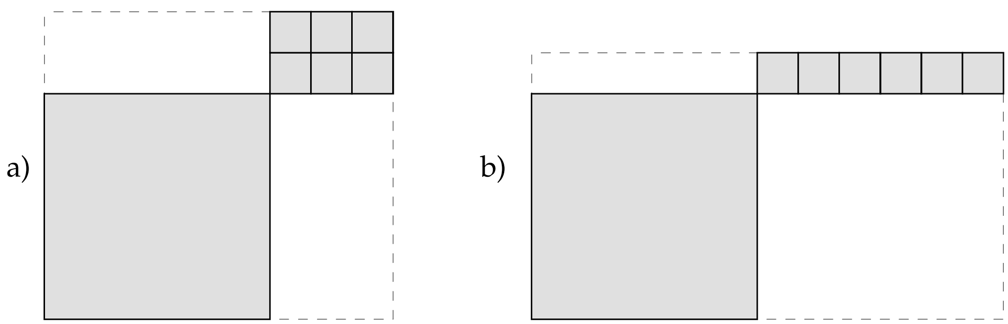


How can you tell before trying different rectangles which ones will work? A clue is in the number of units. Both of the groups shown have six unit squares; let's just look at the units.

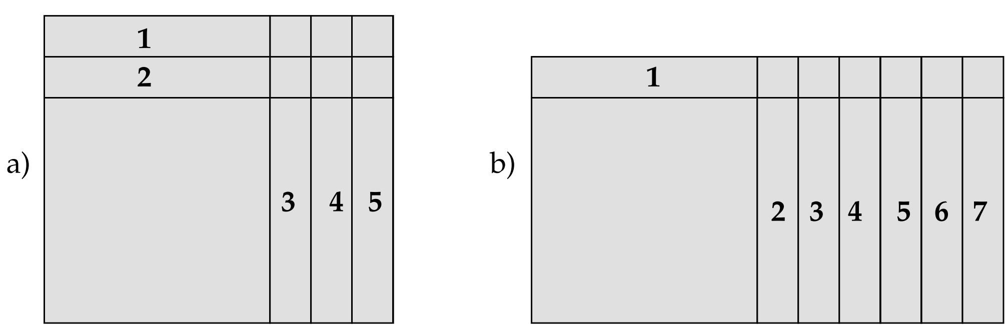
How many ways can you make rectangles using just six units?



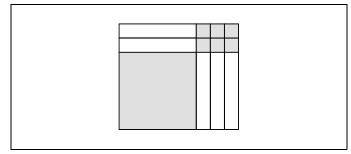
If we think of placing either of these smaller rectangles of units at the corner of the larger rectangle (the total amount), we see two different possible shapes for the larger rectangle.

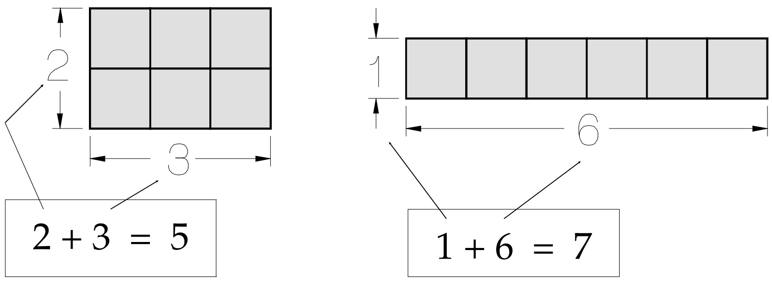


In picture (a) we could fill in the rectangle using two bars on the top and three bars on the side, for a total of five bars. In picture (b) we would need to fill in with one bar on top and six bars on the side, for a total of seven bars.



In each case the number of bars we need to complete the figure depends on the dimensions of the small rectangle of units.





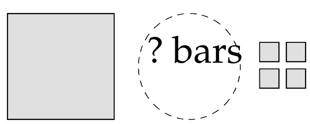
If the units rectangle is (2) by (3) we need 2 + 3 or 5 bars to complete the figure. If the units rectangle is (1) by (6) we need 1 + 6 or 7 bars to complete the figure.

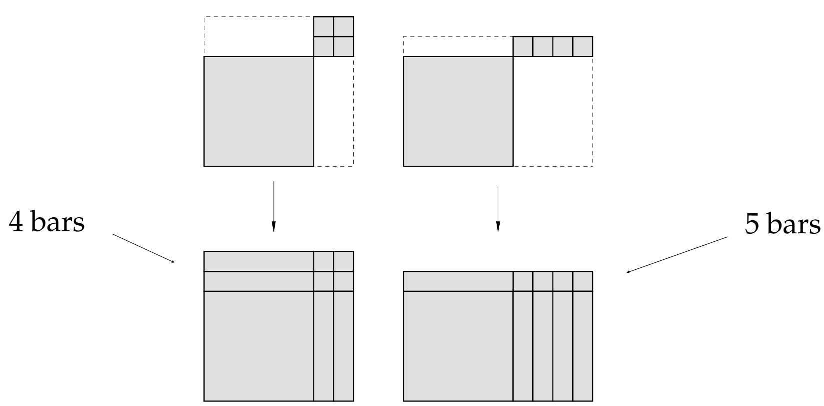
There are only these two ways to make small rectangles using six unit chips. So if we start with one big square and six unit chips we must have either five bars or seven bars in order to make a rectangle which has no holes and no pieces left over. (You can try making rectangles using one big square and six unit chips to see if any are possible with numbers other than five or seven bars.)

Let's Try Predicting

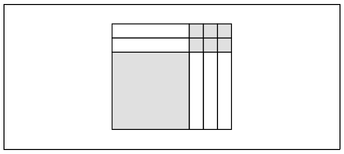
If you are given one big square and any specific number of unit chips, you can learn to predict how many bars you will need to complete each figure.

What if you have one big square and four unit chips? How many ways could you make rectangles and how many bars would you need for each? There are two ways to make a small rectangle using four unit chips:





So we could use either four bars or five bars to make a rectangle.



The product is given by the total number of pieces—large squares, x-bars and units—while the dimensions of the rectangle are the factors:

large square	four bars	four units		1 bar and 2 units	by	1 bar and 2 units
x^2	$+ \; 4x$	$+ \; 4$	$=$	$(x + 2)$	$\cdot$	$(x + 2)$

large square	five bars	four units		1 bar and 4 units	by	1 bar and 1 unit
x^2	$+ \; 5x$	$+ \; 4$	$=$	$(x + 4)$	$\cdot$	$(x + 1)$

Exercises

Set up the following polynomials with chips and factor:

1. $x^2 + 4x$

2. $x^2 + 5x$

3. $x^2 + 6x + 9$

4. $x^2 + 5x + 4$

5. $x^2 + 8x + 15$

6. $x^2 + 8x + 12$

7. $x^2 + 7x + 12$

8. $x^2 + 9x + 14$

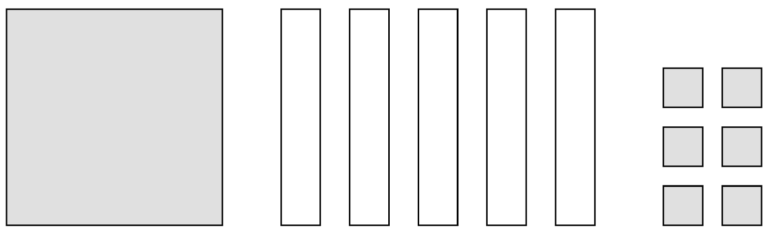
9. $x^2 + 8x + 16$

10. $x^2 + 9x + 20$

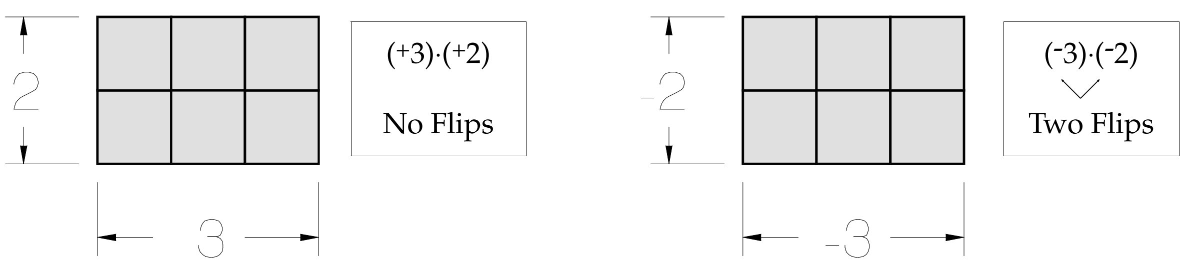
Section 2
Positive Units, Negative Bars

Factoring with Negative Bars

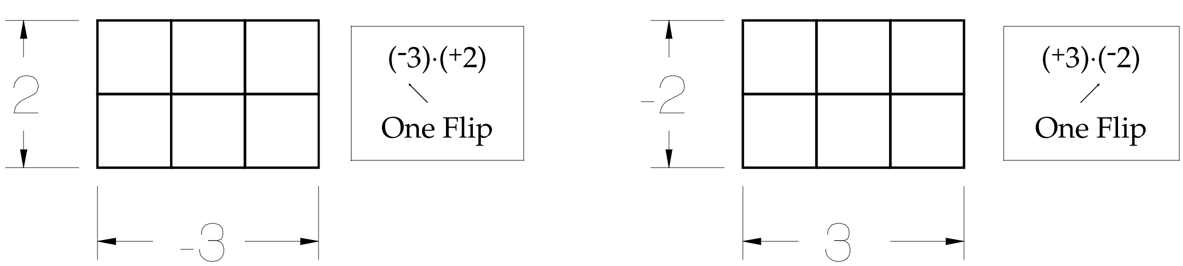
How can we factor polynomials with negative bars?

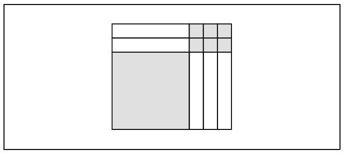


To make a rectangle from pieces having positive units but negative bars we need to remember how to multiply two numbers having signs (see POSITIVE AND NEGATIVE NUMBERS, Section 5 or POLYNOMIALS, Section 1). We can get a positive answer (colored rectangle) from multiplying two positive numbers, or from multiplying two negative numbers.

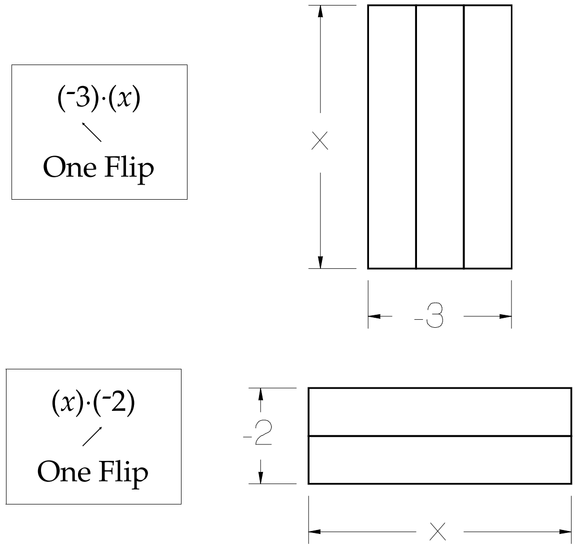


Similarly, we get a negative answer (white rectangle) from multiplying two units having different signs.

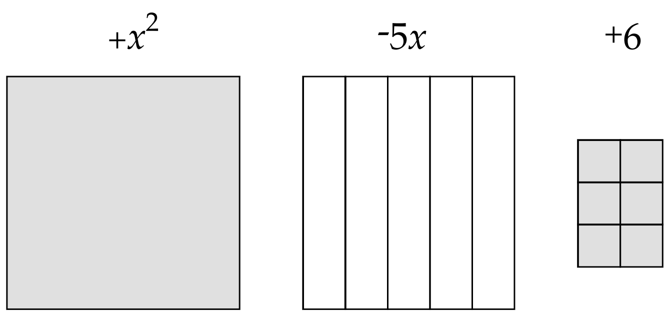




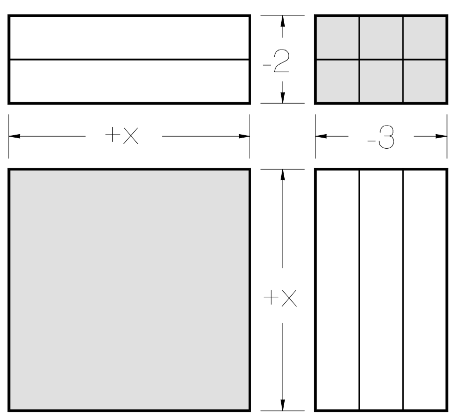
In the same way, we get rectangles made from white (negative) bars whenever one dimension (factor) of the rectangle is positive but the other dimension (factor) is negative.



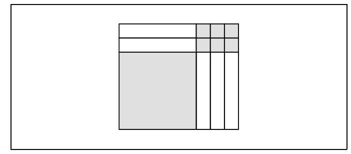
For example, look at the pieces below:

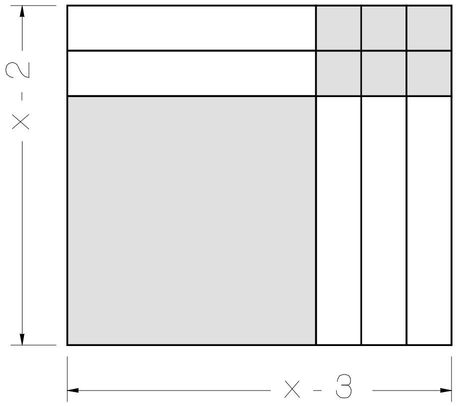


We can make a large rectangle (using four smaller rectangles) in the following way:



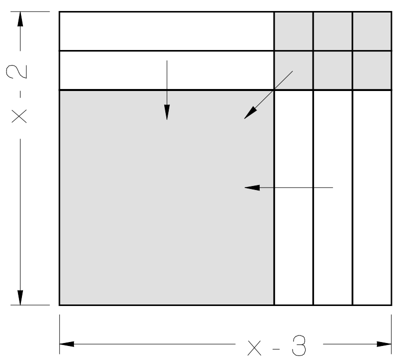
Each of the four small rectangles has its color (sign) determined by the signs of its two dimensions. Then the composite large rectangle looks like this:



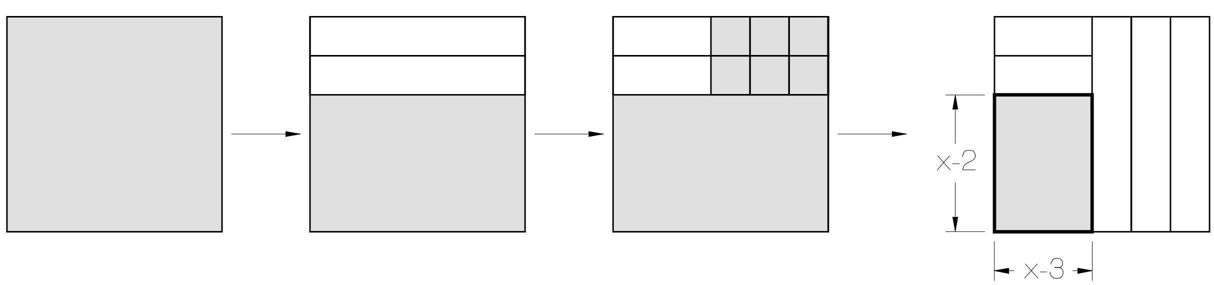


$$x^2 - 5x + 6 = (x-2)(x-3)$$

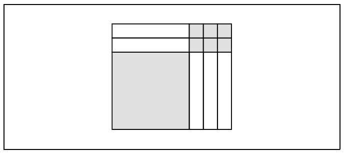
The dimensions of the rectangle are most easily read along the bottom and up the left side. The edges of the large colored square are each $+x$, and the short ends of the white bars are each a negative one ($^-1$). In this figure the white areas can be thought of as canceling out colored areas leaving a rectangle with actual dimensions of $x - 2$ and $x - 3$, as shown below.



