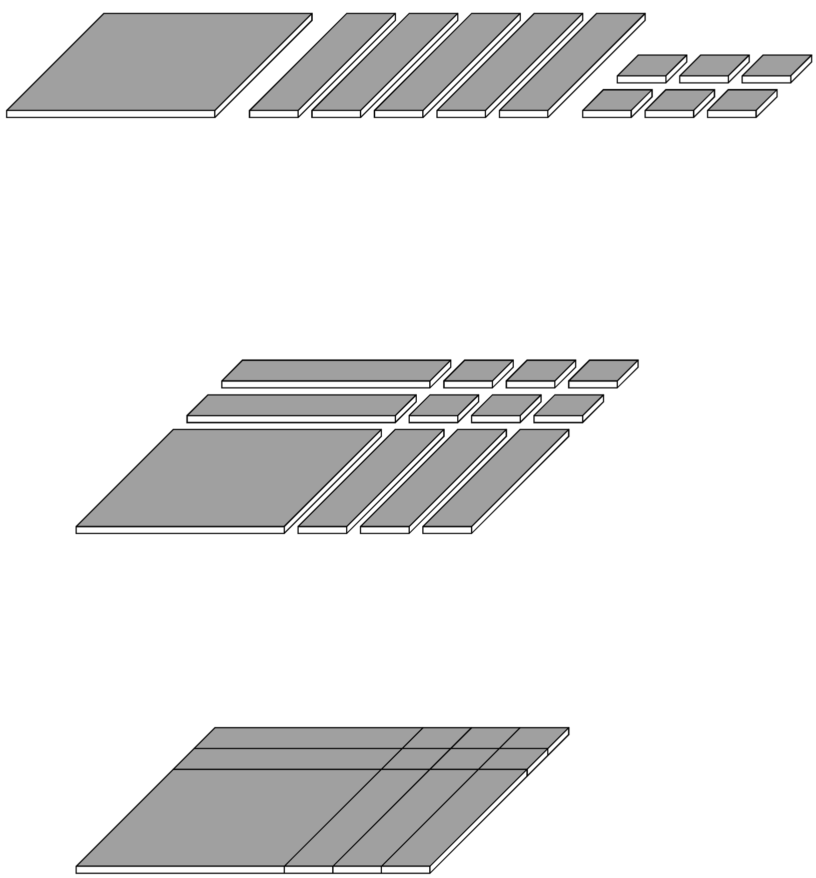
Chapter 5

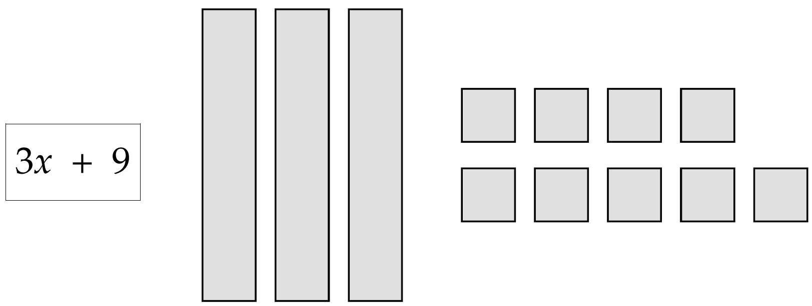
Factoring Polynomials



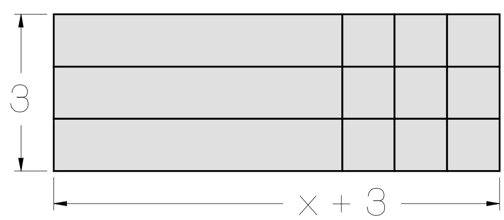
Section 1
Introduction: Rectangles and Factoring

The Meaning of Factoring

Factoring means taking an amount and rewriting the amount as a multiplication problem. Using chips, factoring is the process of taking a group of pieces and arranging them to form a rectangle. The factors are the dimensions (the length and width) of the rectangle. Start with $3x + 9$:

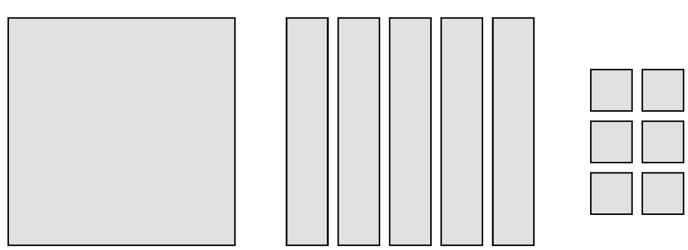


The polynomial $3x + 9$ makes a rectangle that is **3** by $\boldsymbol{x + 3}$:

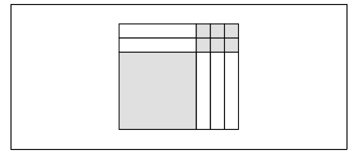


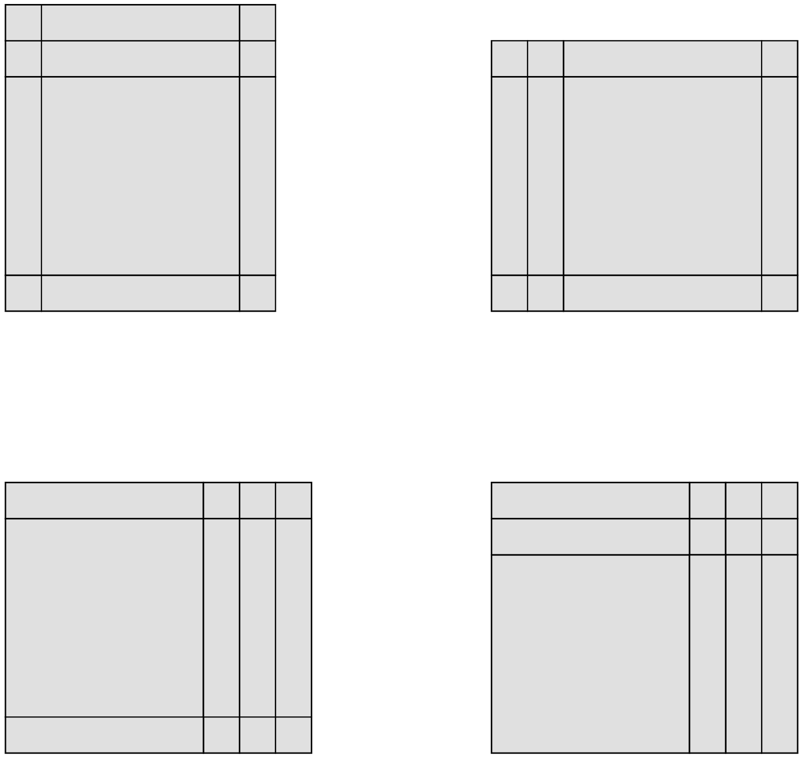
The **factors** of $3x + 9$ are **3** and $\boldsymbol{x + 3}$. This is the same as saying that the product of 3 and $x + 3$ is $3x + 9$. In both forms, *the rectangle means multiplication.*

Sometimes there are several ways to make a rectangle from a group of pieces. Start with the following chips:



From these pieces we could make the following rectangles:



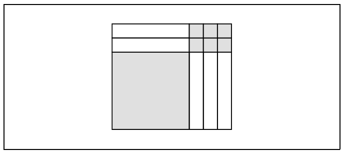


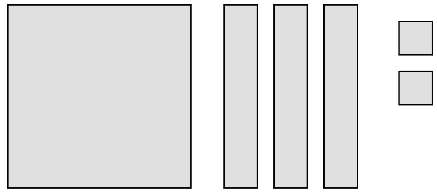
If we check the side lengths of each of these rectangles we find that they all have one direction which is a bar and three unit squares long ($x + 3$), and another which is a bar and two unit squares long ($x + 2$):

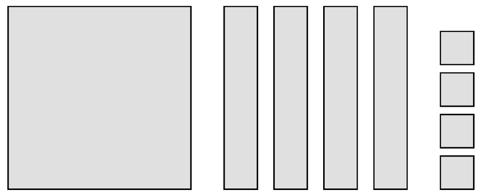
large square		five bars		six units	Factors into a rectangle	1 bar and 3 units	by	1 bar and 2 units
x^2	$+$	$5x$	$+$	6	$=$	$(x + 3)$	$\cdot$	$(x + 2)$

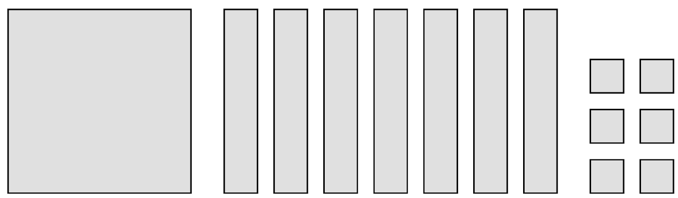
Exercises

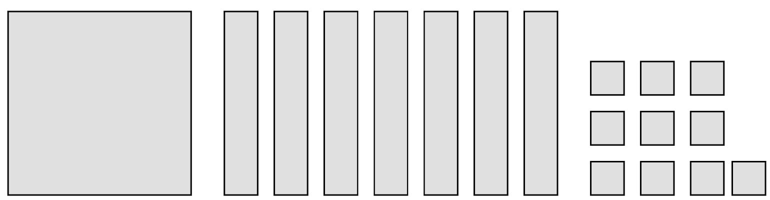
Make rectangles from the following groups of pieces. Remember that there must be no pieces left over and no holes or bumps in the rectangle.



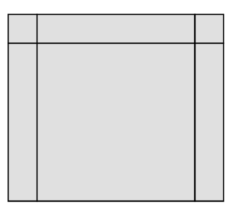
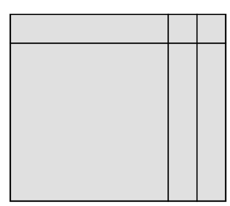
1. 

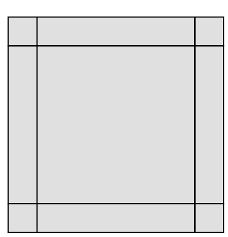
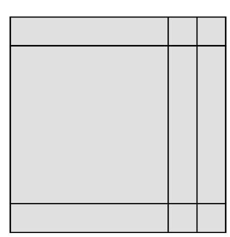
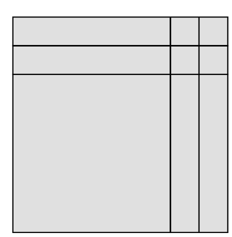
2. 

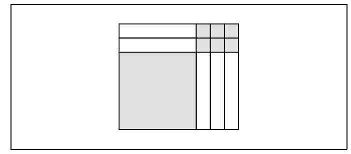
3. 

4. 