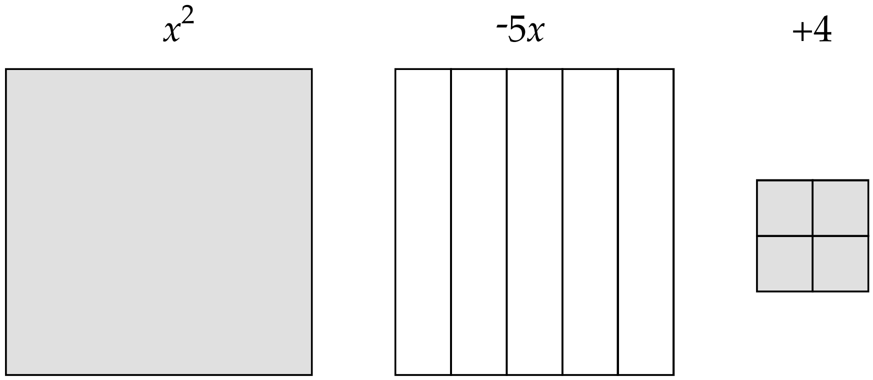
Start with x^2	Put on ^-2x bars. (Cancels)	Put on $^+6$ units. (Adds back)	Put on ^-3x bars (cancels). The result is $(x-2)$ by $(x-3)$



As in the above illustration, the x-bars and units subtract from and add to the original x^2 piece. First, place negative x-bars to cancel out some of the area. Then add back area by placing the positive units on top of the negative bars. Finally, cancel out area with the remaining negative bars. The resulting rectangle is two less than x on one side and three less than x on the other.

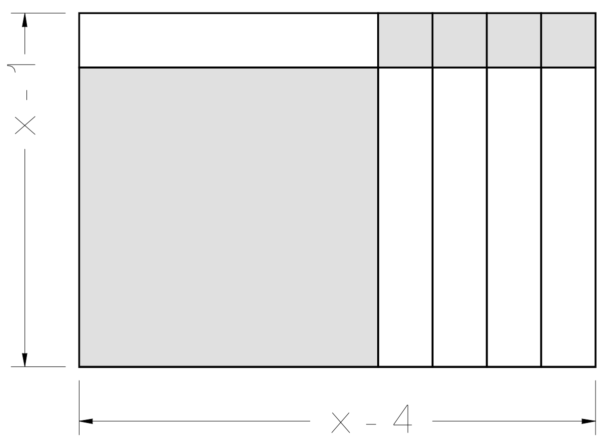


Example: Make a rectangle from the pieces given below:



Solution: The four single chips can only form two possible rectangles—2 by 2 or 1 by 4. Of these two possibilities, only the 1 by 4 corner rectangle would require 5 bars (4+1) which, in this case, are all negative. Looking at the rectangle's dimensions we see that

$$x^2 - 5x + 4 = (x - 4)(x - 1)$$



Exercises

Use chips and factor the following polynomials by making rectangles and noting their dimensions.

1. $x^2 - 4x + 3$

2. $x^2 - 6x + 8$

3. $x^2 - 8x + 12$

4. $x^2 - 7x + 12$

5. $x^2 - 7x + 10$

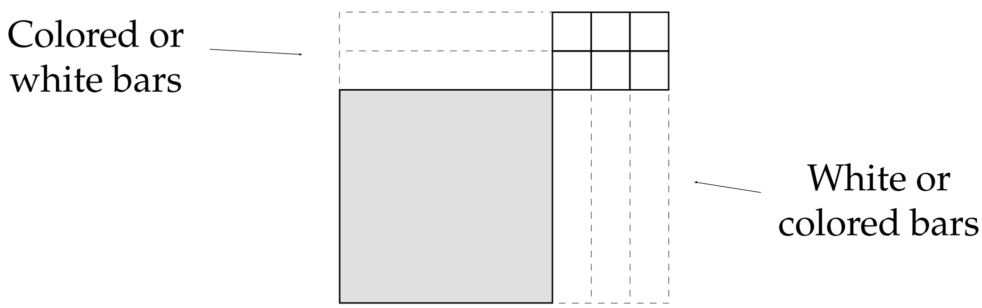
6. $x^2 - 10x + 16$

Section 3
Rectangles Having Negative Units

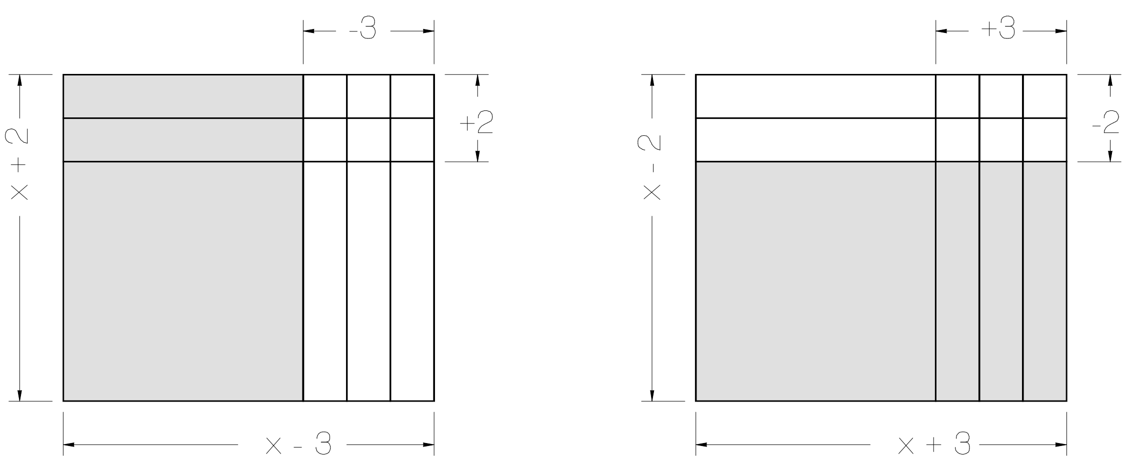
Factoring with Negative Units

As we just reviewed, if a rectangle has a negative value (white side up), it means that one dimension of the rectangle is positive while the other dimension is negative.

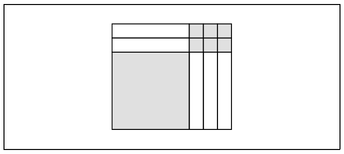
In the case of a polynomial, this means that if the large square (x^2) is colored (positive), and the small units (single chips) are white (negative), then when we make a complete rectangle we will need some colored bars and some white bars. (You may want to review POLYNOMIALS, Section 3.)



The color and number of the bars will match the positive or negative values of the dimensions of the units rectangle. For example we could imagine two different rectangles having $+x^2$ and $^-6$ units:

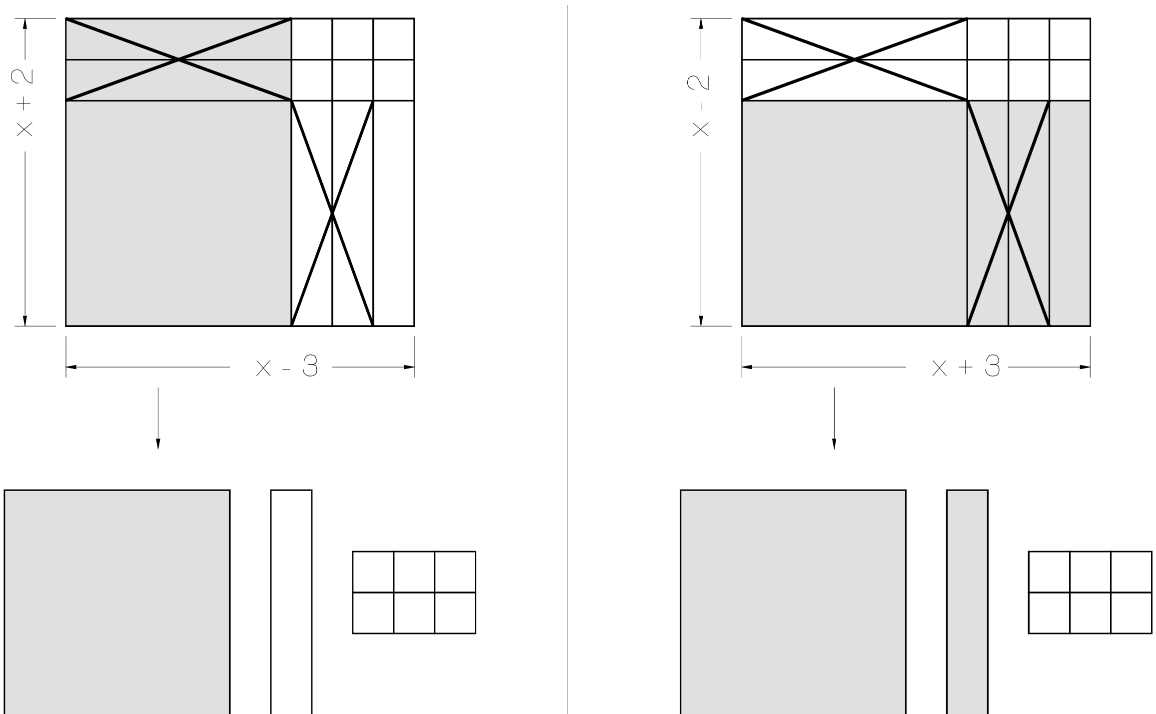


One of these rectangles has two positive bars and three negative bars; the other rectangle has two negative bars and three positive bars.



How Can We Tell Which to Use?

If we begin with a polynomial where some of the bars are positive and some are negative, when we combine like terms (put all the bars together), some of them are going to cancel out.

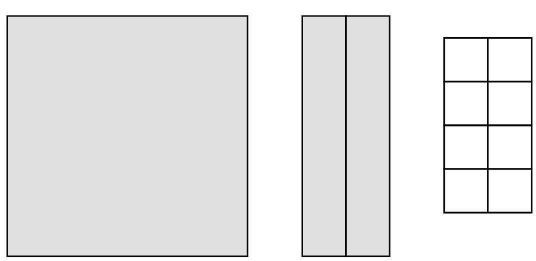


$x^2 - x - 6$ $\qquad$ $x^2 + x - 6$

The sign of the bars which are left over after canceling will match the sign of the larger dimension of the units rectangle.

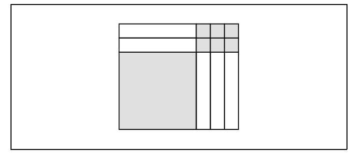
Working Backwards

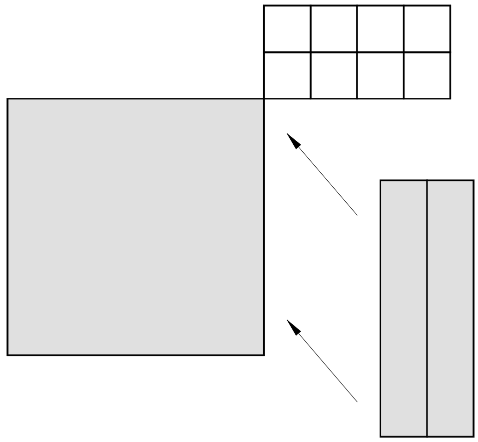
In order to factor a polynomial having negative units, like the following one,



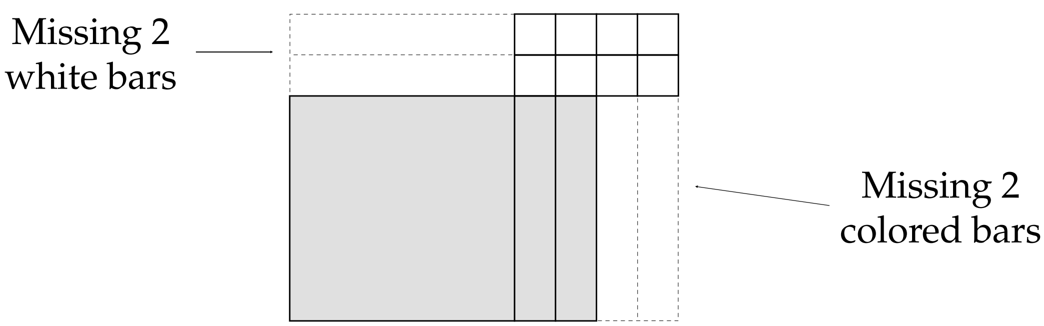
$$x^2 + 2x - 8$$

we begin by putting the unit rectangle at the corner of the big square and putting the bars we have along the *longer side* of the units rectangle.

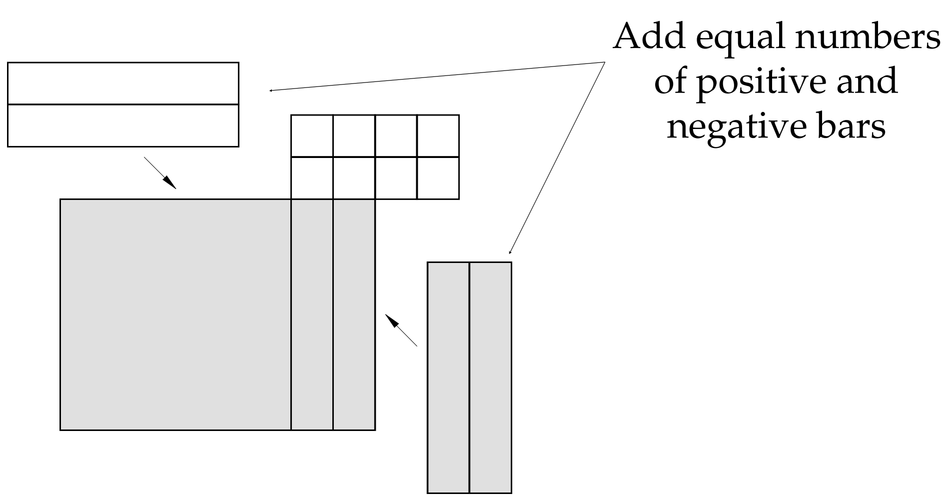


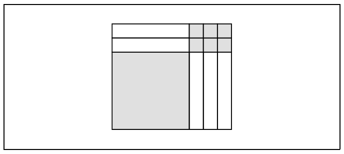


When we look at the result we should see that we are missing *equal numbers* of positive and negative bars.

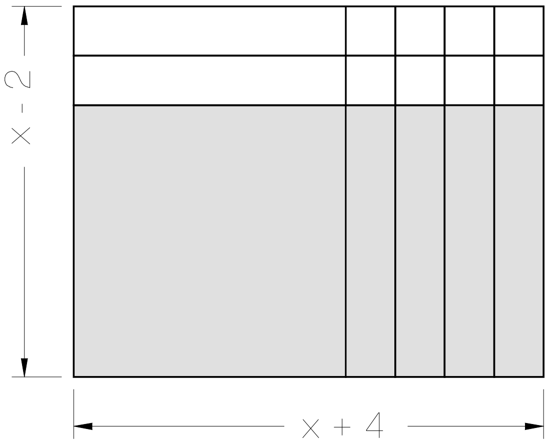


We know that if we add positive and negative bars to the figure in equal numbers we are adding zero, because these pairs of white and colored bars would cancel out.



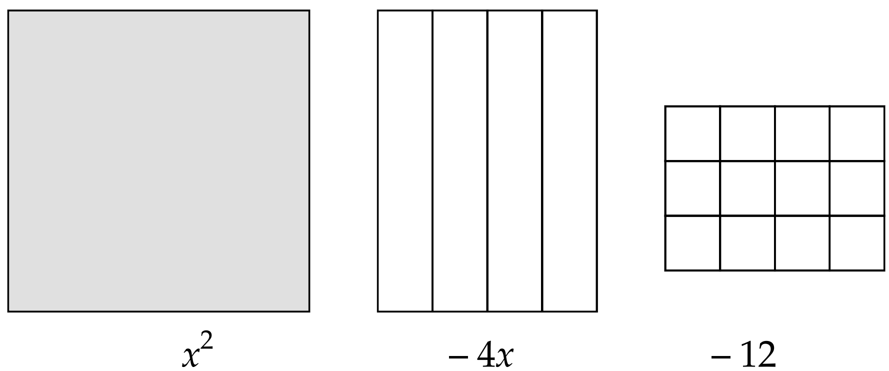


So our final figure looks like this:

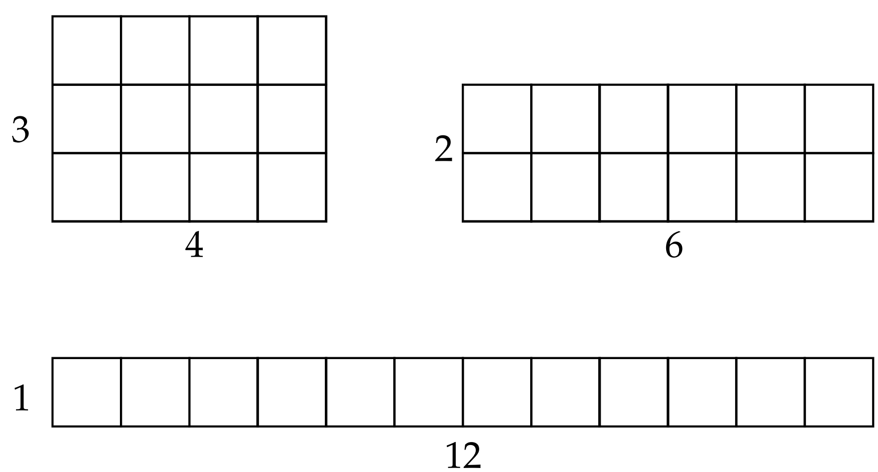


$$x^2 + 2x - 8 = (x-2)(x+4)$$

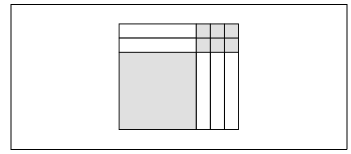
Let's look at another example. Factor $x^2 - 4x - 12$:



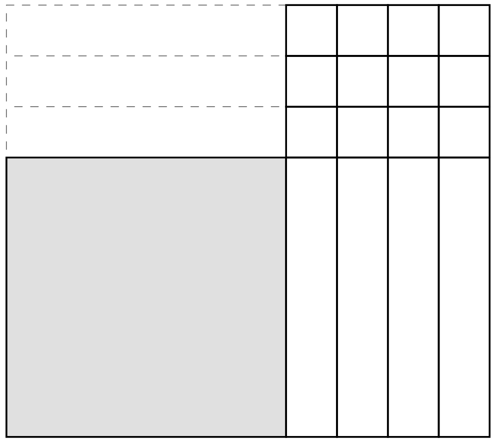
In this case the $^-12$ units can be put into three different possible rectangles:



Putting each of these three small rectangles into the larger complete rectangle we have the following options:

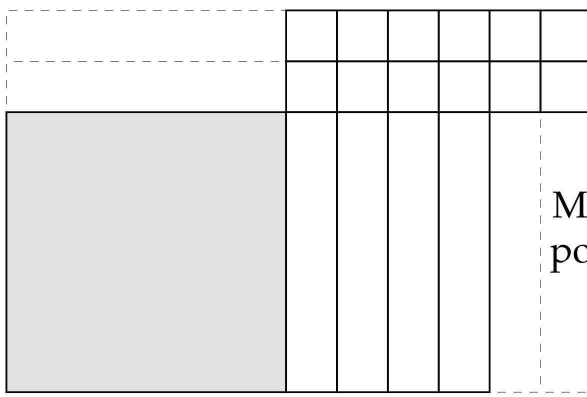


No



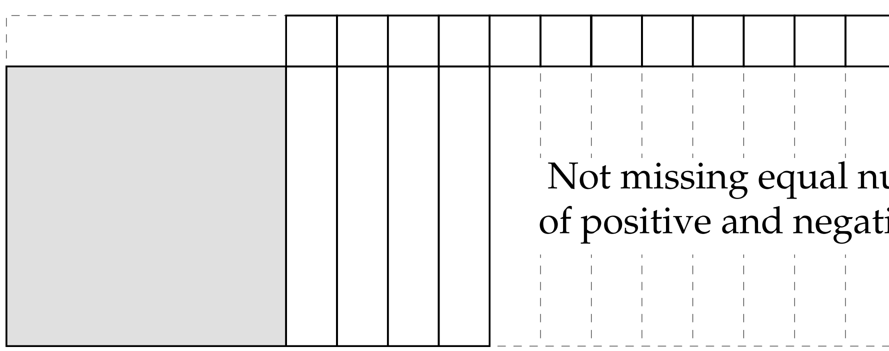
Not missing equal numbers of positive and negative bars

YES

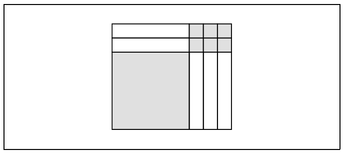


Missing equal numbers of positive and negative bars

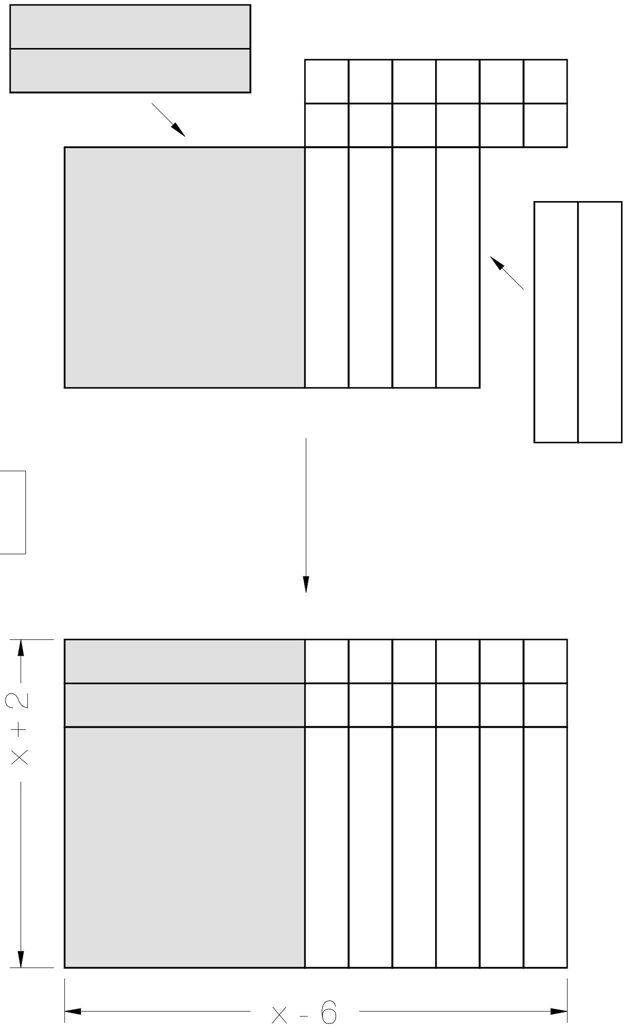
No



Not missing equal numbers of positive and negative bars



To complete the figure we must add equal numbers of white and colored bars, so only the middle figure will work. The solution to our example is



$$x^2 - 4x - 12 \;=\; (x - 6)(x + 2)$$

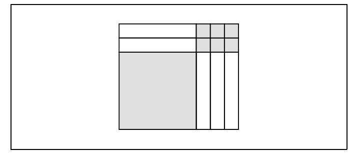
Summary

When factoring a polynomial, remember to take all the pieces and fit them into a large rectangle made up of four smaller rectangles. Each of the smaller rectangles has its sign or color determined by the signs of its two dimensions, and in all, they must match both the signs and the numbers of the pieces you start with.

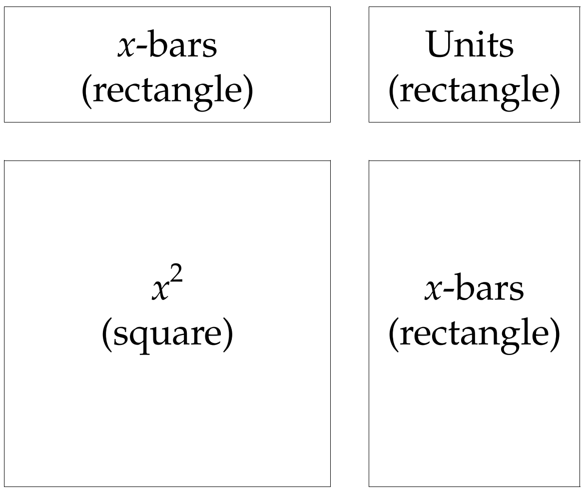
Here are the steps in the factoring process:

- **Consider the possible factors of the units term. (Note the required signs.)**

- **Pick the pair of factors which add together to give the required number of *x*'s.**



- **If the units rectangle is positive, then the two factors add, and all of the *x*-bars should just fit along its left and bottom edges.**
- **If the units rectangle is negative, then equal numbers of positive and negative *x*-bars will be missing when the given *x*-bars are placed along the long side of the units rectangle. In such a case, fill in both the missing positive and negative *x*-bars, remembering that adding equal numbers of positive and negative bars is really adding zero.**
- **When you have finished this process, the dimensions of the large rectangle you have made are the factors of the polynomial with which you began.**



Exercises

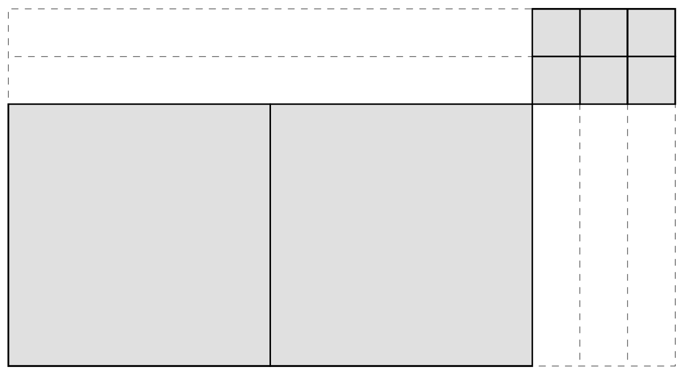
Factor the following polynomials:

1. $x^2 + 5x - 6$
2. $x^2 - 2x - 8$
3. $x^2 - 7x - 8$
4. $x^2 - 11x - 12$
5. $x^2 - 5x - 6$
6. $x^2 + x - 12$
7. $x^2 + 8x - 9$
8. $x^2 - 2x - 15$
9. $x^2 + 2x - 15$
10. $x^2 - 6x - 16$

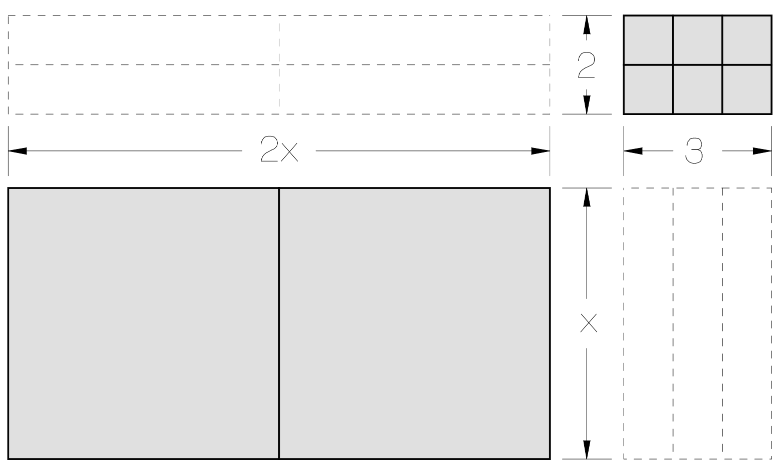
Section 4
Factoring Trinomials with More than One x^2

More than one x^2

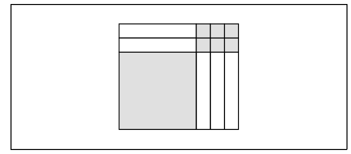
If we make a rectangle out of pieces including two large squares ($2x^2$)



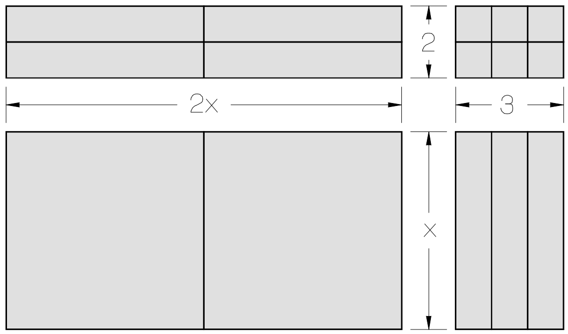
then we can see (from the example shown above) that the number of x-bars needed to complete the figure is more than we would need if we had only one large square.



The top rectangle of x-bars is now twice as long as before because it has to run along the top of two large squares instead of just one. To factor a trinomial having more than one x^2, we make one rectangle out of the large squares, and a second rectangle out of the unit chips, then the dimensions of these two smaller rectangles multiply together to determine the number of x-bars needed to complete the figure.