Some possible solutions look like this:

1.  *or* 

2.  *or*  *or* 



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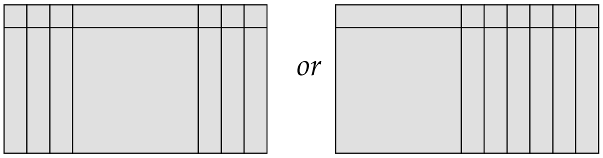

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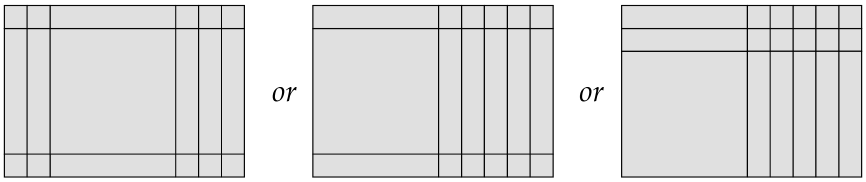
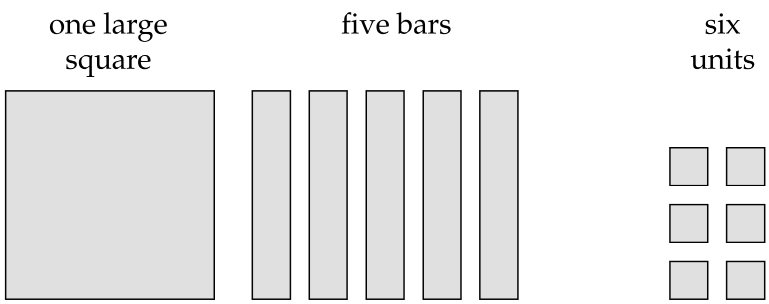
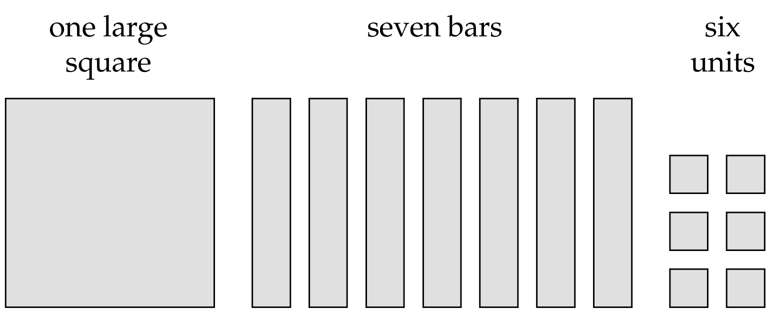


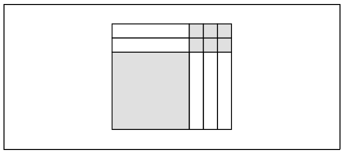
A Clue is in the Units

Look at these two similar examples (both shown before):



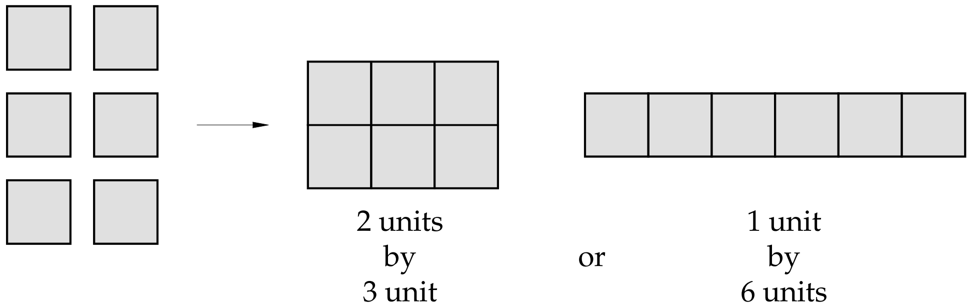


These two groups of pieces differ only in the number of bars. Obviously, since the two groups shown have different numbers of bars, the rectangles they make must have different dimensions.

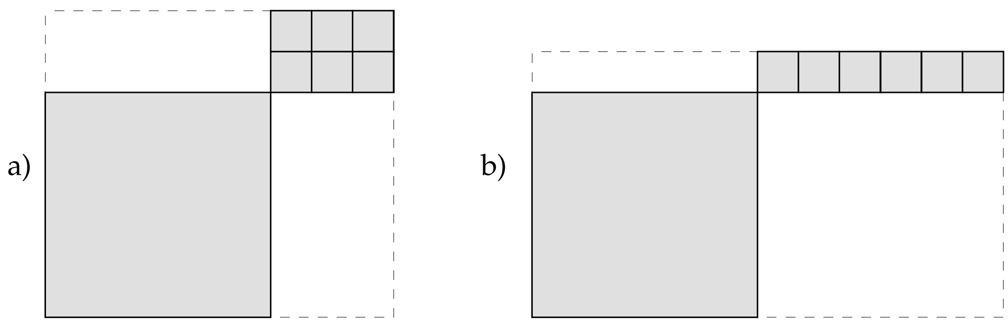


How can you tell before trying different rectangles which ones will work? A clue is in the number of units. Both of the groups shown have six unit squares; let's just look at the units.

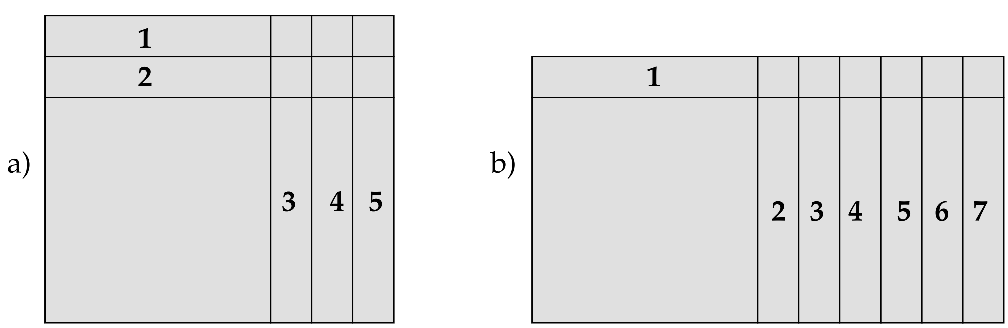
How many ways can you make rectangles using just six units?

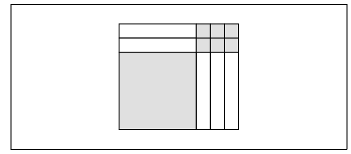


If we think of placing either of these smaller rectangles of units at the corner of the larger rectangle (the total amount), we see two different possible shapes for the larger rectangle.

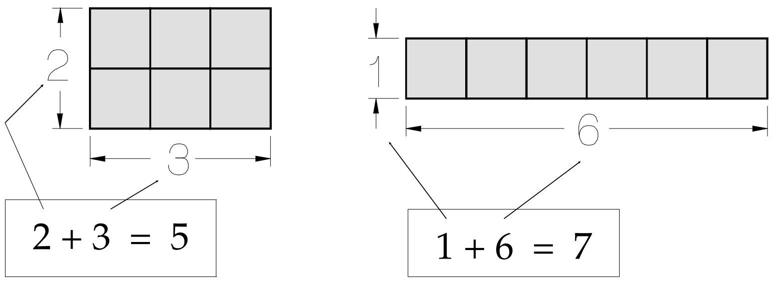


In picture (a) we could fill in the rectangle using two bars on the top and three bars on the side, for a total of five bars. In picture (b) we would need to fill in with one bar on top and six bars on the side, for a total of seven bars.





In each case the number of bars we need to complete the figure depends on the dimensions of the small rectangle of units.



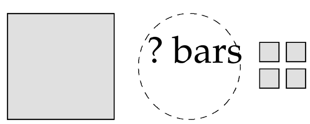
If the units rectangle is (2) by (3) we need 2 + 3 or 5 bars to complete the figure. If the units rectangle is (1) by (6) we need 1 + 6 or 7 bars to complete the figure.

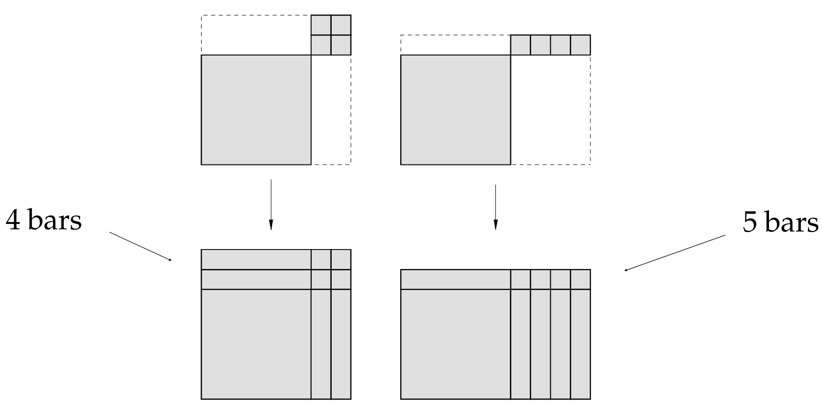
There are only these two ways to make small rectangles using six unit chips. So if we start with one big square and six unit chips we must have either five bars or seven bars in order to make a rectangle which has no holes and no pieces left over. (You can try making rectangles using one big square and six unit chips to see if any are possible with numbers other than five or seven bars.)

Let's Try Predicting

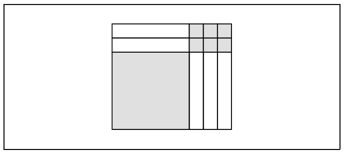
If you are given one big square and any specific number of unit chips, you can learn to predict how many bars you will need to complete each figure.

What if you have one big square and four unit chips? How many ways could you make rectangles and how many bars would you need for each? There are two ways to make a small rectangle using four unit chips:





So we could use either four bars or five bars to make a rectangle.



The product is given by the total number of pieces—large squares, x-bars and units—while the dimensions of the rectangle are the factors:

large square		four bars		four units		1 bar and 2 units	by	1 bar and 2 units
x^2	$+$	$4x$	$+$	4	$=$	$(x + 2)$	$\cdot$	$(x + 2)$

large square		five bars		four units		1 bar and 4 units	by	1 bar and 1 unit
x^2	$+$	$5x$	$+$	4	$=$	$(x + 4)$	$\cdot$	$(x + 1)$

Exercises

Set up the following polynomials with chips and factor:

1. $x^2 + 4x$

2. $x^2 + 5x$

3. $x^2 + 6x + 9$

4. $x^2 + 5x + 4$

5. $x^2 + 8x + 15$

6. $x^2 + 8x + 12$

7. $x^2 + 7x + 12$

8. $x^2 + 9x + 14$

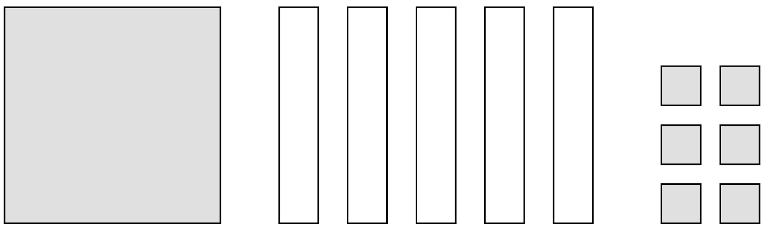
9. $x^2 + 8x + 16$

10. $x^2 + 9x + 20$

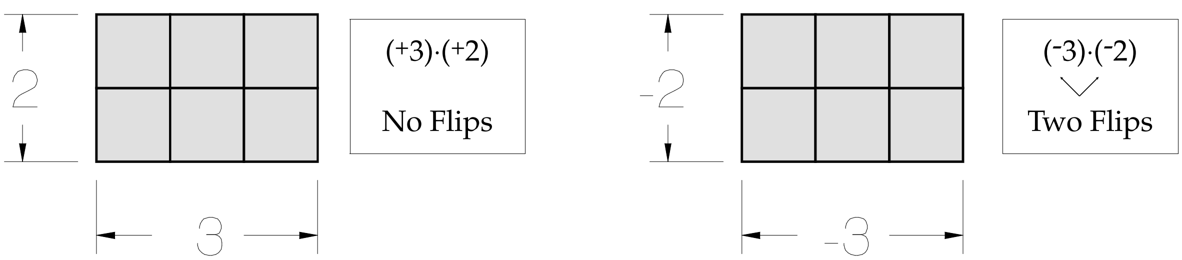
Section 2
Positive Units, Negative Bars

Factoring with Negative Bars

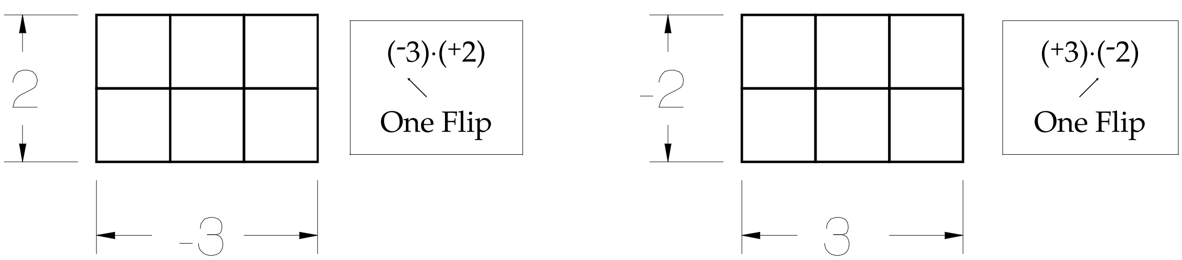
How can we factor polynomials with negative bars?