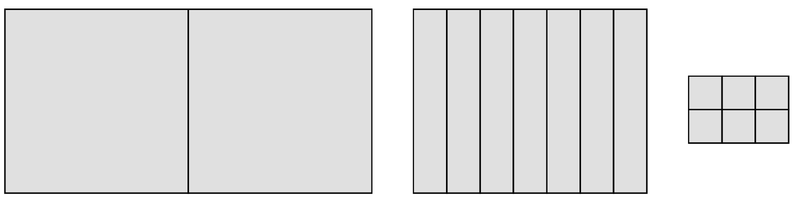


$(2x)(2) = 4x$

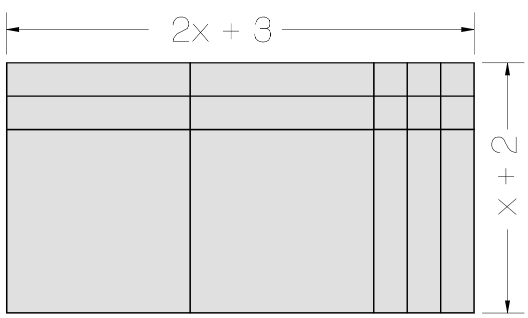


$(x)(3) = 3x$

Working backwards we see the following:

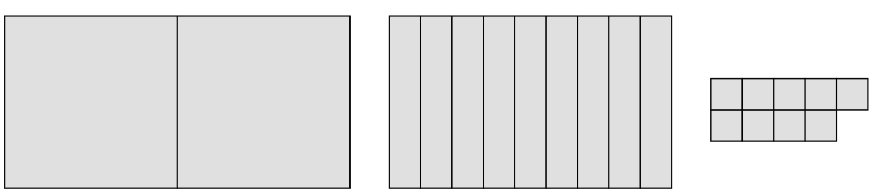


$2x^2$ $7x$ 6

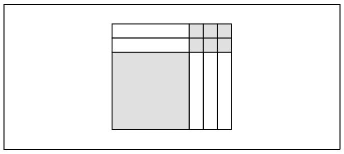


$2x^2 + 7x + 6$
equals
$(2x + 3)(x + 2)$

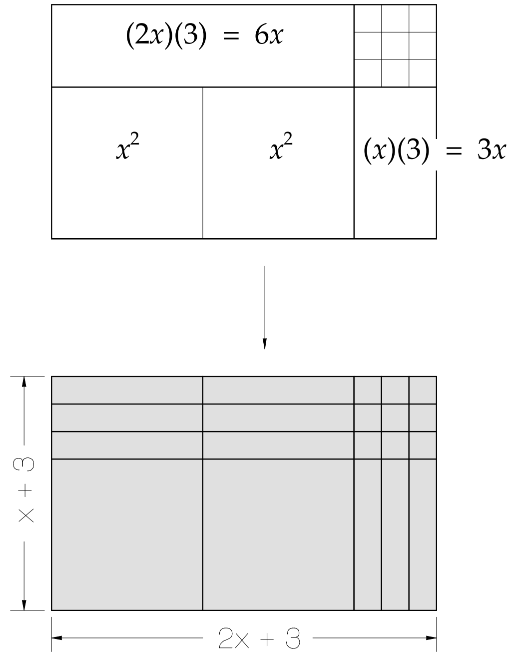
Example: Make a rectangle from the following pieces and use it to determine the factors of the given trinomial.



$2x^2$ $9x$ 9

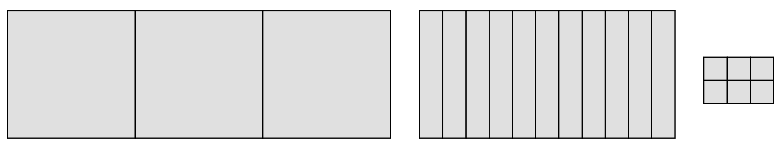


Solution:

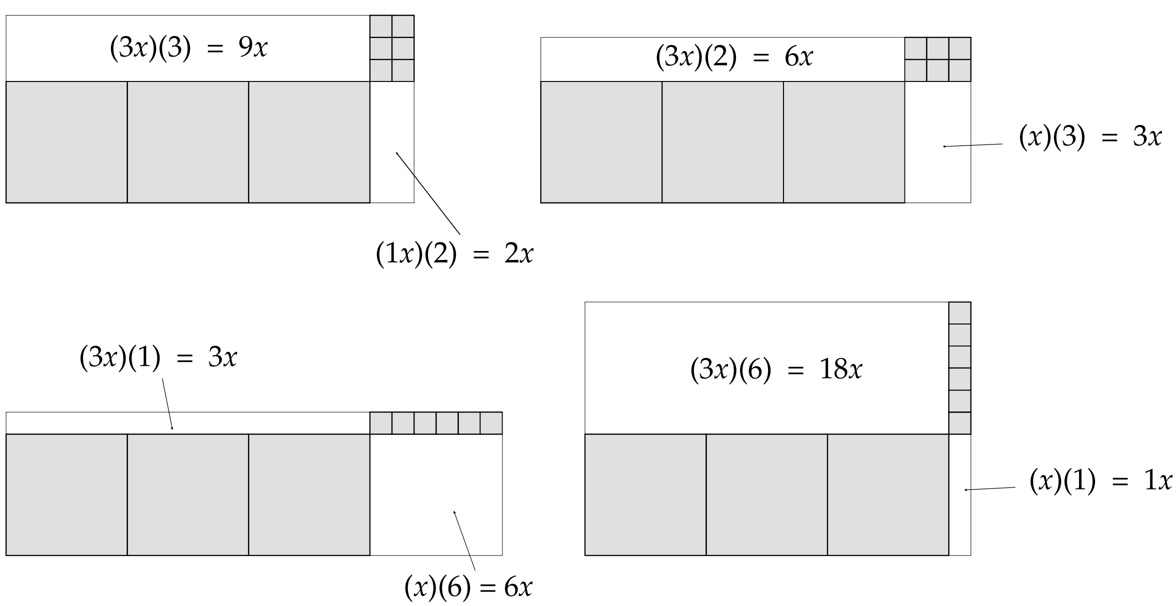


$$2x^2 + 9x + 9 = (2x + 3)(x + 3)$$

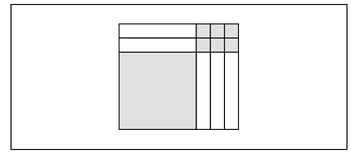
Example 2: Make a rectangle from $3x^2 + 11x + 6$:

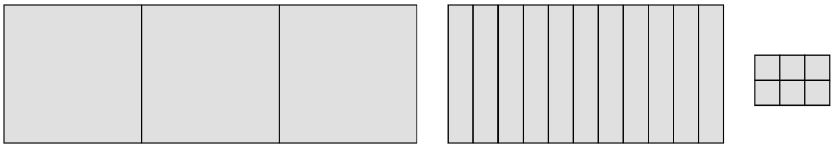


Solution: There are four possible ways to orient rectangles made from the large squares and the unit chips. Each of these will require a particular number of x-bars, as shown below:

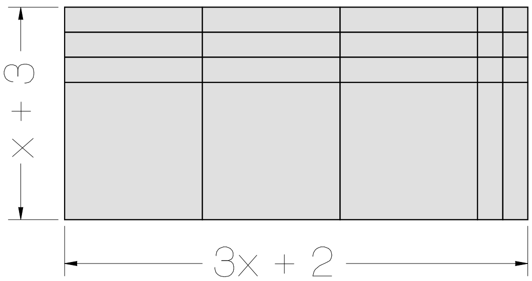


Which of these four possibilities requires 11 x-bars? What are the dimensions of this rectangle?





$$3x^2 + 11x + 6 = (3x + 2)(x + 3)$$



Exercises

Factor these polynomials:

1. $4x^2 + 4x + 1$
2. $3x^2 + 7x + 2$
3. $2x^2 + 7x + 3$
4. $3x^2 + 10x + 3$
5. $2x^2 + 5x + 2$
6. $2x^2 + 3x + 1$
7. $6x^2 + 11x + 3$
8. $6x^2 + 7x + 2$
9. $6x^2 + 11x + 4$
10. $4x^2 + 8x + 3$
11. $12x^2 + 31x + 20$ (Draw a picture instead of using chips)

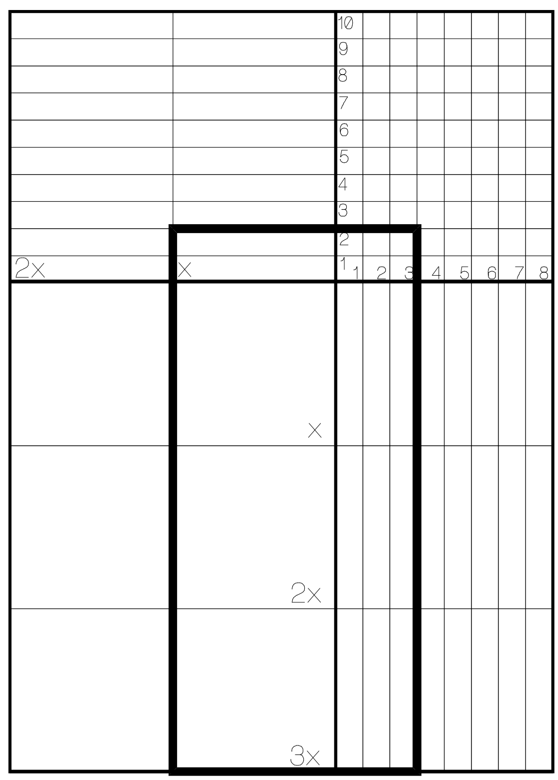
Section 5
Factoring Using the Grid

The Plastic Grid

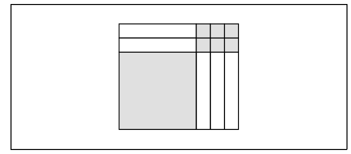
The plastic polynomial grid provided with this book can make factoring trinomials much easier than making rectangles out of the chips themselves. You can make a rectangle over your grid which has the proper dimensions for a given factoring problem. The previous example of

$$3x^2 + 11x + 6 = (3x + 2)(x + 3)$$

would look like this:



Notice that it doesn't matter which direction the rectangle is turned, as long as the correct number of pieces is used. Because the polynomial grid is plastic, it is possible to write on it using *water-soluble* marking pens.(Be sure the marking pens you use have ink which will wash off or you can ruin your plastic grid.) Just outline the areas you want with a heavy line. You can try different arrangements of units and squares in the same way that you move chips around.

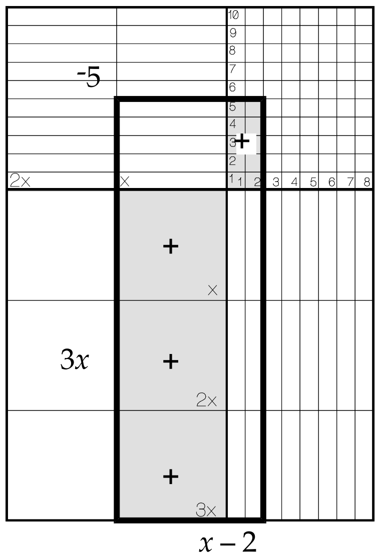


Positive and Negative Areas of the Grid

With the water soluble-marking pen you can mark positive and negative areas on the grid with a plus (+) or a minus (–) sign, and in this way keep them straight. (Of course you will remove the + and – marks after completing each problem). Just as mentioned before, the sign of each portion of the rectangle is determined by the signs of both its dimensions.

For example, let's use the grid to factor the trinomial

$$3x^2 - 11x + 10$$

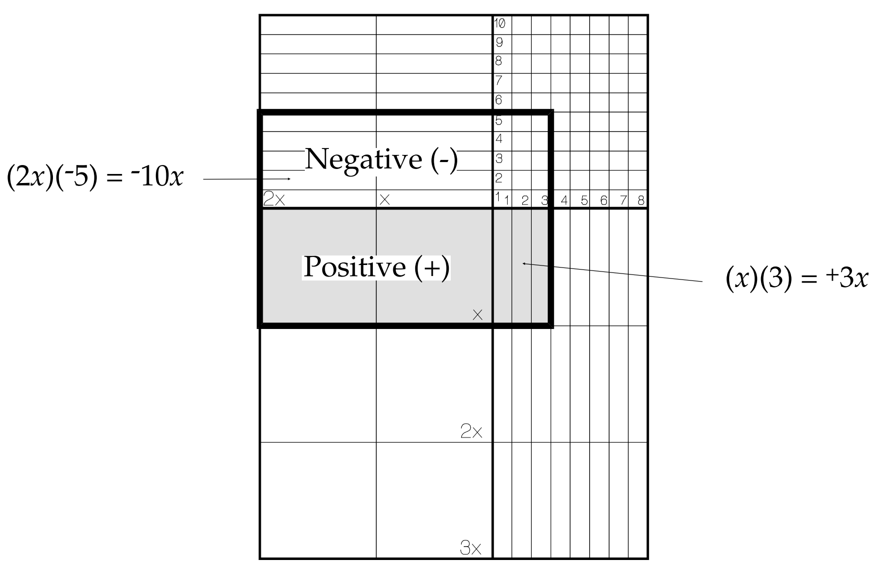


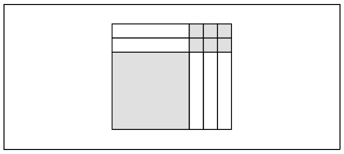
The result is:

$$(3x - 5)(x - 2)$$

Next, use the grid to factor

$$2x^2 - 7x - 15$$



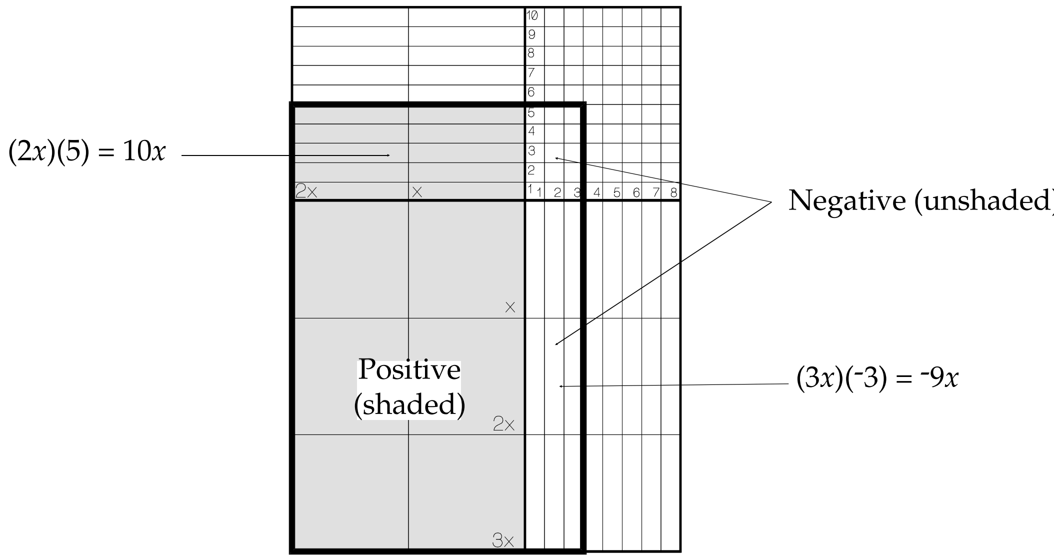


The result is:

$$(2x + 3)(x - 5)$$

Use the grid to factor

$$6x^2 + 1x - 15$$



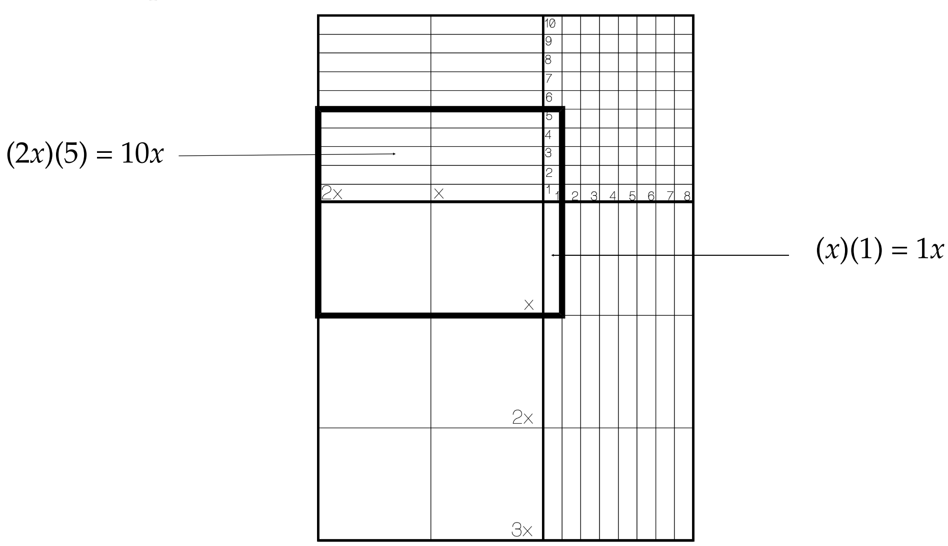
The result is:

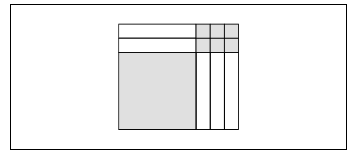
$$(3x + 5)(2x - 3)$$

Exercises

Use your grid to factor the following trinomials:

Example: $2x^2 + 11x + 5$





1. $3x^2 + 8x + 5$
2. $2x^2 + 11x + 12$
3. $3x^2 + 20x + 12$
4. $3x^2 + 10x + 8$
5. $3x^2 + 14x + 8$
6. $3x^2 + 25x + 8$
7. $2x^2 + 13x + 15$

(Remember, start by considering possible rectangles for the large x^2-squares, and for the small unit squares, then figure out which possibility gives the correct number of x-bars.)

Complete the following factoring problems using the plastic grid:

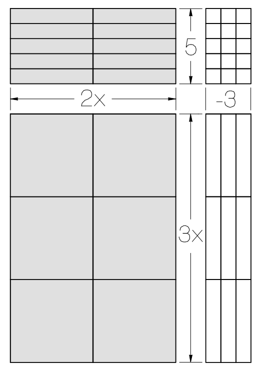
8. $x^2 - x - 6$
9. $x^2 + 4x - 12$
10. $2x^2 + 3x - 5$
11. $2x^2 - 7x + 6$
12. $4x^2 - 4x - 15$
13. $2x^2 + 7x - 15$
14. $6x^2 - x - 15$
15. $6x^2 + 11x - 10$
16. $2x^2 - 13x + 15$
17. $3x^2 - 2x - 5$
18. $2x^2 - x - 6$
19. $6x^2 + x - 2$

Section 6
A Shortcut Method

A Shortcut for Factoring

Let's look closely at the solution to the last example.

$$6x^2 + 1x - 15 = (3x + 5)(2x - 3)$$

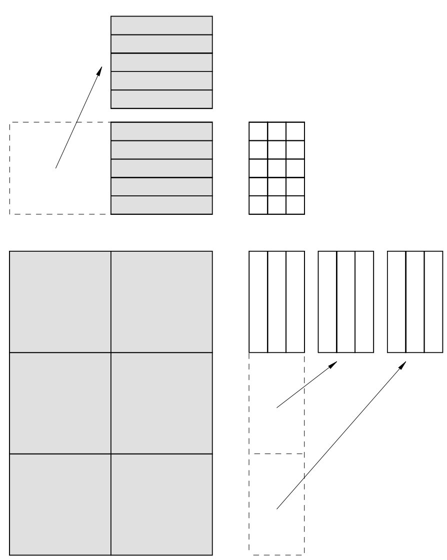


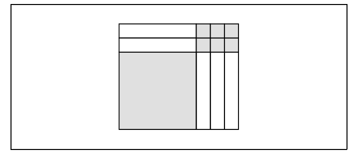
The two rectangles which have the x-bars in this figure have dimensions

$$(2x)(5) = 10x \quad \text{and} \quad (^-3)(3x) = {^-9x}$$

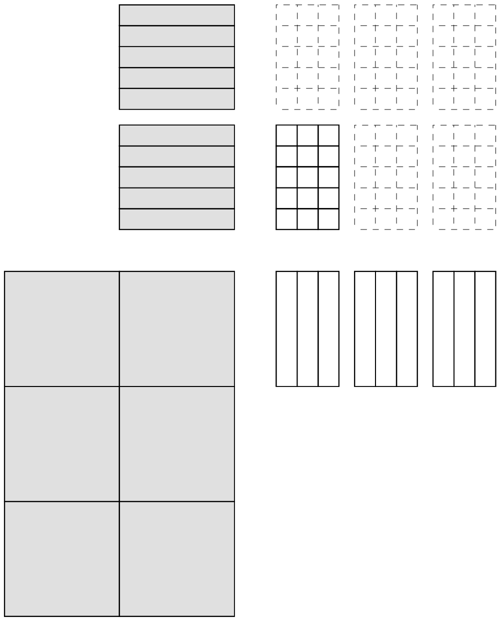
Notice that each of these rectangles of x-bars has one dimension which is a factor of $6x^2$ and another dimension which is a factor of $^-15$.

Mentally move the x-bars to the new positions shown here:



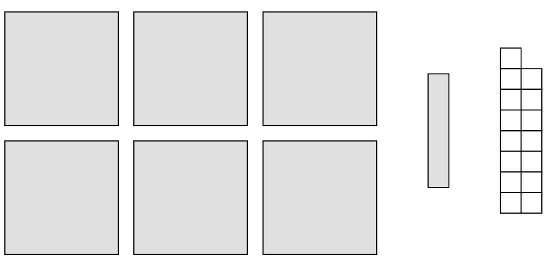


This configuration suggests imagining six rectangles, each having $^{-}15$ chips, as shown in the next diagram.

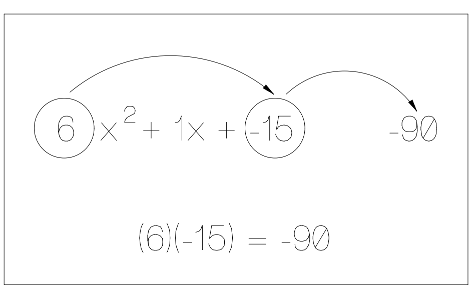


This arrangement will be the key to a shortcut factoring method for polynomials having more than one large square (x^2). *For a more detailed explanation of why this method works, please see the* APPENDIX.

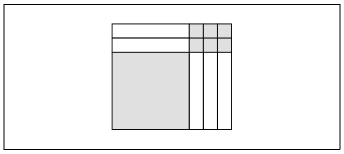
Let's begin with the original polynomial $6x^2 + 1x - 15$ and work through the shortcut factoring method.



First, multiply the 6 times the $^{-}15$. (Note that although we cannot know in advance how the chips are to be arranged, *any* arrangement of $6x^2$ and $^{-}15$ units will give 6 groups of $^{-}15$, or $^{-}90$ imagined unit chips in the corner.



This step corresponds to the picture we "imagined" above (when we started from knowing the solution).



Second, we list all of the ways we could possibly factor $^{-}90$, with the negative sign meaning that one factor will be positive (+) and the other negative (-).

Factors of $^{-}90$	
Factors	**Difference**
$90 \cdot 1$	89
$45 \cdot 2$	43
$30 \cdot 3$	27
$18 \cdot 5$	13
$15 \cdot 6$	9
	1

One factor is negative
One factor is positive.
The difference is positive
or negative

This list shows the dimensions of all the possible rectangles we could make using 90 white chips. But remember that besides multiplying to give $^{-}90$, the factors we are interested in must add together to give us the total number of x-bars we need. The expression

$$6x^2 + 1x - 15$$

has only $^{+}1$ x-bar, so we must find a pair of factors which add together to give a $^{+}1$. This requires that we use the factors

$$(^{+}10) \text{ and } (^{-}9),$$

and tells us that the two rectangles made from x-bars *must* have

$$^{+}10x \text{ and } ^{-}9x$$

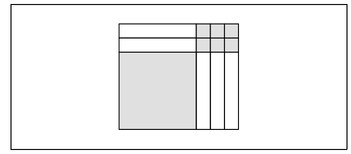
Knowing this we rewrite our original polynomial and replace the term $^{+}1x$ with the two terms $^{+}10x - 9x$, as shown below:

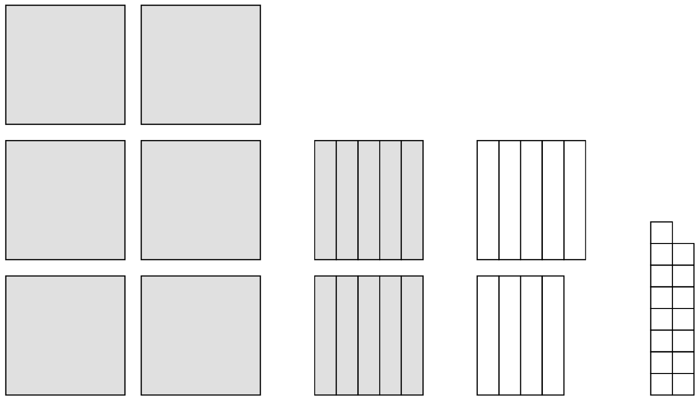
$$6x^2 + 1x - 15$$

$$6x^2 + (10x - 9x) - 15$$

Notice that these four terms correspond to the four parts of the rectangle which we know will be our final factored solution.

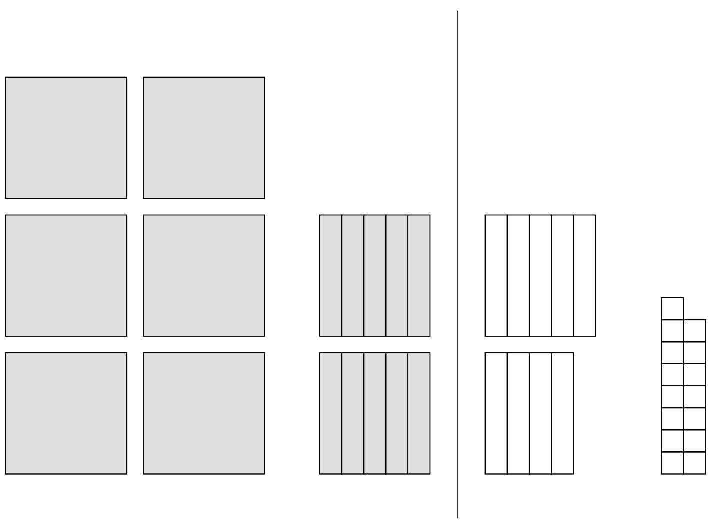
Now we have the following pieces to use:





$6x^2$ $\quad$ $10x$ $\quad$ $^{-}9x$ $\quad$ $^{-}15$

The third step in the process separates these four terms into two groups. Move the first two terms (the $6x^2$ and the $^{+}10x$ pieces) to one place, and move the last two terms (the $^{-}9x$ and the $^{-}15$ pieces) to a different place.

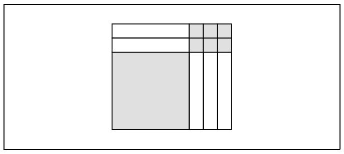


From each of these two groups take the largest common factor.

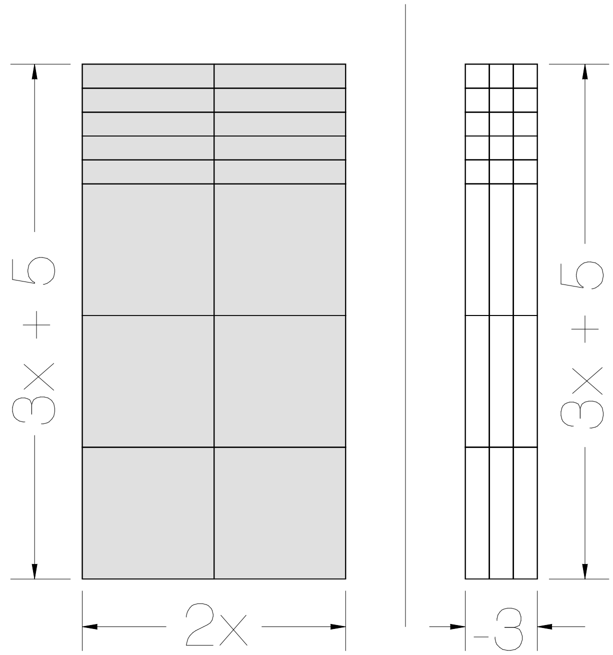
$6x^2 + 10x$	$-9x - 15$
$2x(3x + 5)$	$^{-}3(3x + 5)$

Largest Common Factors

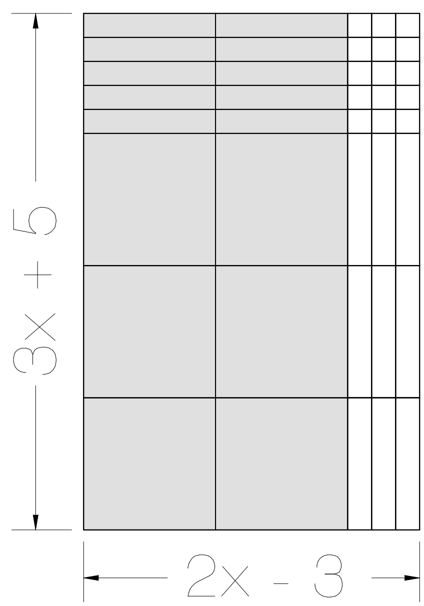
In each case the largest common factor is the width of a rectangle which can be made from the group of pieces, and the parentheses holding two terms is the length of the same rectangle.



This idea is illustrated below:

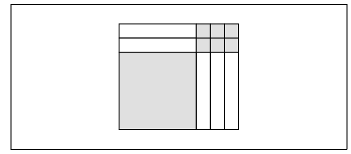


The surprise, which you may have already noticed, is that the rectangles we have made from the two separate groups of pieces have *the same length* ! We can put them side by side—they will form one large rectangle.



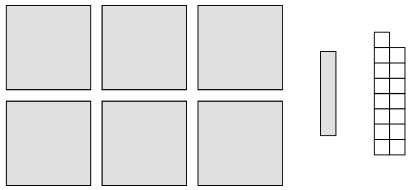
The dimensions of this rectangle are the factors of the original expression.

$$6x^2 + 1x - 15 = (2x - 3)(3x + 5)$$

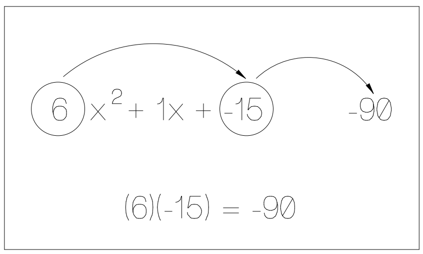


Shortcut Method: Summary

Begin with the original expression: $6x^2 + 1x - 15$:



- **Step 1: Multiply the first coefficient times the last number.**

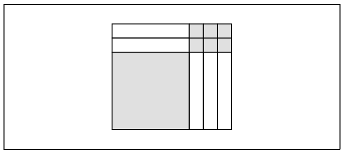


- **Step 2: List all the possible factors of the product.**

Factors of -90	
Factors	**Difference**
$90 \cdot 1$	89
$45 \cdot 2$	43
$30 \cdot 3$	27
$18 \cdot 5$	13
$15 \cdot 6$	9
	1

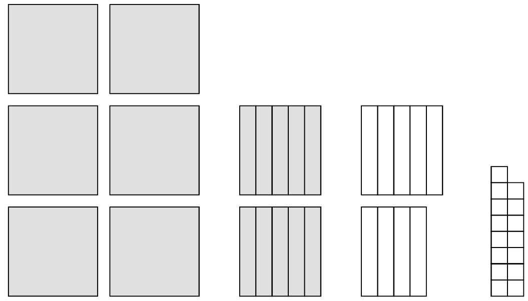
- **Step 3: Select the pair of factors which adds together to give the needed number of *x*'s.**

$$+10x - 9x = +1x$$



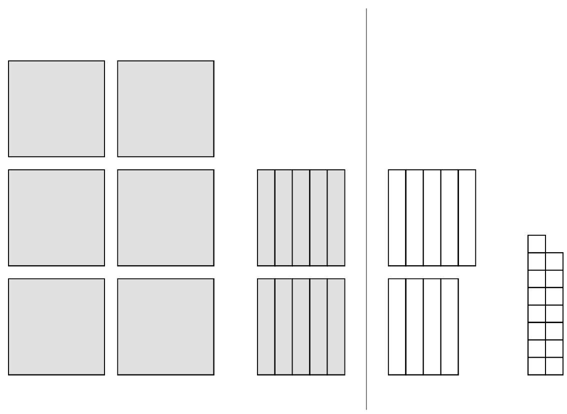
- **Step 4: Rewrite the given expression using four terms instead of three.**