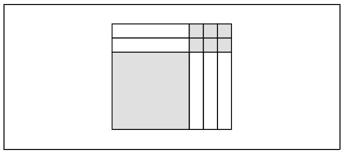


To make a rectangle from pieces having positive units but negative bars we need to remember how to multiply two numbers having signs (see POSITIVE AND NEGATIVE NUMBERS, Section 5 or POLYNOMIALS, Section 1). We can get a positive answer (colored rectangle) from multiplying two positive numbers, or from multiplying two negative numbers.

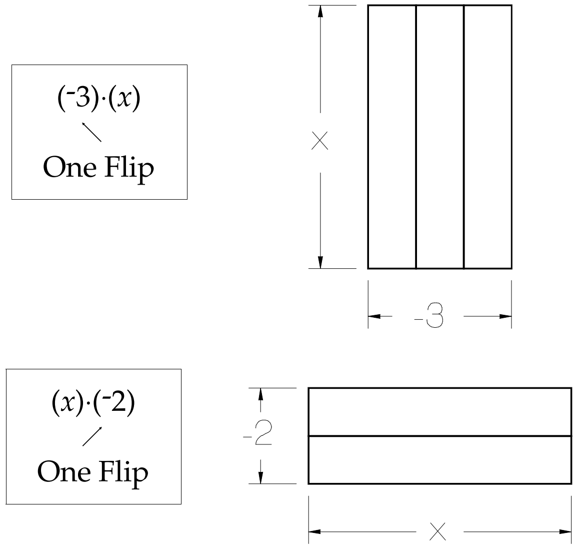


Similarly, we get a negative answer (white rectangle) from multiplying two units having different signs.

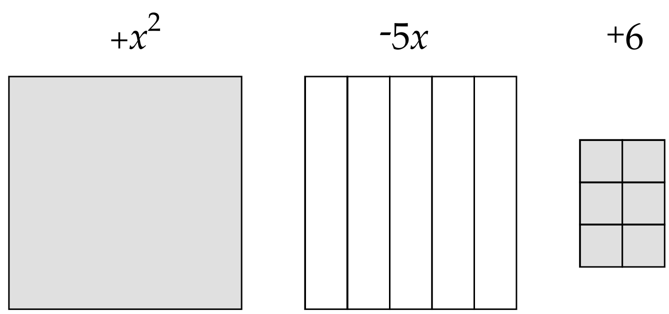




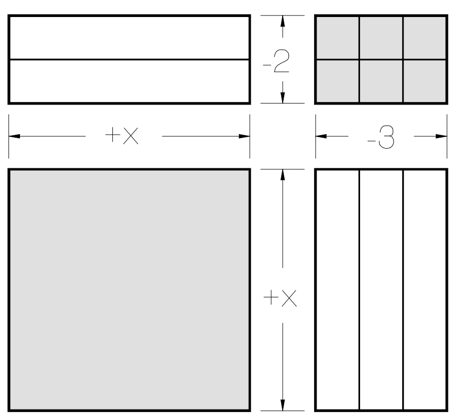
In the same way, we get rectangles made from white (negative) bars whenever one dimension (factor) of the rectangle is positive but the other dimension (factor) is negative.



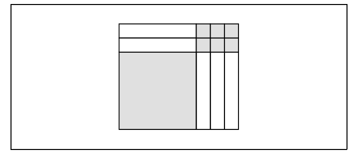
For example, look at the pieces below:

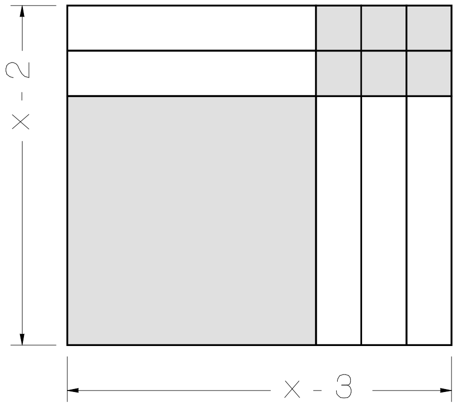


We can make a large rectangle (using four smaller rectangles) in the following way:



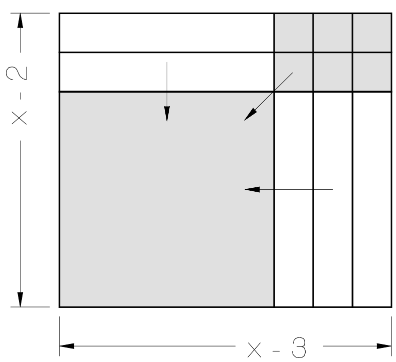
Each of the four small rectangles has its color (sign) determined by the signs of its two dimensions. Then the composite large rectangle looks like this:



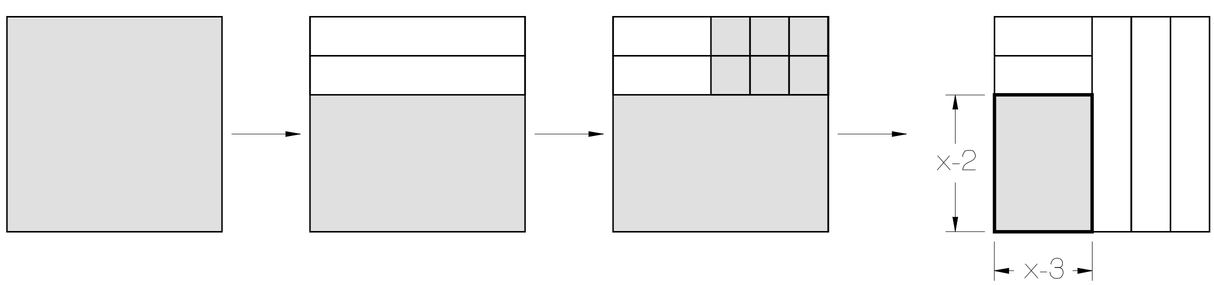


$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

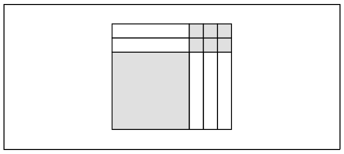
The dimensions of the rectangle are most easily read along the bottom and up the left side. The edges of the large colored square are each ^{+}x, and the short ends of the white bars are each a negative one ($^{-}1$). In this figure the white areas can be thought of as canceling out colored areas leaving a rectangle with actual dimensions of $x - 2$ and $x - 3$, as shown below.



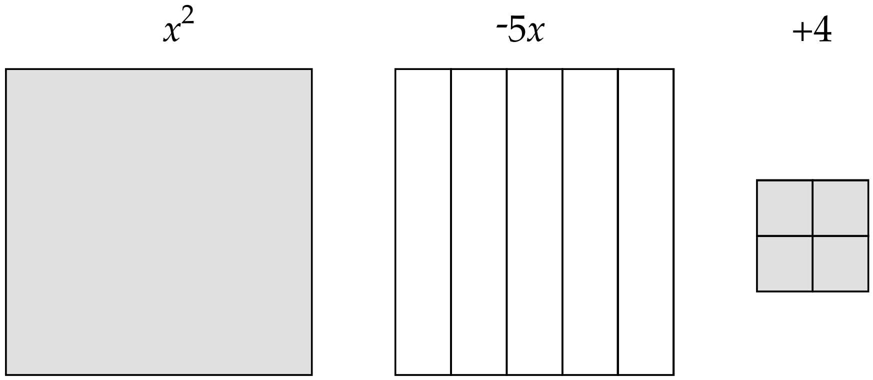
Start with x^2	Put on $^{-}2x$ bars. (Cancels)	Put on $^{+}6$ units. (Adds back)	Put on $^{-}3x$ bars (cancels). The result is $(x - 2)$ by $(x - 3)$



As in the above illustration, the x-bars and units subtract from and add to the original x^2 piece. First, place negative x-bars to cancel out some of the area. Then add back area by placing the positive units on top of the negative bars. Finally, cancel out area with the remaining negative bars. The resulting rectangle is two less than x on one side and three less than x on the other.

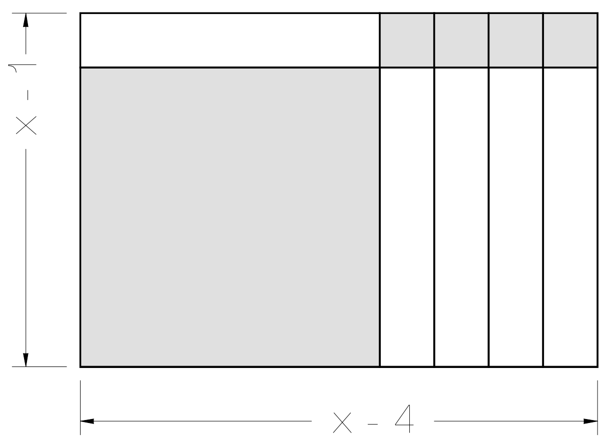


Example: Make a rectangle from the pieces given below:



Solution: The four single chips can only form two possible rectangles—2 by 2 or 1 by 4. Of these two possibilities, only the 1 by 4 corner rectangle would require 5 bars (4+1) which, in this case, are all negative. Looking at the rectangle's dimensions we see that

$$x^2 - 5x + 4 = (x - 4)(x - 1)$$



Exercises

Use chips and factor the following polynomials by making rectangles and noting their dimensions.

1. $x^2 - 4x + 3$

2. $x^2 - 6x + 8$

3. $x^2 - 8x + 12$

4. $x^2 - 7x + 12$

5. $x^2 - 7x + 10$

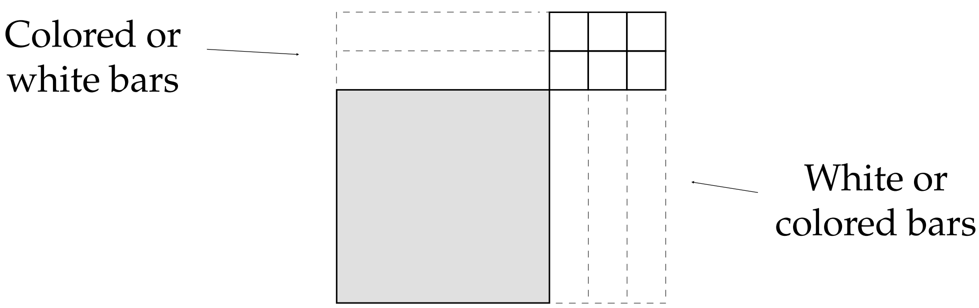
6. $x^2 - 10x + 16$

Section 3
Rectangles Having Negative Units

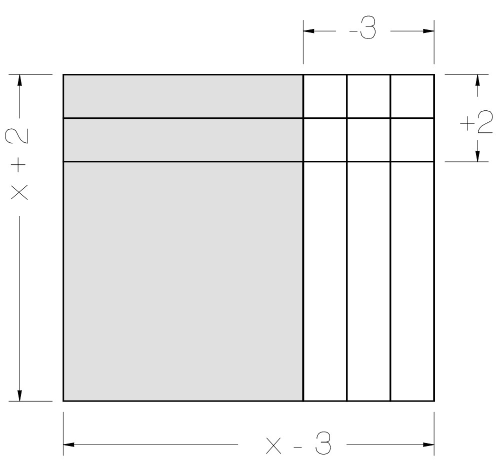
Factoring with Negative Units

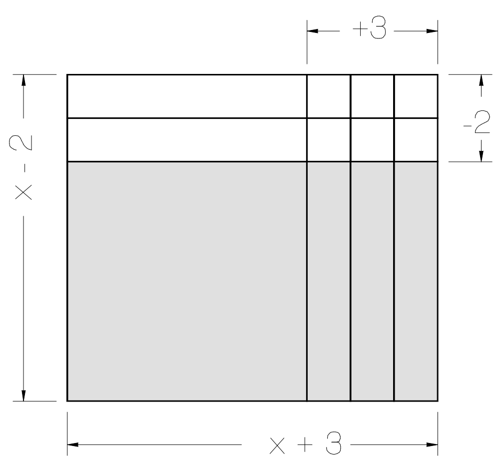
As we just reviewed, if a rectangle has a negative value (white side up), it means that one dimension of the rectangle is positive while the other dimension is negative.

In the case of a polynomial, this means that if the large square (x^2) is colored (positive), and the small units (single chips) are white (negative), then when we make a complete rectangle we will need some colored bars and some white bars. (You may want to review POLYNOMIALS, Section 3.)