$6x^2 + 10x - 9x - 15$



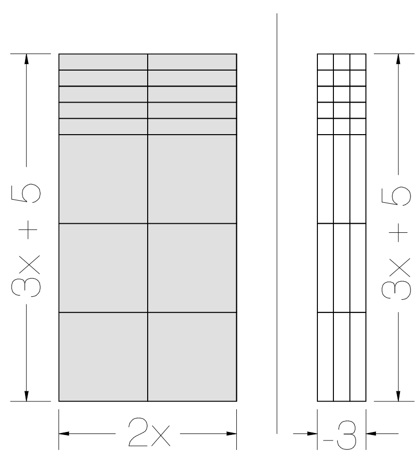
- **Step 5: Separate the first two terms and the last two terms. This makes two groups.**

$(6x^2 + 10x) + (^-9x - 15)$

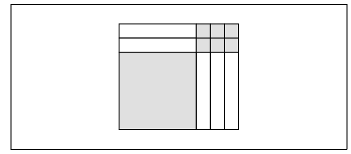


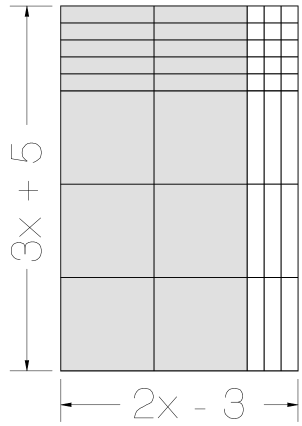
- **Step 6: Take the largest common factor out of each pair of terms. Make two rectangles.**

$2x(3x + 5) + {}^-3(3x + 5)$



- **Step 7: Put the two pieces together. (The two common factors go together in one new factor.)**





$(2x - 3)(3x + 5)$

Here's what you write down without using pictures:

$$6x^2 + 1x - 15$$

$$6x^2 + (10x - 9x) - 15$$

$$(6x^2 + 10x) + (^-9x - 15)$$

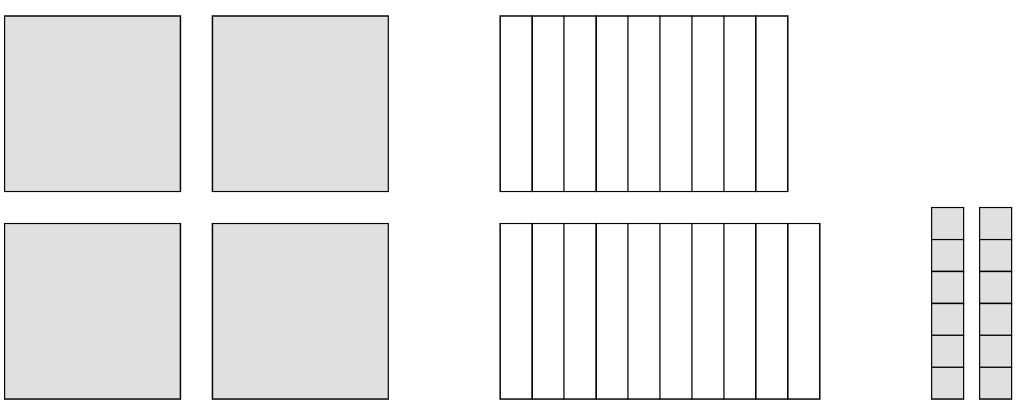
$$2x(3x + 5) + {}^-3(3x + 5)$$

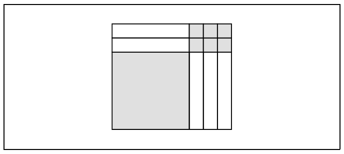
$$(2x - 3)(3x + 5)$$

$(6)(-15) = {}^-90$

90	1
45	2
30	3
18	5
15	6
10	9

Let's try one more example. Factor: $4x^2 - 19x + 12$:





Solution:

$$4x^2 - 19x + 12$$

$$4x^2 + ({}^{-}16x + {}^{-}3x) + 12$$

$$(4x^2 - 16x) + ({}^{-}3x + 12)$$

$$4x(x - 4) + {}^{-}3(x - 4)$$

$$(4x - 3)(x - 4)$$

(4)(12) = 48

48	1
24	2
16	3
12	4
8	6

Notes: Since our product is positive 48, the two factors will add. Since we need two factors that add to be $^{-}19$, we use $^{-}16$ and $^{-}3$. Also, when there is a negative sign on the third term of the four terms, *always* use this negative as part of the common factor. If you do not do this, there will be no shared factor to join the two products together in the last step.

Exercises

Use the shortcut method to factor the following polynomials:

1. $2x^2 - 7x - 15$

2. $2x^2 - 3x - 5$

3. $2x^2 + 3x - 5$

4. $2x^2 - 7x + 6$

5. $4x^2 - 4x - 15$

6. $2x^2 + 7x - 15$

7. $6x^2 - x - 15$

8. $6x^2 + 11x - 10$

9. $2x^2 - 13x + 15$

10. $12x^2 + 25x + 12$

11. $20x^2 - 26x - 6$

12. $15x^2 + 8x + 1$

13. $25x^2 + 30x + 9$

14. $12x^2 - 7x - 12$

15. $3x^2 + 2x - 5$

16. $4x^2 + 8x + 3$

17. $2x^2 + x - 6$

Section 7
Recognizing Special Products

Introduction

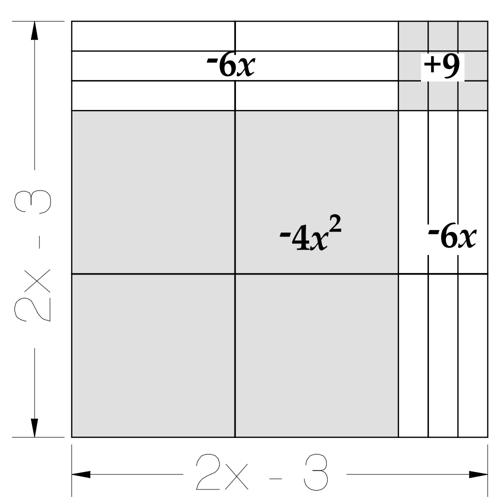
The factoring methods discussed so far in this chapter will work for any quadratic expression *which can be factored*. Many quadratic expressions cannot be factored, and they will be discussed briefly in this section. It may be useful to learn to recognize some special types of quadratic expressions so that factoring them will be even easier. The special expressions we are talking about are **perfect squares** and the **difference of two perfect squares**, both of which were discussed at the end of the previous chapter.

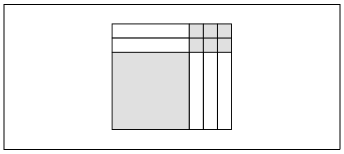
Recognizing Perfect Squares

As you will recall from our earlier discussion, perfect square trinomials have some very specific characteristics which make them relatively easy to recognize. An example of a perfect square can be generated by multiplying a binomial times itself, such as

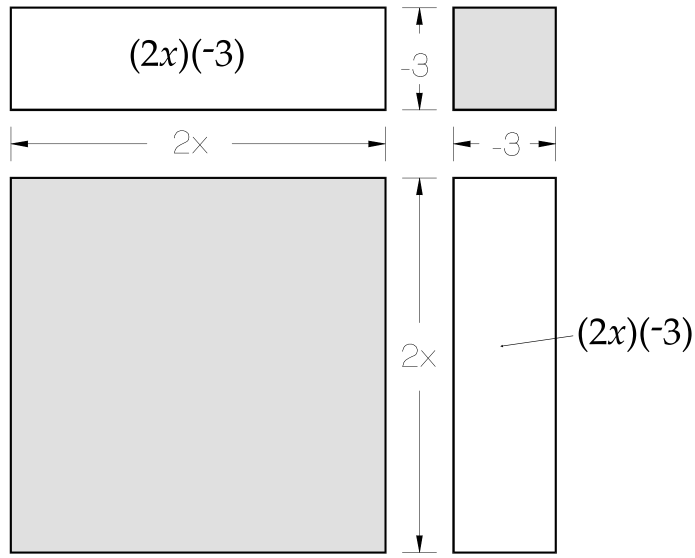
$$\begin{aligned}(2x-3)^2 &= (2x-3)(2x-3)\\ &= 4x^2-6x-6x+9\\ &= 4x^2-12x+9\end{aligned}$$

We can illustrate this product with the following diagram.





From the diagram we can see that both the x^2 term and the units term are themselves positive perfect squares. (Do you recognize the perfect square numbers?) Also we see that there are two equal groups of negative x-bars, each group being the product of the square roots of the squares.



The fact that the x-bars are all negative tells us that both dimensions of one of our squares (x^2 pieces or units) must be negative. (We generally put the negative signs on the units square, giving dimensions of $(2x - 3)$, but both dimensions could also be written $({}^{-}2x + 3)$ and the result would still be correct.)

From this we see that perfect square trinomials *always* have the following characteristics:

- **The x^2 term and the units term are always positive perfect squares. Look for numbers associated with each of these terms which are perfect square numbers.**

- **The x term may be either positive or negative, but its value is always twice the product of the square roots of the other two terms.**

If you look for these characteristics when factoring you will recognize a perfect square trinomial.

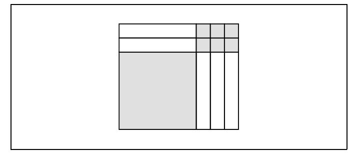
$4x^2 - 12x + 9$

$(2x)^2 \qquad (3)^2$ — Perfect squares

Square Roots

$2 \cdot (2x)(3)$ — Twice the product of the square roots

$12x$



Once a perfect square trinomial is recognized, factoring it is very easy. The terms in each of the binomial factors are the square roots of the x^2 term and the units term, separated by the sign of the x term.

$$4x^2 - 12x + 9$$

$$(2x)^2 \qquad (3)^2$$

$$(2x - 3)^2$$

$$4x^2 - 12x + 9 \;=\; (2x - 3)^2$$

Let's look at another example. Factor:

$$9x^2 + 6x + 1$$

Is this a perfect square?

$$9x^2 + 6x + 1$$ Perfect squares

$$(3x)^2 \qquad (1)^2$$ Square Roots

$$2 \cdot (3x)(1)$$ Twice the product of the square roots

$$6x$$

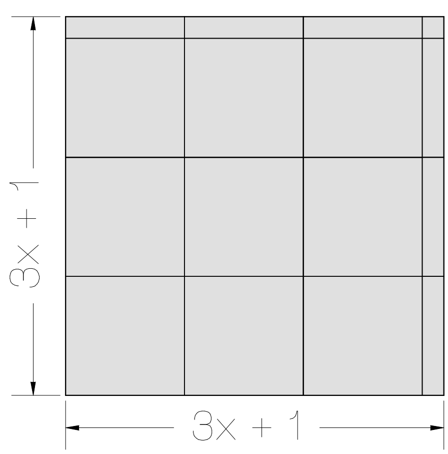
Yes, this is a perfect square. What are its factors?

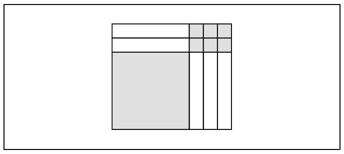
$$9x^2 + 6x + 1$$

$$(3x)^2 \qquad (1)^2$$

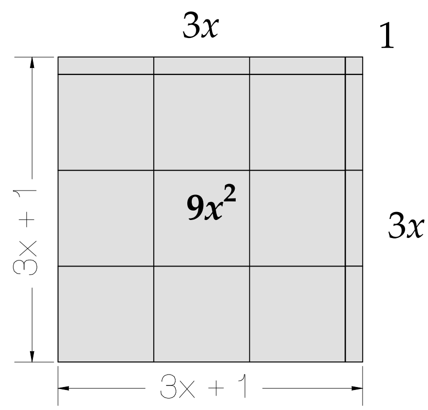
$$(3x + 1)^2$$

$$9x^2 + 6x + 1 \;=\; (3x + 1)^2$$





To check your work draw a diagram and/or multiply out your answer using the FOIL method to verify that the product equals the given trinomial.



$$(3x+1)^2 = (3x+1)(3x+1)$$

$$= 9x^2 + 3x + 3x + 1$$

$$= 9x^2 + 6x + 1$$

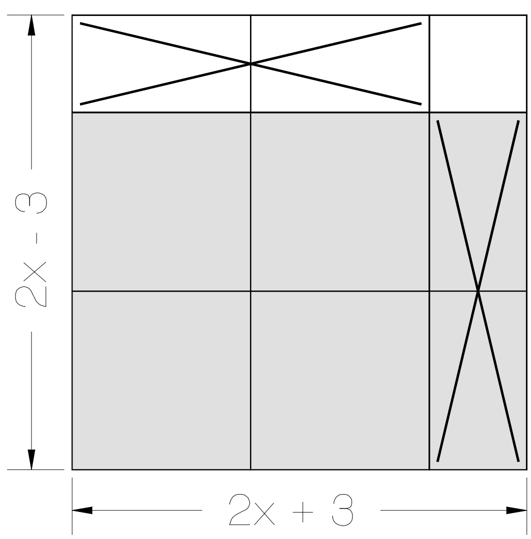
Recognizing the Difference of Two Perfect Squares

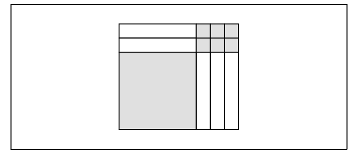
The difference of two perfect squares is the result of multiplying two binomials which are the same except for the signs on their second terms.

Different Signs

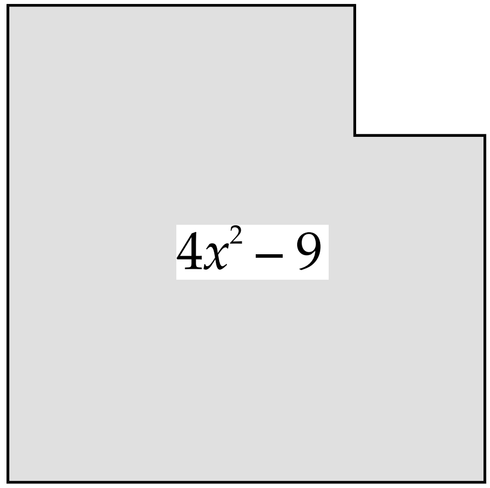
$$(2x + 3)(2x - 3)$$

$$= 4x^2 - 6x + 6x - 9$$
$$= 4x^2 - 9$$





Our result is one square (the units) taken away from another square (the x^2's), with all the x-bars canceling out.



From this we see that the difference of two perfect squares should be easy to recognize when factoring. This is due to several specific characteristics:

- **The x^2 term is a positive perfect square.**
- **The units term is a negative perfect square.**
- **The x term is missing altogether.**

There are other expressions which look a little like the difference of two squares, but if you look carefully you can always tell them apart.

For example:

$$4x^2 - 9x$$

or

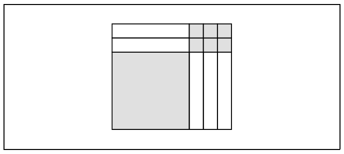
$$16x - 25$$

or

$$x - 25$$

Both of these expressions have two terms separated by a minus sign, and the number associated with each term is a perfect square number. Still these examples are **not** the difference of two perfect squares, because each expression has an x term, and since x is a bar, not a square, the x term cannot be a perfect square. (The top example can still be factored, however, by taking out the common factor of x.)