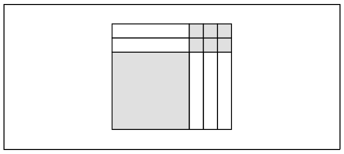


The color and number of the bars will match the positive or negative values of the dimensions of the units rectangle. For example we could imagine two different rectangles having $+x^2$ and $^-6$ units:



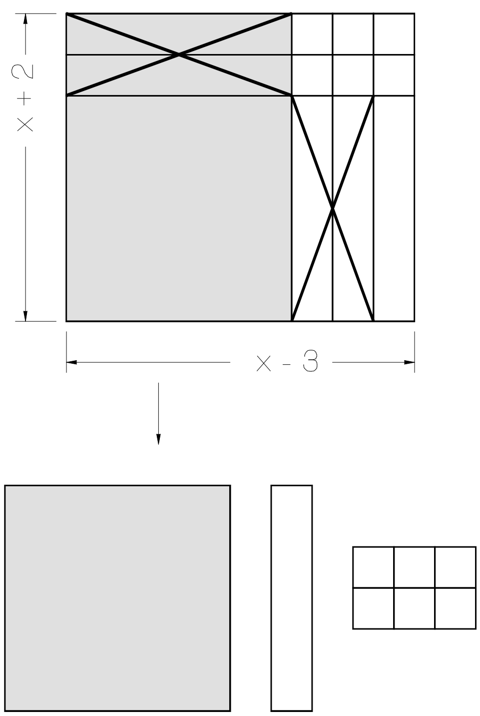


One of these rectangles has two positive bars and three negative bars; the other rectangle has two negative bars and three positive bars.

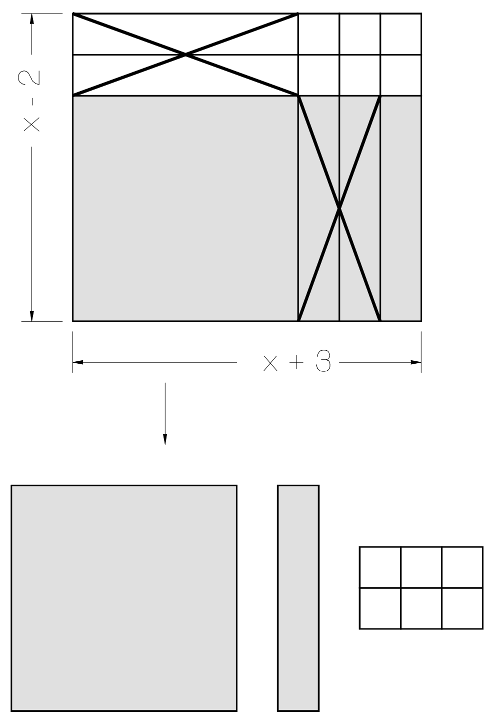


How Can We Tell Which to Use?

If we begin with a polynomial where some of the bars are positive and some are negative, when we combine like terms (put all the bars together), some of them are going to cancel out.



$x^2 - x - 6$

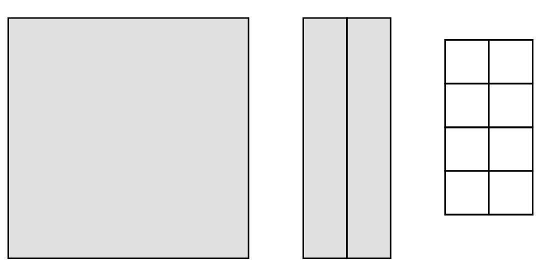


$x^2 + x - 6$

The sign of the bars which are left over after canceling will match the sign of the larger dimension of the units rectangle.

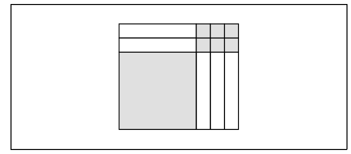
Working Backwards

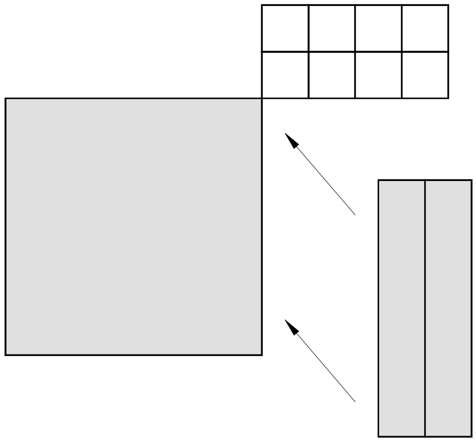
In order to factor a polynomial having negative units, like the following one,



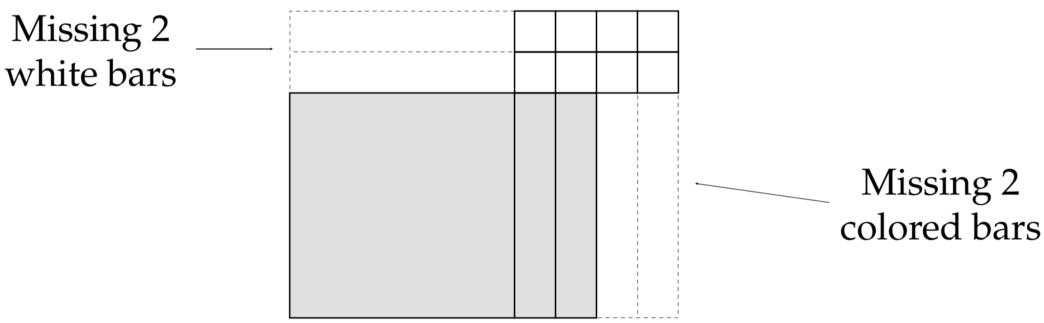
$x^2 + 2x - 8$

we begin by putting the unit rectangle at the corner of the big square and putting the bars we have along the *longer side* of the units rectangle.

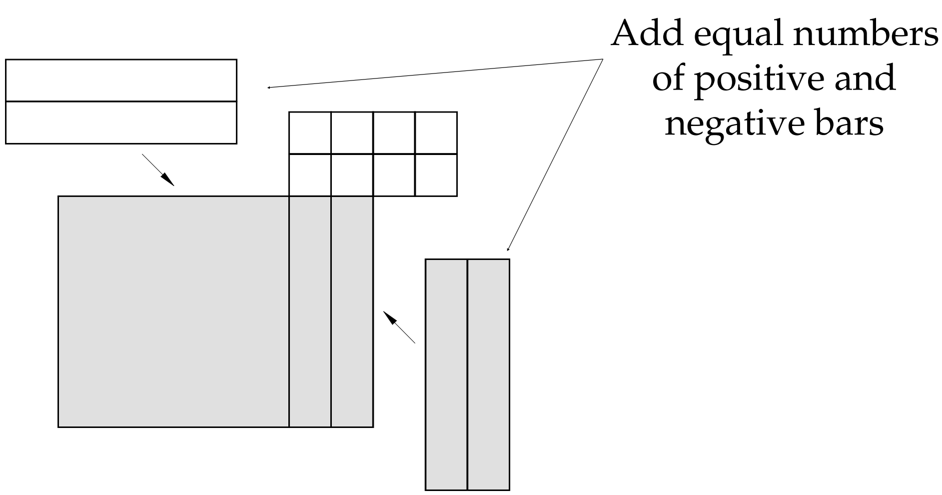


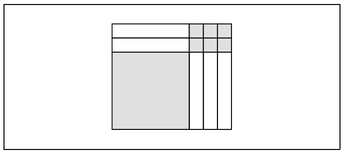


When we look at the result we should see that we are missing *equal numbers* of positive and negative bars.

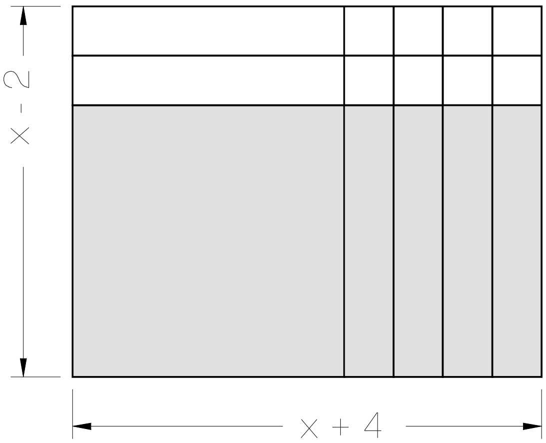


We know that if we add positive and negative bars to the figure in equal numbers we are adding zero, because these pairs of white and colored bars would cancel out.



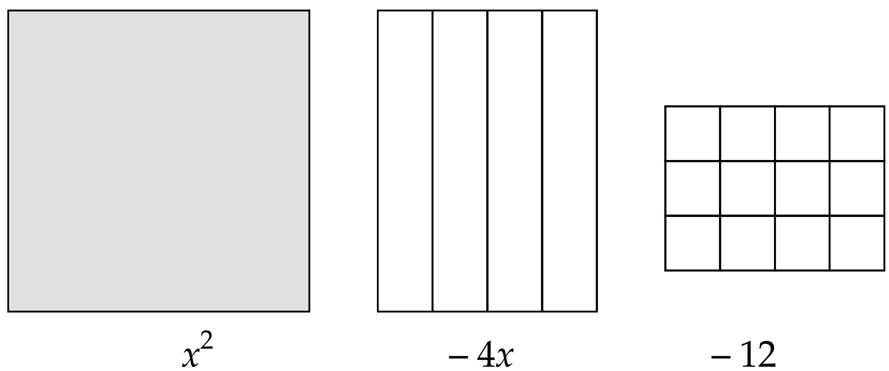


So our final figure looks like this:

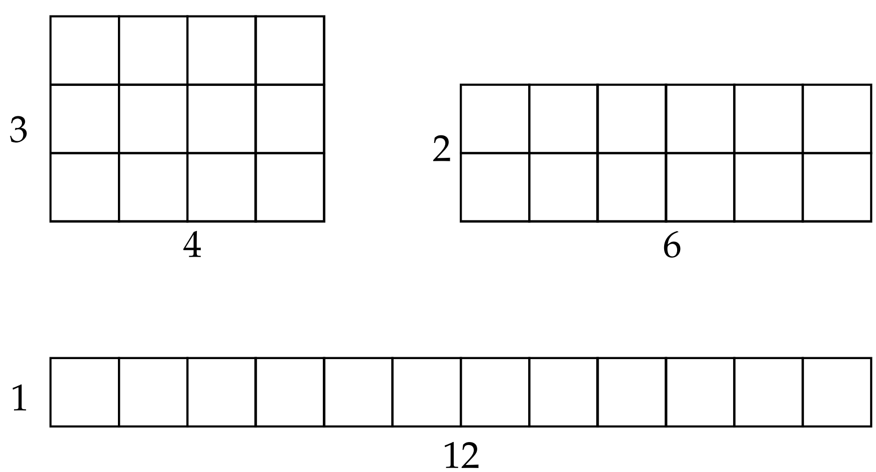


$$x^2 + 2x - 8 = (x - 2)(x + 4)$$

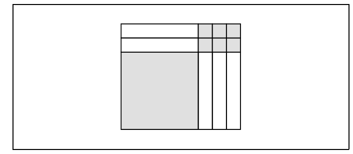
Let's look at another example. Factor $x^2 - 4x - 12$:



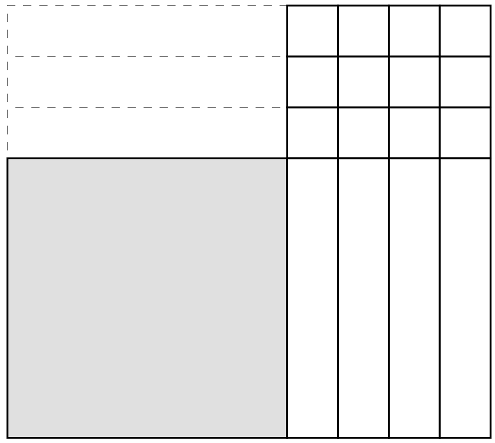
In this case the ⁻12 units can be put into three different possible rectangles:



Putting each of these three small rectangles into the larger complete rectangle we have the following options:

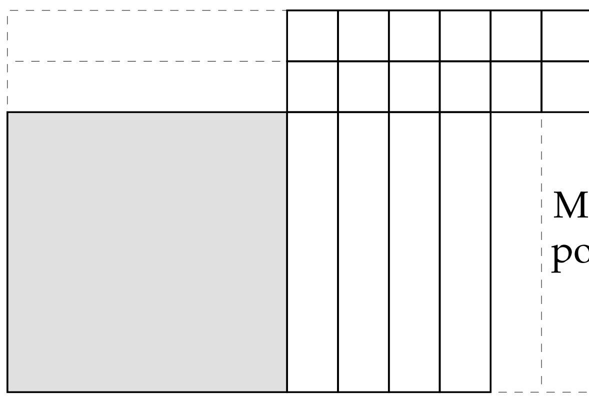


No



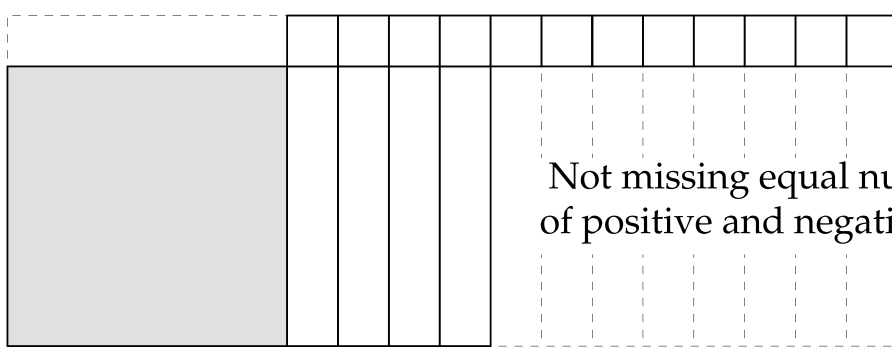
Not missing equal numbers of positive and negative bars

YES

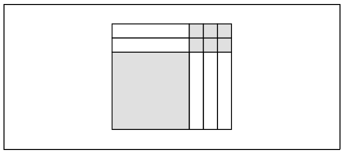


Missing equal numbers of positive and negative bars

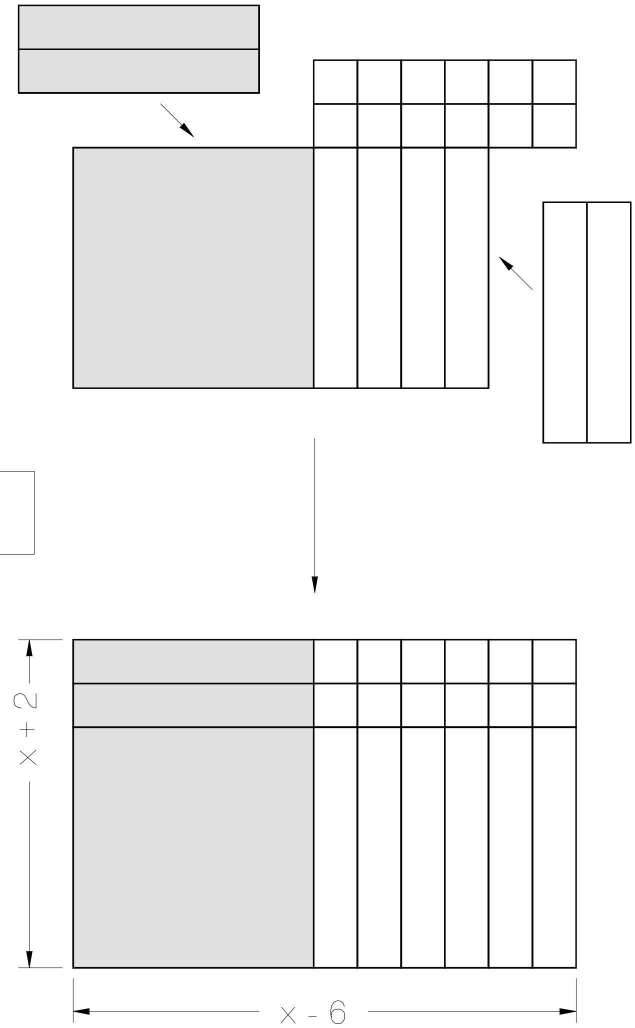
No



Not missing equal numbers of positive and negative bars



To complete the figure we must add equal numbers of white and colored bars, so only the middle figure will work. The solution to our example is



$$x^2 - 4x - 12 = (x - 6)(x + 2)$$

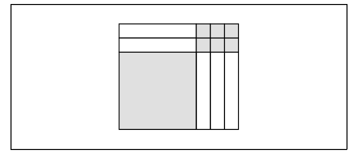
Summary

When factoring a polynomial, remember to take all the pieces and fit them into a large rectangle made up of four smaller rectangles. Each of the smaller rectangles has its sign or color determined by the signs of its two dimensions, and in all, they must match both the signs and the numbers of the pieces you start with.

Here are the steps in the factoring process:

- **Consider the possible factors of the units term. (Note the required signs.)**

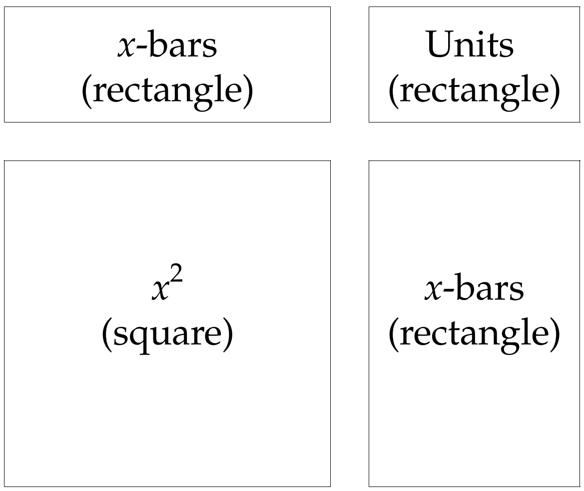
- **Pick the pair of factors which add together to give the required number of x's.**



- **If the units rectangle is positive, then the two factors add, and all of the *x*-bars should just fit along its left and bottom edges.**

- **If the units rectangle is negative, then equal numbers of positive and negative *x*-bars will be missing when the given *x*-bars are placed along the long side of the units rectangle. In such a case, fill in both the missing positive and negative *x*-bars, remembering that adding equal numbers of positive and negative bars is really adding zero.**

- **When you have finished this process, the dimensions of the large rectangle you have made are the factors of the polynomial with which you began.**



Exercises

Factor the following polynomials:

1. $x^2 + 5x - 6$

2. $x^2 - 2x - 8$

3. $x^2 - 7x - 8$

4. $x^2 - 11x - 12$

5. $x^2 - 5x - 6$

6. $x^2 + x - 12$

7. $x^2 + 8x - 9$

8. $x^2 - 2x - 15$

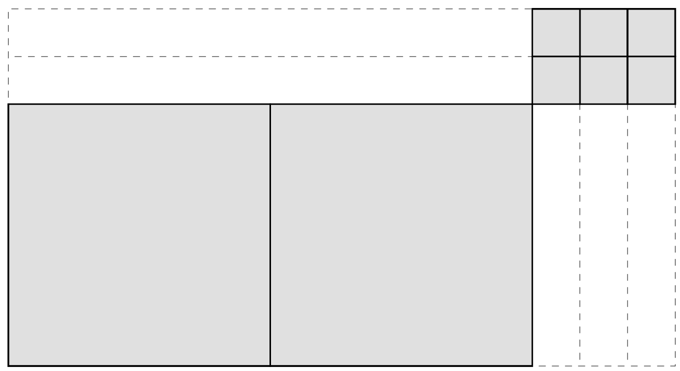
9. $x^2 + 2x - 15$

10. $x^2 - 6x - 16$

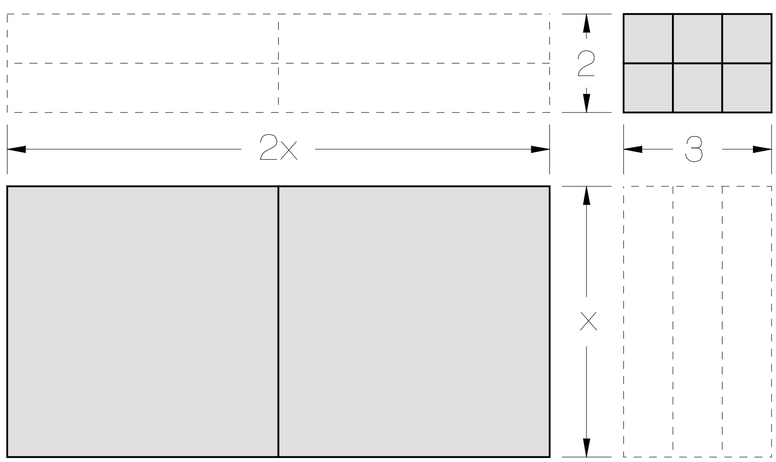
Section 4
Factoring Trinomials with More than One x^2

More than one x^2

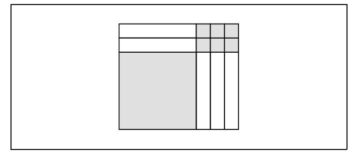
If we make a rectangle out of pieces including two large squares ($2x^2$)



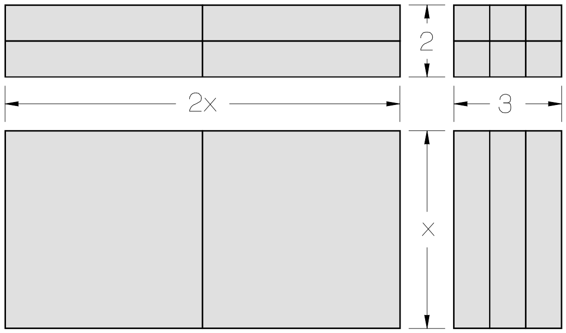
then we can see (from the example shown above) that the number of x-bars needed to complete the figure is more than we would need if we had only one large square.



The top rectangle of x-bars is now twice as long as before because it has to run along the top of two large squares instead of just one. To factor a trinomial having more than one x^2, we make one rectangle out of the large squares, and a second rectangle out of the unit chips, then the dimensions of these two smaller rectangles multiply together to determine the number of x-bars needed to complete the figure.

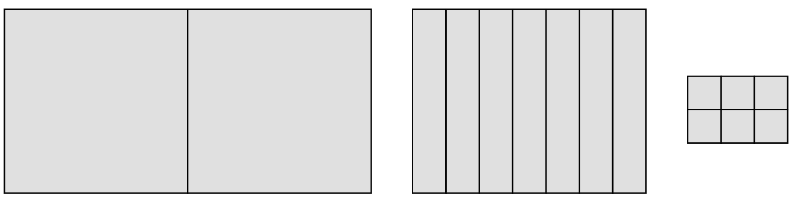


$(2x)(2) = 4x$

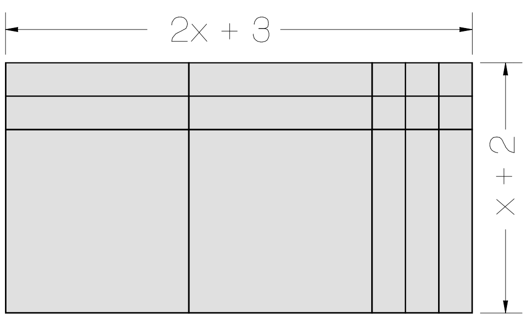


$(x)(3) = 3x$

Working backwards we see the following:

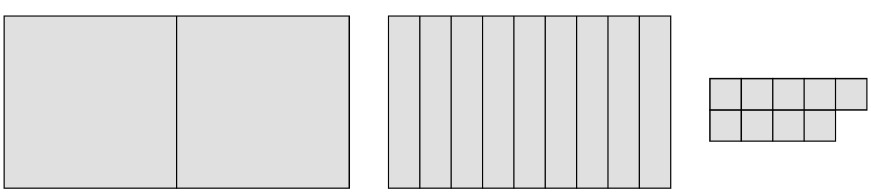


$2x^2$ $7x$ 6

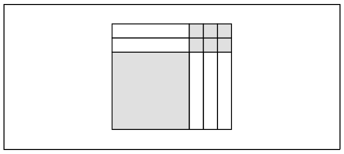


$2x^2 + 7x + 6$
equals
$(2x + 3)(x + 2)$

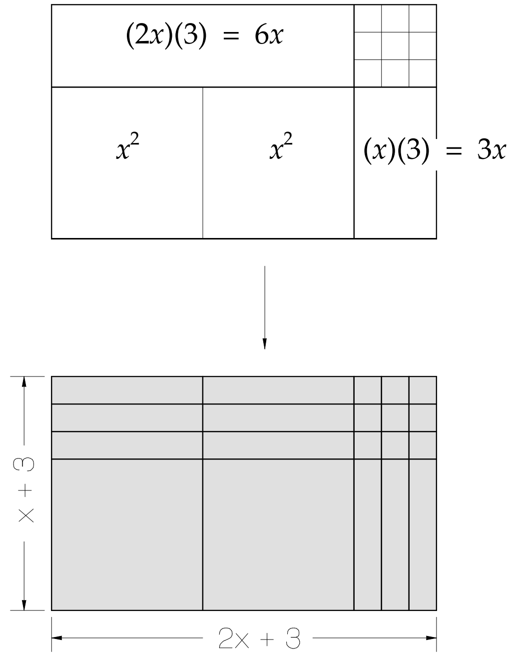
Example: Make a rectangle from the following pieces and use it to determine the factors of the given trinomial.