When you are asked to factor an expression having only two terms separated by a minus sign, look to see if one term is x^2 pieces and the other is units, with no x term; and then see if both the x^2 and the units terms are perfect squares. If they are, the expression is the difference of two perfect squares, and the factorization will be quite easy.



$4x^2 - 9$

$16x^2 - 49$

$x^2 - 1$

$25x^2 - 1$

These are all the difference of two perfect squares

Once you have identified an expression as the difference of two perfect squares, factoring is a breeze.

$4x^2 - 9$ — Two perfect squares

$(2x)^2 \quad (3)^2$ — Square Roots

$(2x + 3)(2x - 3)$ — Use different signs

Different signs

Further factoring examples:

$16x^2 - 49$

$(4x)^2 \quad (7)^2$

$(4x + 7)(4x - 7)$

$x^2 - 1$

$(x)^2 \quad (1)^2$

$(x + 1)(x - 1)$

As with perfect squares, recognizing when you have an expression with a difference of two perfect squares is more than half of the work involved in factoring the expression. If you don't recognize an expression right away as a special type, but you see that it has no x term, place a zero-x in as the middle term, and then look again:

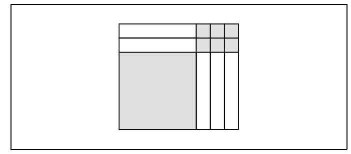
$$25x^2 - 1$$

$$25x^2 + 0x - 1$$

Ask yourself, "How can I get zero for the middle term?". The answer is to multiply two binomials which are the same except that they have opposite signs on the second term. The result must be

$$(5x + 1)(5x - 1)$$

Exercises

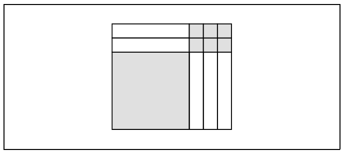


Identify which of the following are perfect square trinomials. Label each example as YES or NO. Factor only the perfect square trinomials.

1. $x^2 + 6x + 9$
2. $x^2 + 5x + 6$
3. $2x^2 + 3x - 9$
4. $4x^2 + 20x + 25$
5. $9x^2 + 6x - 1$
6. $4x^2 - 4x + 1$
7. $6x^2 + 11x + 5$
8. $x^2 + 8x - 9$
9. $3x^2 - 5x + 2$
10. $16x^2 - 24x + 9$
11. $4x^2 + 21x - 25$
12. $4x^2 - 28x + 4$

Label the following expressions either **PS** for perfect squares, **DTPS** for the difference of two perfect squares, or **neither**. Factor those labeled PS or DTPS. Do not attempt to factor the examples that are not PS or DTPS

13. $4x^2 - 1$
14. $x^2 + 1$
15. $x^2 + 6x + 9$
16. $x^2 - 9$
17. $4x^2 - 6x$
18. $9x^2 + 12x - 4$
19. $4x^2 - 12x + 9$
20. $9x - 1$
21. $16x^2 + 8x + 1$
22. $25x^2 - 4$
23. $x^2 - 5x + 6$
24. $x^2 - 10x + 25$
25. $4x^2 + 9$
26. $4x^2 - 25$



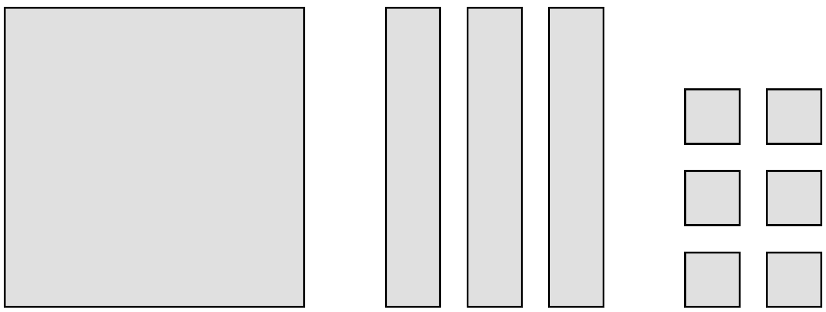
27. $x - 4$

28. $x^2 + 6x - 16$

Section 8
Expressions Which Cannot Be Factored

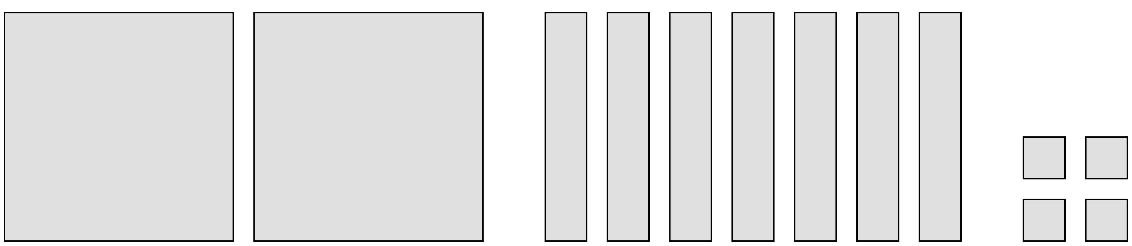
Introduction

Using the chips, factoring means to form a rectangle from the given pieces, with no missing pieces and no pieces left over. *For many groups of pieces, making such a rectangle is not possible*. For example, try making a rectangle out of these pieces:



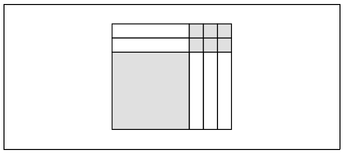
$x^2 + 3x + 6$

Or these



$2x^2 + 7x + 4$

Actually there are many more expressions which *cannot* be factored than those that *can* be factored. So if you are faced with a tough factoring problem, try all the approaches you have learned, but realize that *not factorable* is a possible answer.



Remember: Look for Common Factors First

Perhaps the most often forgotten step in factoring is to *always* look for common factors first. Removing a common factor will always simplify an expression and will sometimes turn an apparently impossible problem into an easy problem.

For example, factor:

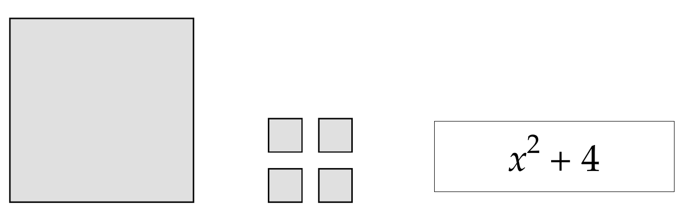
$18x^2 - 8$	
$2(9x^2 - 4)$	Common Factor
$2(3x + 2)(3x - 2)$	Difference of Squares

$3x^2 - 24x + 48$	
$3(x^2 - 8x + 16)$	Common Factor
$3(x - 4)^2$	Perfect Square

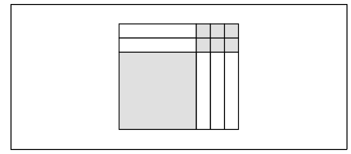
$3x^3 + 15x^2 + 18x$	
$3x(x^2 + 5x + 6)$	Common Factor
$3x(x + 2)(x + 3)$	Factor

The Sum of Two Squares

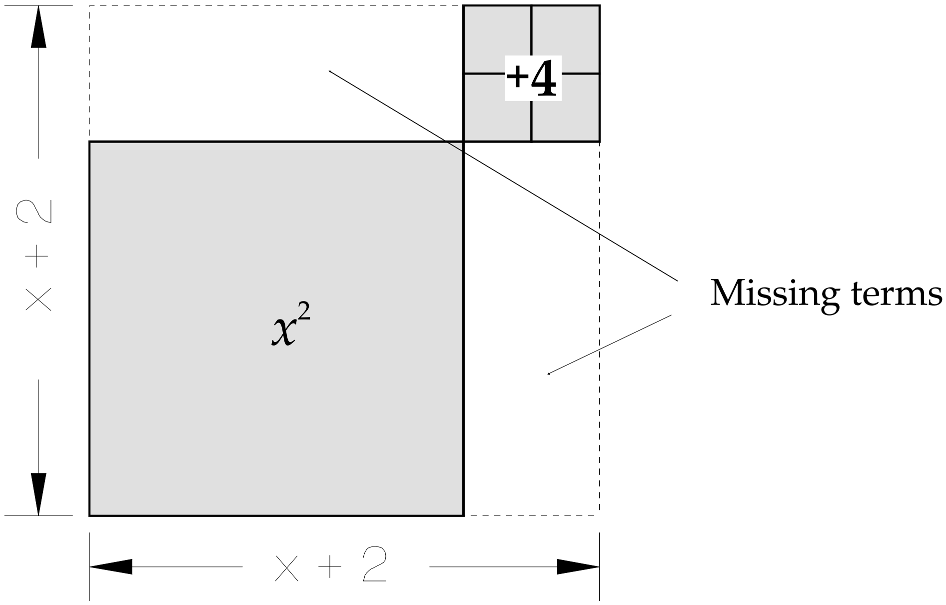
Perhaps the type of expression most often mis-factored is the sum of two squares.



Using chips it may be obvious that no rectangle can be made from the pieces given. But students often try to suggest the following:



$$x^2 + 4 = (x + 2)(x + 2) \qquad \textbf{(Not True !!)}$$



Although the above suggestion may *seem* reasonable, the picture illustrates that there are terms missing which are needed to make a perfect square. If the units square (the $^+4$) were negative, then the two missing terms would have had opposite signs and would have canceled out. But if the units square is positive, the missing terms must both have the same sign, and therefore they can't cancel.

This is why we *can't factor* the *sum* of two squares, but we *can* factor the *difference* of two squares.

Exercises

Factor completely *if possible*.

1. $3x^2 + 15x + 18$

2. $4x^2 + 9$

3. $2x^2 - 18$

4. $3x^2 + 18x + 27$

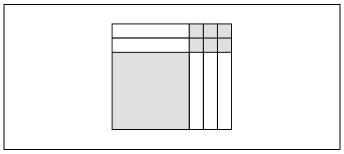
5. $x^2 - 3x + 5$

6. $x^2 + 4x - 5$

7. $3x^2 + 2x - 5$

8. $2x^2 + 5x + 6$

9. $4x^2 - 24x + 9$



10. $2x^2 + 16x + 32$

11. $5x^2 - 20$

12. $4x^2 - 9x$

13. $3x^2 + 12$

14. $x^3 + 2x^2 + x$

15. $x^2 + 6x + 5$

16. $x^2 + 5x - 6$

17. $x^2 + 7x - 6$

18. $18x^2 - 8x$

Answers to Exercises

Chapter 1: Positive and Negative Numbers

Section 1: Positive and Negative Numbers

3. $^{+}11$
4. $^{-}3$
5. $^{-}9$
6. $^{+}10$
12. $^{+}17$
13. 0
14. 0

Section 2: Addition of Signed Numbers

1. 0
2. $^{+}14$
3. $^{-}6$
4. $^{-}6$
5. $^{-}2$
6. $^{+}12$
7. $^{-}7$
8. $^{-}7$
9. $^{-}13$
10. $^{-}13$
11. $^{-}1$
12. $^{+}1$
13. $^{-}9$
14. $^{+}9$
15. $^{-}1$
16. $^{+}1$
17. $^{-}1$
18. $^{-}4$
19. $^{+}2$
20. $^{-}5$
21. $^{+}2$
22. $^{-}8$
23. $^{+}2$
24. $^{+}8$
25. $^{+}4$
26. $^{-}4$
27. $^{-}8$
28. $^{+}9$
29. $^{-}1$
30. $^{+}1$

Section 3: Subtraction of Signed Numbers

1. $^{+}8$
2. $^{-}8$
3. $^{-}2$
4. $^{-}3$
5. $^{-}2$
6. $^{+}8$
7. $^{-}8$
8. $^{+}2$
9. $^{+}17$
10. $^{+}4$
11. $^{+}6$
12. $^{+}2$
13. $^{+}17$
14. $^{-}7$
15. $^{-}16$
16. $^{+}2$
17. $^{-}8$
18. 0
19. $^{+}8$
20. $^{-}10$
21. $^{-}4$
22. $^{+}4$
23. $^{+}10$
24. $^{+}3$
25. $^{-}3$
26. $^{-}3$
27. $^{+}3$
28. $^{-}7$
29. $^{+}7$
30. 12

Section 4: Addition and Subtraction

2. $^{-}5$
3. $^{+}2$
4. $^{-}5$
5. $^{-}1$
6. $^{+}16$
7. 0
14. $^{+}5$
15. $^{-}3$
16. $^{+}4$
17. $^{+}4$

18. $^+2$
19. $^-2$
20. $^-8$
21. $^+8$
22. $^-8$
23. $^-4$
24. $^-10$
25. $^+4$
26. $^+10$
27. $^-14$

Section 5: Multiplication

3. $^-10$
4. $^+15$
5. $^+12$
6. $^-3$
7. $^+4$
8. $^-4$
9. $^+28$
10. $^+28$
11. $^-28$
12. $^+1$
13. $^-1$
14. $^+1$
15. $^+17$
16. $^-17$
17. $^+0$
18. $^+30$
19. $^-6$
20. $^+15$
21. $^-12$
22. $^-14$
23. $^-14$
24. $^+14$
25. $^-18$
26. $^-18$
27. $^+18$
28. $^+12$
29. $^+9$
30. $^-25$

Section 6: Division

1. $^-6$
2. $^-6$
3. $^+6$
4. $^-2$
5. $^+4$
6. $^+1$
7. $^-1$
8. $^-1$
9. $^-1$
10. $^+1$
11. 0
12. 0
13. $^-2$
14. $^+8$
15. $^-6$
16. $^+2$
17. $^-4$
18. $^-4$
19. $^+4$
20. $^+1$
21. $^-4$
22. $^+3$
23. $^-2$
24. $^+4$
25. $^-5$
26. $^+5$
27. $\frac{-8}{3}$ *or* $-2\frac{2}{3}$
28. $\frac{8}{3}$ *or* $2\frac{2}{3}$
29. $\frac{3}{2}$ *or* $1\frac{1}{2}$
30. $\frac{-12}{5}$ *or* $-2\frac{2}{5}$

Section 7: The Number Line

1. $^+6$
2. **$^-4$**
3. **$^-1$**
4. $^+4$
5. $^-1$
6. **$^-5$**
7. 0
8. $^-2$
9. $^+4$
10. $^-8$

$\frac{2}{3}$

7. 8
8. $\frac{8}{9}$
9. 1
10. $\frac{1}{2}$

Chapter 2: Expressions

Section 1: Simple Expressions

1. 9, 6, 4
2. $^{-}1$, 2, 4
3. 14, 5, $^{-}1$
4. 0, 0, 0
5. 4, 7, 9
6. 4, $^{-}5$, $^{-}11$
7. 12, $^{-}3$, $^{-}13$
8. 7, 4, 2
9. 0, 0, 0
10. 5, 5, 5
11. 3, 0, -2
12. $^{-}9$, $^{-}6$, $^{-}4$
13. 0, 3, 5
14. 0, 3, 5
15. 0, $^{-}3$, $^{-}5$