$2x^2$ $9x$ 9

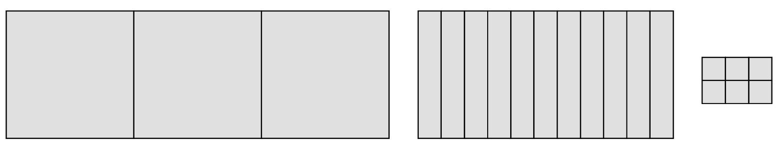


Solution:

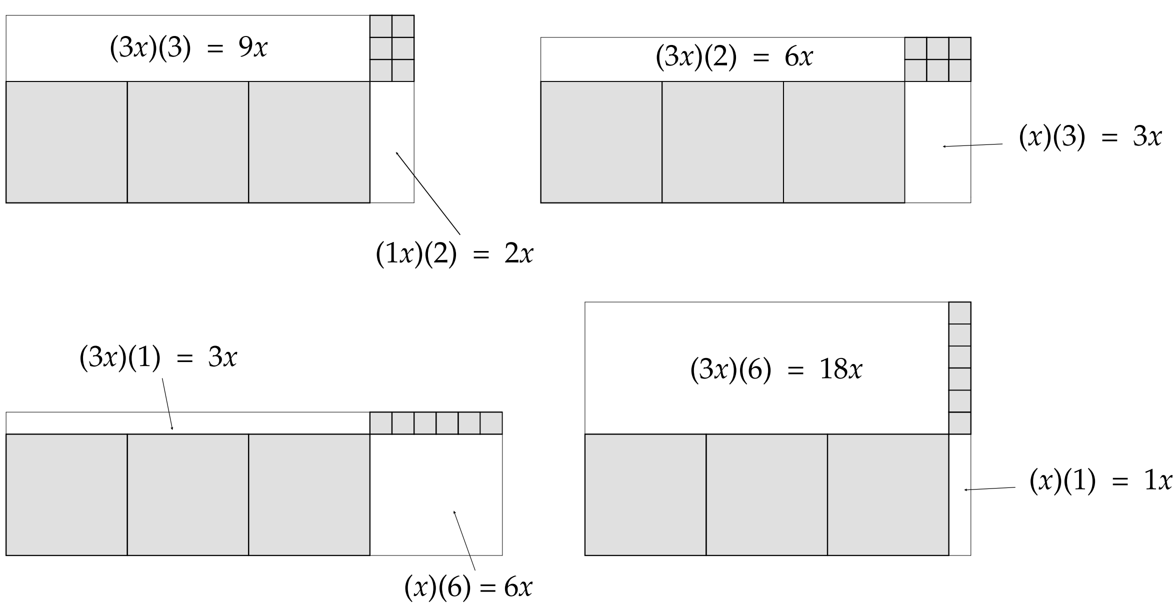


$$2x^2 + 9x + 9 = (2x + 3)(x + 3)$$

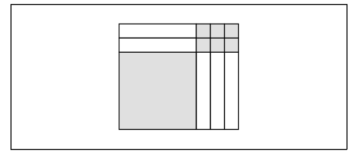
Example 2: Make a rectangle from $3x^2 + 11x + 6$:

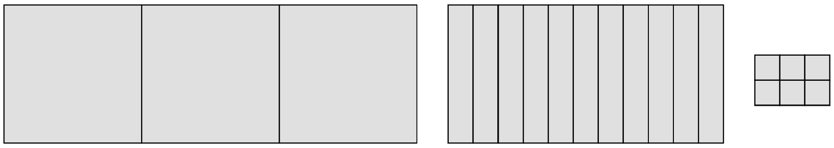


Solution: There are four possible ways to orient rectangles made from the large squares and the unit chips. Each of these will require a particular number of x-bars, as shown below:

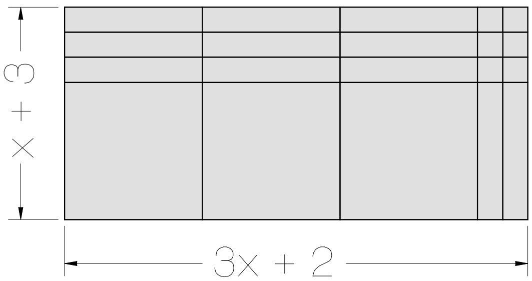


Which of these four possibilities requires 11 x-bars? What are the dimensions of this rectangle?





$$3x^2 + 11x + 6 = (3x + 2)(x + 3)$$



Exercises

Factor these polynomials:

1. $4x^2 + 4x + 1$
2. $3x^2 + 7x + 2$
3. $2x^2 + 7x + 3$
4. $3x^2 + 10x + 3$
5. $2x^2 + 5x + 2$
6. $2x^2 + 3x + 1$
7. $6x^2 + 11x + 3$
8. $6x^2 + 7x + 2$
9. $6x^2 + 11x + 4$
10. $4x^2 + 8x + 3$
11. $12x^2 + 31x + 20$ (Draw a picture instead of using chips)

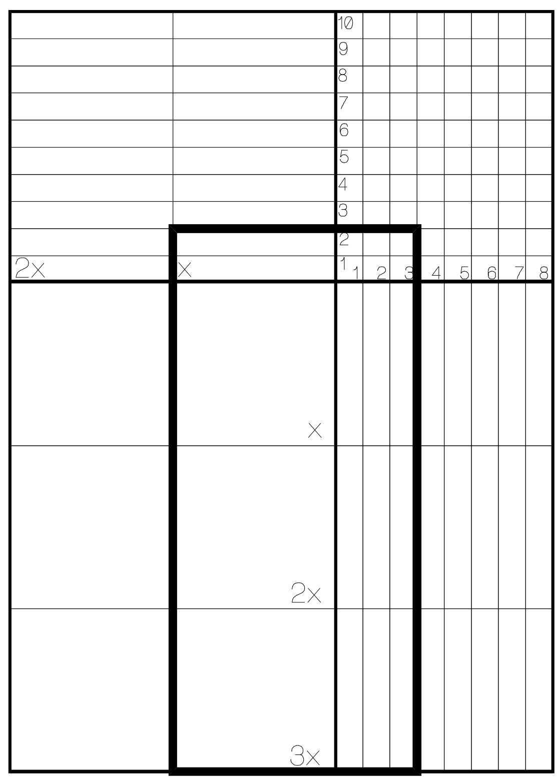
Section 5
Factoring Using the Grid

The Plastic Grid

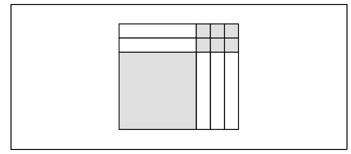
The plastic polynomial grid provided with this book can make factoring trinomials much easier than making rectangles out of the chips themselves. You can make a rectangle over your grid which has the proper dimensions for a given factoring problem. The previous example of

$$3x^2 + 11x + 6 = (3x + 2)(x + 3)$$

would look like this:



Notice that it doesn't matter which direction the rectangle is turned, as long as the correct number of pieces is used. Because the polynomial grid is plastic, it is possible to write on it using *water-soluble* marking pens.(Be sure the marking pens you use have ink which will wash off or you can ruin your plastic grid.) Just outline the areas you want with a heavy line. You can try different arrangements of units and squares in the same way that you move chips around.

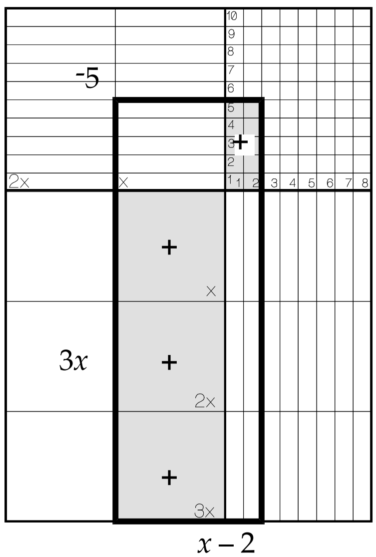


Positive and Negative Areas of the Grid

With the water soluble-marking pen you can mark positive and negative areas on the grid with a plus (+) or a minus (–) sign, and in this way keep them straight. (Of course you will remove the + and – marks after completing each problem). Just as mentioned before, the sign of each portion of the rectangle is determined by the signs of both its dimensions.

For example, let's use the grid to factor the trinomial

$$3x^2 - 11x + 10$$

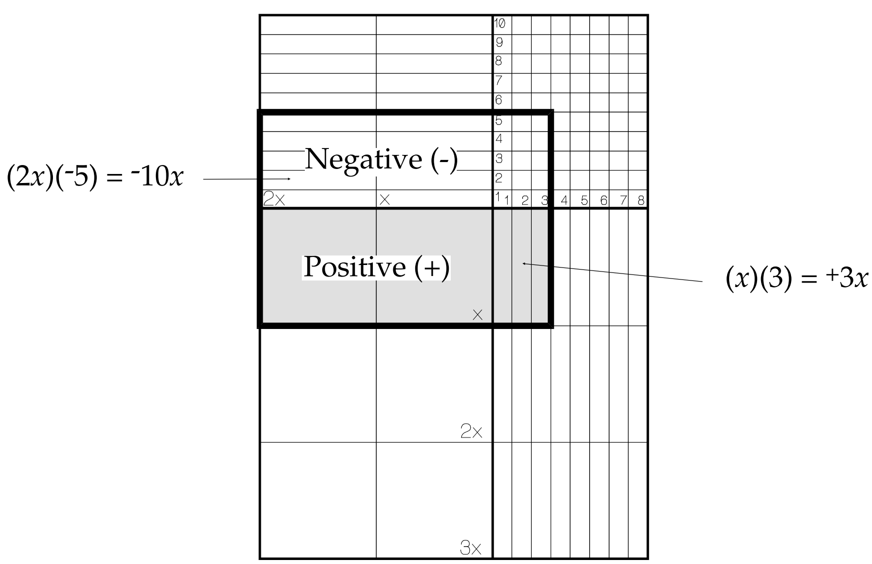


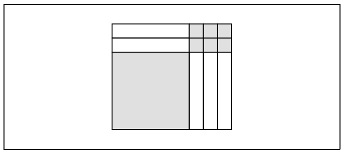
The result is:

$$(3x - 5)(x - 2)$$

Next, use the grid to factor

$$2x^2 - 7x - 15$$



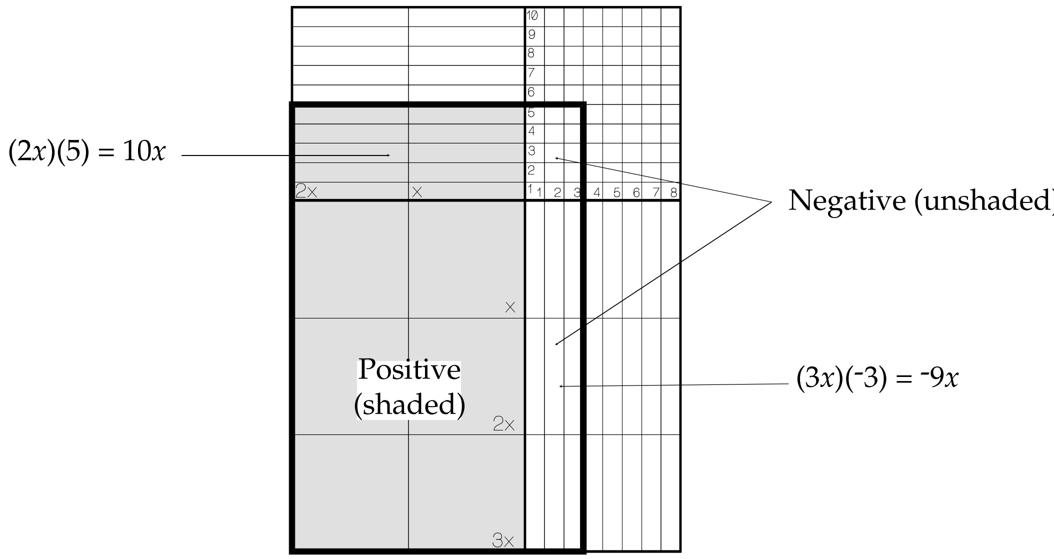


The result is:

$$(2x + 3)(x - 5)$$

Use the grid to factor

$$6x^2 + 1x - 15$$



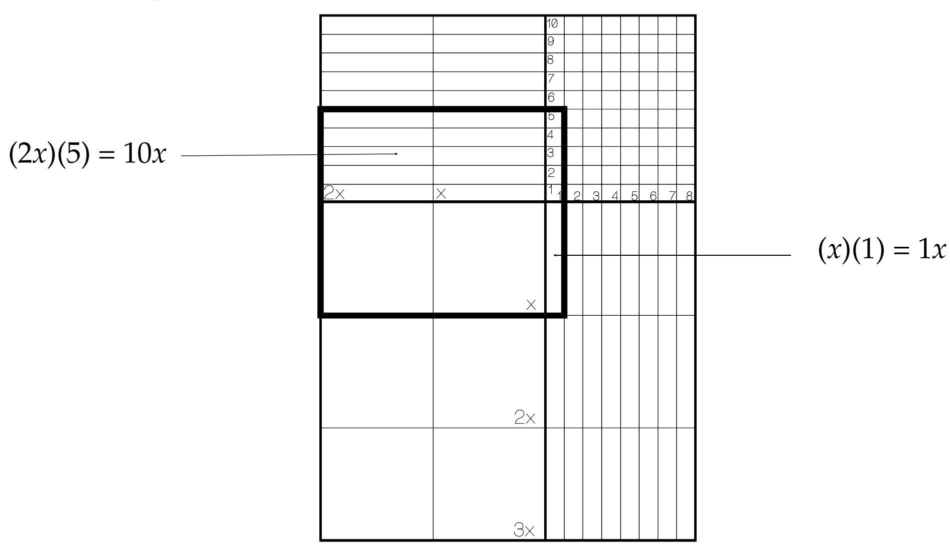
The result is:

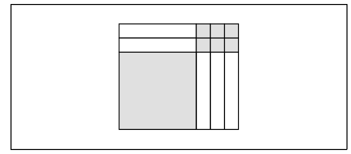
$$(3x + 5)(2x - 3)$$

Exercises

Use your grid to factor the following trinomials:

Example: $2x^2 + 11x + 5$





1. $3x^2 + 8x + 5$

2. $2x^2 + 11x + 12$

3. $3x^2 + 20x + 12$

4. $3x^2 + 10x + 8$

5. $3x^2 + 14x + 8$

6. $3x^2 + 25x + 8$

7. $2x^2 + 13x + 15$

(Remember, start by considering possible rectangles for the large x^2-squares, and for the small unit squares, then figure out which possibility gives the correct number of x-bars.)

Complete the following factoring problems using the plastic grid:

8. $x^2 - x - 6$

9. $x^2 + 4x - 12$

10. $2x^2 + 3x - 5$

11. $2x^2 - 7x + 6$

12. $4x^2 - 4x - 15$

13. $2x^2 + 7x - 15$

14. $6x^2 - x - 15$

15. $6x^2 + 11x - 10$

16. $2x^2 - 13x + 15$

17. $3x^2 - 2x - 5$

18. $2x^2 - x - 6$

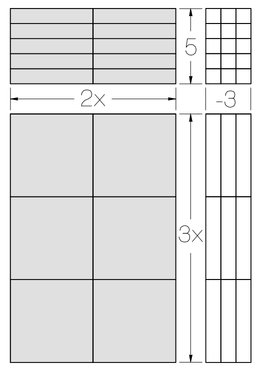
19. $6x^2 + x - 2$

Section 6
A Shortcut Method

A Shortcut for Factoring

Let's look closely at the solution to the last example.

$$6x^2 + 1x - 15 = (3x + 5)(2x - 3)$$

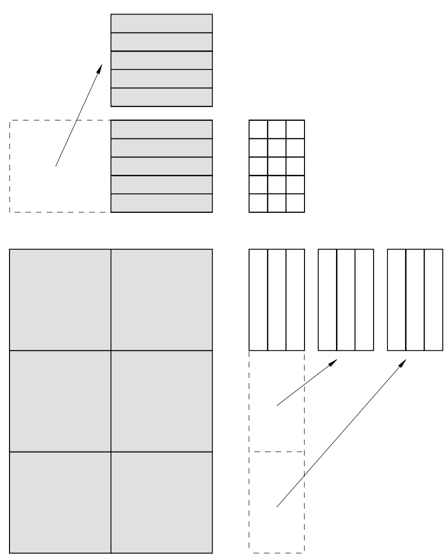


The two rectangles which have the x-bars in this figure have dimensions

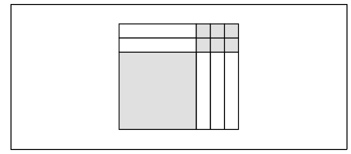
$$(2x)(5) = 10x \quad \text{and} \quad (^-3)(3x) = {}^-9x$$

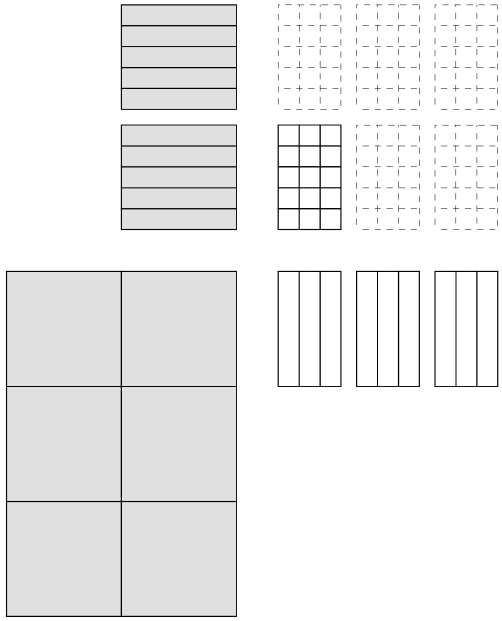
Notice that each of these rectangles of x-bars has one dimension which is a factor of $6x^2$ and another dimension which is a factor of $^-15$.

Mentally move the x-bars to the new positions shown here:



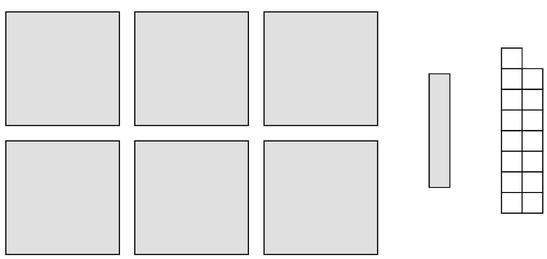
This configuration suggests imagining six rectangles, each having ⁻15 chips, as shown in the next diagram.



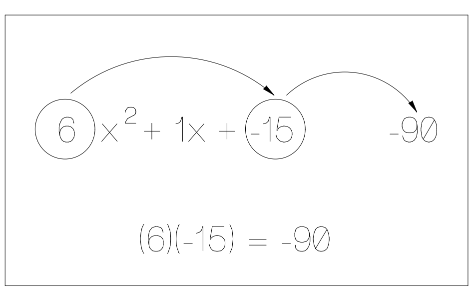


This arrangement will be the key to a shortcut factoring method for polynomials having more than one large square (x^2). *For a more detailed explanation of why this method works, please see the APPENDIX.*

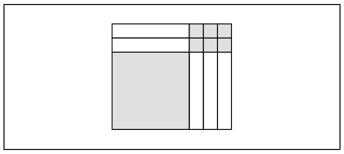
Let's begin with the original polynomial $6x^2 + 1x - 15$ and work through the shortcut factoring method.



First, multiply the 6 times the ⁻15. (Note that although we cannot know in advance how the chips are to be arranged, *any* arrangement of $6x^2$ and ⁻15 units will give 6 groups of ⁻15, or ⁻90 imagined unit chips in the corner.



This step corresponds to the picture we "imagined" above (when we started from knowing the solution).



Second, we list all of the ways we could possibly factor $^{-}90$, with the negative sign meaning that one factor will be positive (+) and the other negative (-).

Factors of $^{-}90$	
Factors	**Difference**
$90 \cdot 1$	89
$45 \cdot 2$	43
$30 \cdot 3$	27
$18 \cdot 5$	13
$15 \cdot 6$	9
	1

One factor is negative
One factor is positive.
The difference is positive or negative

This list shows the dimensions of all the possible rectangles we could make using 90 white chips. But remember that besides multiplying to give $^{-}90$, the factors we are interested in must add together to give us the total number of x-bars we need. The expression

$$6x^2 + 1x - 15$$

has only $^{+}1$ x-bar, so we must find a pair of factors which add together to give a $^{+}1$. This requires that we use the factors

$$(^{+}10) \text{ and } (^{-}9),$$

and tells us that the two rectangles made from x-bars *must* have

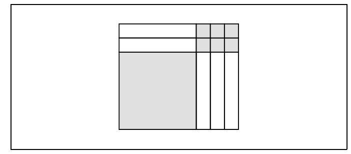
$$^{+}10x \text{ and } ^{-}9x$$

Knowing this we rewrite our original polynomial and replace the term $^{+}1x$ with the two terms $^{+}10x - 9x$, as shown below:

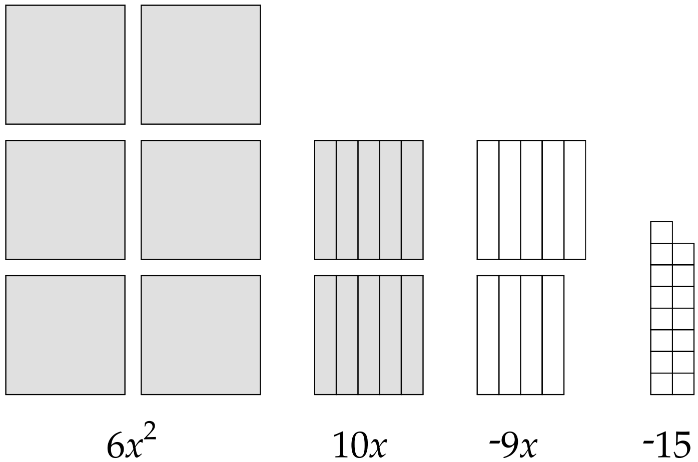
$$6x^2 + 1x - 15$$

$$6x^2 + (10x - 9x) - 15$$

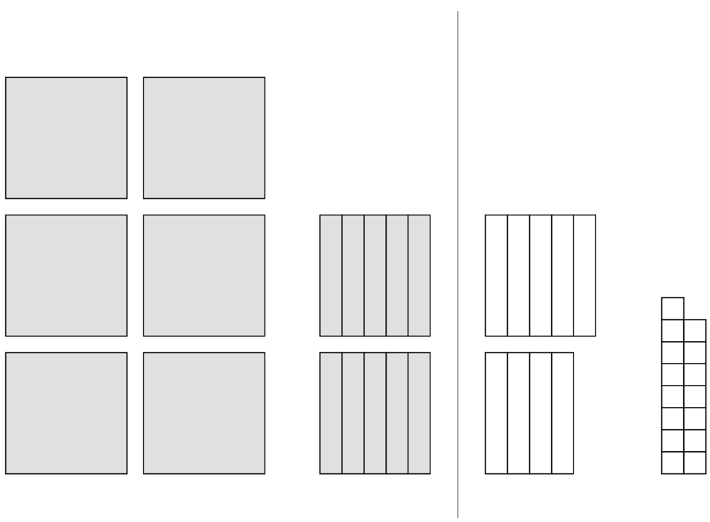
Notice that these four terms correspond to the four parts of the rectangle which we know will be our final factored solution.