Section 2: Multiples of *x*

1. $^{-}12$, 0, $^{-}15$, $^{-}18$
2. $^{-}15$, $^{-}27$, $^{-}12$, $^{-}9$
3. 4, 0, 5, 6
4. $^{-}3$, $^{-}11$, $^{-}1$, 1
5. 1, 1, 1, 1
6. 4, 0, 5, 6
7. $^{-}1$, $^{-}13$, 2, 5
8. 15, 11, 16, 17
9. 5, 1, 6, 7
10. 5, 17, 2, $^{-}1$
11. $^{-}5$, 7, $^{-}8$, $^{-}11$
12. 0, 8, $^{-}2$, $^{-}4$
13. 4, 28, $^{-}2$, $^{-}8$
14. 3, 7, 2, 1
15. 10, 22, 7, 4
16. $^{-}6$, $^{-}14$, $^{-}4$, $^{-}2$
17. 8, 28, 3, $^{-}2$
18. 0, $^{-}8$, 2, 4
19. 0, 20, $^{-}5$, $^{-}10$
20. $^{-}4$, $^{-}32$, 3, 10

Section 3: Combining Similar Terms

1. $3x$, $-x$, 5
2. 0
3. $4x$, 1, 1
4. 1, $^{-}2$, x
5. $13x + 1$
6. $10x + 5$
7. $11x + 4$
8. $-x + 1$ *or* $1 - x$
9. $2x - 4$ *or* $^{-}4 + 2x$
10. 21, $^{-}4$
11. 29, $^{-}1$
12. 48, $^{-}7$
13. $^{-}3$, 2
14. 4, $^{-}6$

Section 4: Expressions and Parentheses

1. $11x - 13$ *or* $11x + {}^{-}13$
2. $-x - 2$ *or* $-x + {}^{-}2$
3. $13 - 2x$ *or* $^{-}2x + 13$
4. $2x + 13$
5. 20
6. $4x - 8$, $^{-}4$
7. $11 - 4x$ *or* $^{-}4x + 11$, 15
8. x, 256
9. x, $^{-}17$

Section 5: Expressions Containing Fractions

1. $\frac{3}{4}x$ or $\frac{3x}{4}$
2. $-\frac{x}{4}$ *or* $\frac{-x}{4}$
3. $\frac{1}{24}x$ or $\frac{x}{24}$
4. $\frac{1}{6}x$ or $\frac{x}{6}$
5. $x + 2$
6. $x + \frac{1}{4}$
7. $3x$
8. x
9. x
10. $20x$
11. $\frac{13x}{15}$
12. $\frac{13x}{6}$
13. $\frac{3x}{2}$

14. $\frac{13x}{6}$

15. $\frac{x}{9}$

16. $\frac{2x}{15}$

17. $2x-\frac{5}{3}$

18. $6x+3$

Section 6: Properties of Expressions

1. $6x + 30$
2. $^-1 - 2x$
3. $22 + x$
4. 0
5. x
6. $^-5x - 13$
7. $6x - 20$
8. 0
9. $3x$
10. $2x + 2$
11. $7x$
12. $3x + 2$
13. $3x$
14. $3x - 12$
15. $x + 1$
16. $x - 1$
17. $x+\frac{1}{6}$
18. ^-5x
19. $35x$
20. $35x$
21. $35x$
22. ^-35x
23. 0
24. 0
25. 0
26. $5x-70$
27. $^-12x+24$
28. $13x-23$
29. $21x-84$
30. $^-8x+38$

Chapter 3: Equations

Section 1: Introduction to Equations

1. Expression
2. Expression
3. Equation
4. Equation
5. Equation
6. Expression
7. Equation
8. Expression
9. Expression
10. Expression
11. Equation
12. Equation
13. Expression
14. Equation
15. Equation

Section 2: The Equation Game

1. $x = 5$
2. $x = 8$
3. $x = 8$
4. $x = 7$
6. $x = 7$
7. $x = 9$
8. $x = 5$
9. $x = 4$
10. $x = 6$
11. $x = 5$
12. $x = 12$
13. $x = 7$
14. $x = 9$
15. $x = 13$
16. $x = 21$
17. $x = 7$
18. $x = 8$
19. $x = 8$
20. $x = 9$

Section 3: Equations Using Unknowns

1. $x = 9$
2. $x = {}^{-}1$
3. $y = {}^{-}3$
4. $n = 3$
5. $y = 0$
6. $x = 12$
7. $x = 1$
8. $y = {}^{-}16$
9. $y = {}^{-}1$
10. $x = 23$
11. $x = {}^{-}15$
12. $y = 2$
13. $y = 8$
14. $x = 15$
15. $x = 0$
16. $x = 17$
17. $x = 7$
18. $y = {}^{-}1$
19. $y = {}^{-}5$
20. $x = {}^{-}7$
21. $y = {}^{-}3$
22. $n = {}^{-}1$
23. $n = 8$
24. $x = 0$
25. $x = {}^{-}14$

Section 4: Equations with Multiples of Unknowns

1. $x = 4$
2. $x = {}^{-}7$
3. $x = 2$
4. $x = \frac{9}{5}$ *or* $1\frac{4}{5}$
5. $y = 2$
6. $n = \frac{5}{6}$
7. $b = 0$
8. $x = 2$
9. $x = 11$
10. $x = \frac{10}{3}$ *or* $3\frac{1}{3}$
11. $y = 5$
12. $x = 0$
13. $x = {}^{-}2$
14. $x = {}^{-}2$
15. $y = 8$
16. $n = \frac{1}{7}$

17. $x = \frac{4}{3}$ *or* $1\frac{1}{3}$

18. $x = \frac{3}{5}$

19. $x = \frac{1}{2}$

Section 5: Unknowns in More than One Term

1. $x = {}^{-}11$
2. $x = 3$
3. $y = {}^{-}3$
4. $n = 1$
5. $y = 0$
6. $x = 2$
7. $z = 4$
8. $x = {}^{-}3$
9. $x = 7$
10. $x = {}^{-}2$
11. $y = {}^{-}2$
12. $x = {}^{-}5$
13. $x = 3$
14. $x = 0$
15. $x = 5$
16. $x = {}^{-}4$
17. $x = 0$

Section 6: Equations with Parentheses

1. $x = 6$
2. $x = 0$
3. $x = 4$
4. $x = 4$
5. $x = 0$
6. $x = {}^{-}3$
7. $x = {}^{-}2$
8. $y = 0$
9. $y = 8$
10. $x = {}^{-}1$
11. $x = \frac{1}{2}$
12. $x = 3$
13. $x = 1$
14. $x = 0$
15. $x = \frac{2}{3}$
16. $x = 8$
17. $x = 3$
18. $x = 15$
19. $x = {}^{-}2$
20. $x = 11$

Section 7: Equations with Fractions or Decimals

1. $x = 24$
2. $x = {}^{-}8$
3. $x = {}^{-}2$
4. $x = 1$
5. $x = \frac{21}{4}$ *or* $5\frac{1}{4}$
6. $x = 12$
7. $x = 6$
8. $x = {}^{-}2$
9. $x = 15$
10. $x = 12$
11. $x = \frac{12}{7}$ *or* $1\frac{5}{7}$
12. $x = {}^{-}3.6$
13. $x = 1.8$
14. $x = 4$
15. $x = 1$
16. $x = 100$
17. $x = 6$
18. $x = 1$
19. $x = 0$
20. $x = 20$

Section 8: Special Solutions

1. x can be any number
2. No solution
3. $x = 3$
4. No solution
5. x can be any number
6. No solution
7. $x = 0$
8. x can be any number
9. x can be any number
10. No solution

Chapter 4: Polynomials

Section 1: Using Unknowns

1. $7x$
2. $7x$, $^-2$
3. $4x^2$
4. $3x^2$, $^-6$
5. 6, $^-2x^2$
6. $2x^2$, ^-3x, 12
7. $^-2x^2$, ^-5x, $^-1$
8. $^-0x^2$
9. 5, $^-3x^2$
10. $2x$, 3
11. x^2,^-5x, 6
12. $2x$, $-x^2$, 4
13. $4x$, $3x^2$
14. $2x^2$, $^-7$
15. $3x^2$, ^-5x, 2

Section 2: Adding and Subtracting Polynomials

1. x
2. $^-9x + 5$
3. $^-4x^2 + x + 5$ or $5 - 4x^2 + x$
4. $3x^2 + 4x$
5. $^-4x^2$
6. $2x^2 - x + 2$
7. $x^2 - 3x - 3$
8. $^-3x^2 - x - 2$
9. $x^2 + 5x - 5$
10. $4x + 12$
11. $8x - 8$
12. $3x^2 + 2x + 3$
13. $^-2$
14. $x^2 + 5x - 3$
15. $2x^2 - 3x - 6$
16. $2x^2 - 8x + 3$
17. $4x^2 + 2x - 7$
18. $4x^2 - 3x$
19. $^-2x + 4$
20. $-x^2 + 4x + 3$
21. $^-4x^2 - 2x$
22. $8x - 5$
23. $5x - 6$
24. $-x^2 + 7x + 3$
25. $x^2 - 5x + 1$
26. $2x^2 - 2x + 3$

Section 3: Multiplying Polynomials

1. $2x - 8$
2. $6x + 3$
3. $^-3x + 3$
4. $^-2x + 6$
5. $2x + 2$
6. $6x - 2$
7. $3x - 9$
8. $^-4x + 10$
9. $10x - 15$
10. $^-3x + 15$
11. $^-4x + 2$
12. $10x + 15$
13. $^-15x + 10$
14. $10 - 6x$
15. $^-12 + 4x$
16. $x^2 + 5x + 4$
17. $x^2 + x - 12$
18. $x^2 - 6x + 5$
19. $x^2 + 2x - 15$
20. $x^2 - 6x$
21. $2x^2 - 7x - 4$
22. $^-3x^2 + 2x$
23. $2x^2 - 7x + 6$
24. $x^2 - 2x - 15$
25. $x^2 - 8x + 12$
26. $2x^2 + 5x - 3$
27. $2x^2 + x - 6$
28. $^-3x + 2x^2$
29. $2x^2 - 3x - 2$
30. $4x^2 + 4x - 3$

Section 4: Special Products

1. $x^2 - 4x + 4$
2. $9x^2 + 6x + 1$
3. 49
4. $4x^2 - 28x + 49$
5. $9x^2$
6. $9x^2 + 12x + 4$

7. $x^2 + 8x + 16$

8. $4x^2 - 4x + 1$

9. $x^2 - 18x + 81$

10. $25x^2 + 30x + 9$

11. *No* ($4x^2 + 4x - 15$)

12. $x^2 - 4$

13. *No* ($9x^2 + 24x + 16$)

14. *No* ($15x^2 + 16x - 15$)

15. $9x^2 - 25$

16. $x^2 - 49$

17. *No* ($4x^2 + 4x - 3$)

18. *No* ($2x^2 + x - 1$)

19. $4x^2 - 1$

20. *No* ($6x^2 + 5x - 6$)

21. $25x^2 - 36$

22. $49x^2 - 1$

Chapter 5: Factoring Polynomials

Section 1: Introduction to Rectangles and Factoring

1. $x(x + 4)$
2. $x(x + 5)$
3. $(x + 3)(x + 3)$
4. $(x + 4)(x + 1)$
5. $(x + 5)(x + 3)$
6. $(x + 6)(x + 2)$
7. $(x + 4)(x + 3)$
8. $(x + 7)(x + 2)$
9. $(x + 4)(x + 4)$
10. $(x + 5)(x + 4)$

Section 2: Positive Units, Negative Bars

1. $(x - 3)(x - 1)$
2. $(x - 4)(x - 2)$
3. $(x - 6)(x - 2)$
4. $(x - 4)(x - 3)$
5. $(x - 5)(x - 2)$
6. $(x - 8)(x - 2)$

Section 3: Rectangles Having Negative Units

1. $(x + 6)(x - 1)$
2. $(x - 4)(x + 2)$
3. $(x - 8)(x + 1)$
4. $(x - 12)(x + 1)$
5. $(x - 6)(x + 1)$
6. $(x + 4)(x - 3)$
7. $(x + 9)(x - 1)$
8. $(x - 5)(x + 3)$
9. $(x + 5)(x - 3)$
10. $(x - 8)(x + 2)$

Section 4: Factoring Trinomials with More than One x^2

1. $(2x + 1)(2x + 1)$
2. $(3x + 1)(x + 2)$
3. $(2x + 1)(x + 3)$
4. $(3x + 1)(x + 3)$
5. $(2x + 1)(x + 2)$
6. $(2x + 1)(x + 1)$
7. $(2x + 3)(3x + 1)$
8. $(2x + 1)(3x + 2)$
9. $(2x + 1)(3x + 4)$
10. $(2x + 1)(2x + 3)$
11. $(3x + 4)(4x + 5)$

Section 5: Factoring Using the Grid

1. $(3x + 5)(x + 1)$
2. $(2x + 3)(x + 4)$
3. $(3x + 2)(x + 6)$
4. $(3x + 4)(x + 2)$
5. $(3x + 2)(x + 4)$
6. $(3x + 1)(x + 8)$
7. $(2x + 3)(x + 5)$
8. $(x - 3)(x + 2)$
9. $(x + 6)(x - 3)$
10. $(x - 1)(2x + 5)$
11. $(2x - 3)(x - 2)$
12. $(2x + 3)(2x - 5)$
13. $(2x - 3)(x + 5)$
14. $(2x + 3)(3x - 5)$
15. $(3x - 2)(2x + 5)$
16. $(2x - 3)(x - 5)$
17. $(x + 1)(3x - 5)$
18. $(2x + 3)(x - 2)$
19. $(2x - 1)(3x + 2)$

Section 6: A Shortcut Method

1. $(2x+ 3)(x - 5)$
2. $(x + 1)(2x - 5)$
3. $(x - 1)(2x + 5)$
4. $(2x - 3)(x - 2)$
5. $(2x + 3)(2x - 5)$
6. $(2x - 3)(x + 5)$
7. $(2x + 3)(3x - 5)$
8. $(3x - 2)(2x + 5)$
9. $(2x - 3)(x - 5)$
10. $(4x + 3)(3x + 4)$
11. $(10x + 2)(2x - 3)$ *or* $2(5x + 1)(2x - 3)$
12. $(5x + 1)(3x + 1)$
13. $(5x + 3)(5x + 3)$ *or* $(5x + 3)^2$
14. $(4x + 3)(3x - 4)$
15. $(x - 1)(3x + 5)$
16. $(2x + 1)(2x + 3)$
17. $(2x - 3)(x + 2)$

Section 7: Recognizing Special Products

1. Yes. $(x+3)^2$
2. No.
3. No.
4. Yes. $(2x+5)^2$
5. No.
6. Yes. $(2x-1)^2$
7. No.
8. No.
9. No.
10. Yes. $(4x-3)^2$
11. No.
12. Yes. $(2x-7)^2$
13. DTPS: $(2x+1)(2x-1)$
14. Neither
15. PS: $(x+3)^2$
16. DTPS: $(x+3)(x-3)$
17. Neither
18. Neither
19. PS: $(2x-3)^2$
20. Neither
21. PS: $(4x+1)^2$
22. DTPS: $(5x+2)(5x-2)$
23. Neither
24. PS: $(x-5)^2$
25. Neither
26. DTPS: $(2x-5)(2x+5)$
27. Neither
28. Neither

Section 8: Expressions which Cannot be Factored

1. $3(x+2)(x+3)$
2. Not factorable
3. $2(x-3)(x+3)$
4. $3(x+3)^2$
5. Not factorable
6. $(x+5)(x-1)$
7. $(3x+5)(x-1)$
8. Not factorable
9. Not factorable
10. $2(x+4)^2$
11. $5(x+2)(x-2)$
12. $x(4x-9)$
13. $3(x^2+4)$
14. $x(x+1)^2$
15. $(x+5)(x+1)$
16. $(x+6)(x-1)$
17. Not factorable
18. $2x(9x-4)$