Now we have the following pieces to use:



The third step in the process separates these four terms into two groups. Move the first two terms (the $6x^2$ and the ^+10x pieces) to one place, and move the last two terms (the ^-9x and the $^-15$ pieces) to a different place.

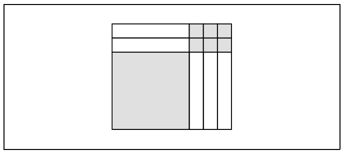


From each of these two groups take the largest common factor.

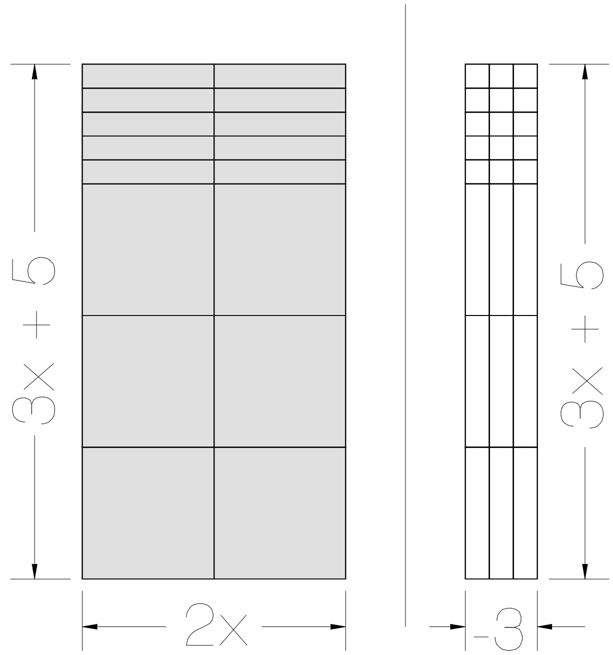
$6x^2 + 10x$	$-9x - 15$
$2x(3x + 5)$	$^-3(3x + 5)$

Largest Common Factors

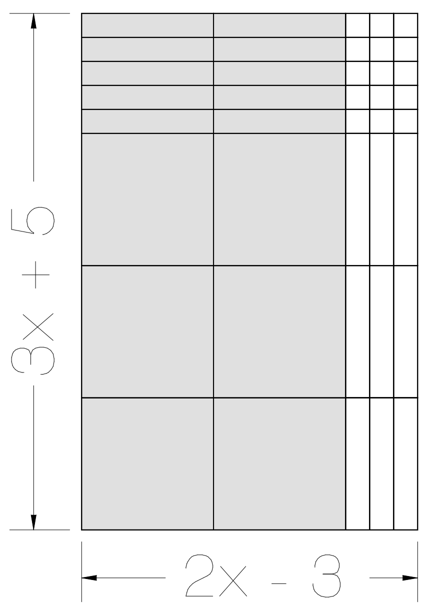
In each case the largest common factor is the width of a rectangle which can be made from the group of pieces, and the parentheses holding two terms is the length of the same rectangle.



This idea is illustrated below:

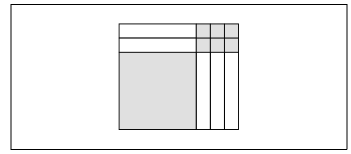


The surprise, which you may have already noticed, is that the rectangles we have made from the two separate groups of pieces have *the same length* ! We can put them side by side—they will form one large rectangle.



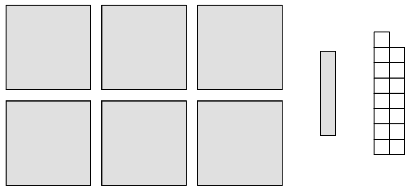
The dimensions of this rectangle are the factors of the original expression.

$$6x^2 + 1x - 15 = (2x - 3)(3x + 5)$$

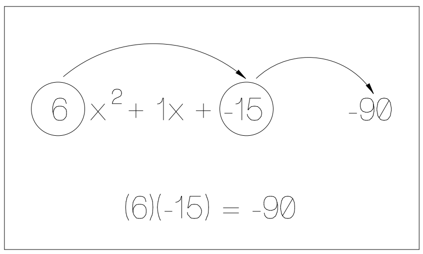


Shortcut Method: Summary

Begin with the original expression: $6x^2 + 1x - 15$:



- **Step 1: Multiply the first coefficient times the last number.**

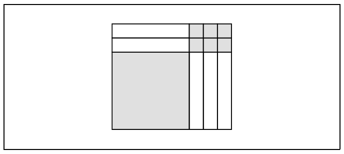


- **Step 2: List all the possible factors of the product.**

Factors of -90	
Factors	**Difference**
$90 \cdot 1$	89
$45 \cdot 2$	43
$30 \cdot 3$	27
$18 \cdot 5$	13
$15 \cdot 6$	9
	1

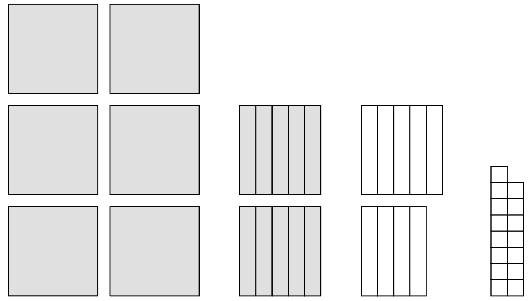
- **Step 3: Select the pair of factors which adds together to give the needed number of *x*'s.**

$$+10x - 9x = +1x$$



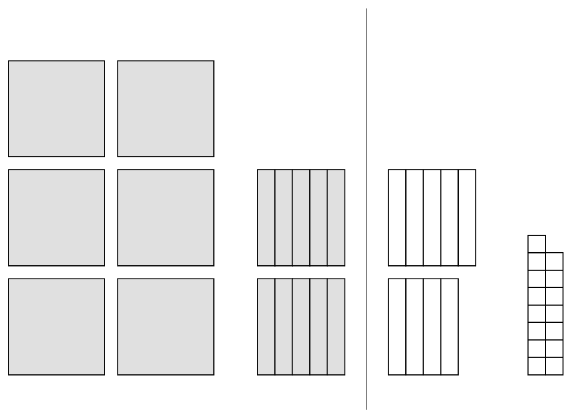
- **Step 4: Rewrite the given expression using four terms instead of three.**

$6x^2 + 10x - 9x - 15$



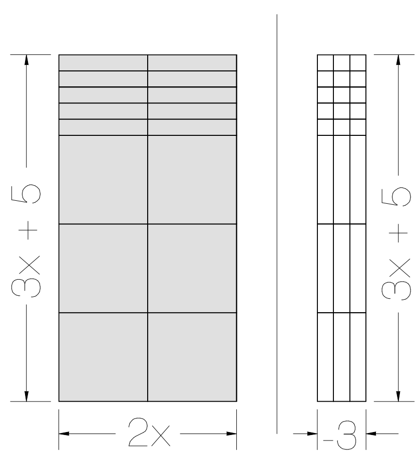
- **Step 5: Separate the first two terms and the last two terms. This makes two groups.**

$(6x^2 + 10x) + (^{-}9x - 15)$

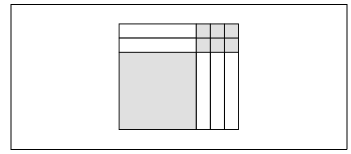


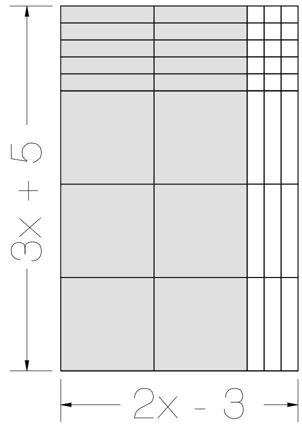
- **Step 6: Take the largest common factor out of each pair of terms. Make two rectangles.**

$2x(3x + 5) + {}^{-}3(3x + 5)$



- **Step 7: Put the two pieces together. (The two common factors go together in one new factor.)**





$(2x - 3)(3x + 5)$

Here's what you write down without using pictures:

$$6x^2 + 1x - 15$$

$$6x^2 + (10x - 9x) - 15$$

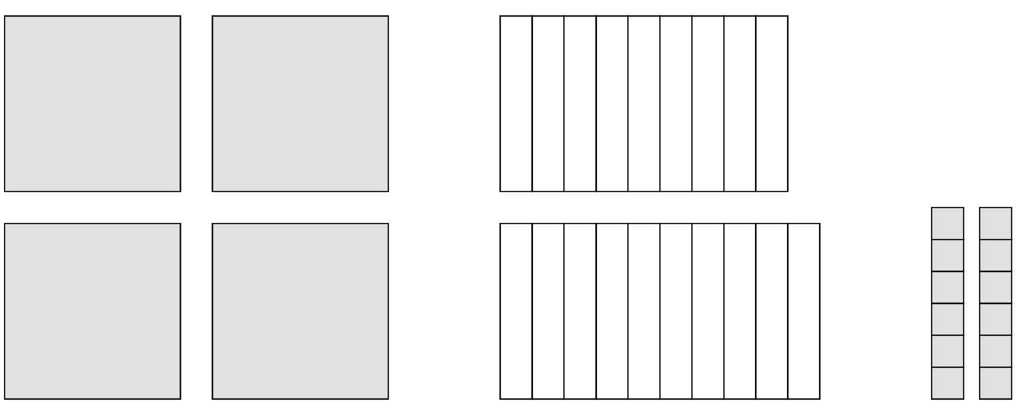
$$(6x^2 + 10x) + (^-9x - 15)$$

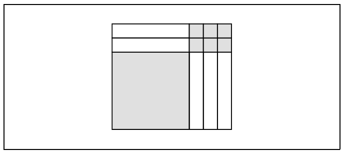
$$2x(3x + 5) + {}^-3(3x + 5)$$

$$(2x - 3)(3x + 5)$$

(6)(-15) = ⁻90	
90	1
45	2
30	3
18	5
15	6
10	9

Let's try one more example. Factor: $4x^2 - 19x + 12$:





Solution:

$$4x^2 - 19x + 12$$

$$4x^2 + ({}^{-}16x + {}^{-}3x) + 12$$

$$(4x^2 - 16x) + ({}^{-}3x + 12)$$

$$4x(x - 4) + {}^{-}3(x - 4)$$

$$(4x - 3)(x - 4)$$

(4)(12) = 48

48	1
24	2
16	3
12	4
8	6

Notes: Since our product is positive 48, the two factors will add. Since we need two factors that add to be ${}^{-}19$, we use ${}^{-}16$ and ${}^{-}3$. Also, when there is a negative sign on the third term of the four terms, *always* use this negative as part of the common factor. If you do not do this, there will be no shared factor to join the two products together in the last step.

Exercises

Use the shortcut method to factor the following polynomials:

1. $2x^2 - 7x - 15$

2. $2x^2 - 3x - 5$

3. $2x^2 + 3x - 5$

4. $2x^2 - 7x + 6$

5. $4x^2 - 4x - 15$

6. $2x^2 + 7x - 15$

7. $6x^2 - x - 15$

8. $6x^2 + 11x - 10$

9. $2x^2 - 13x + 15$

10. $12x^2 + 25x + 12$

11. $20x^2 - 26x - 6$

12. $15x^2 + 8x + 1$

13. $25x^2 + 30x + 9$

14. $12x^2 - 7x - 12$

15. $3x^2 + 2x - 5$

16. $4x^2 + 8x + 3$

17. $2x^2 + x - 6$

Section 7
Recognizing Special Products

Introduction

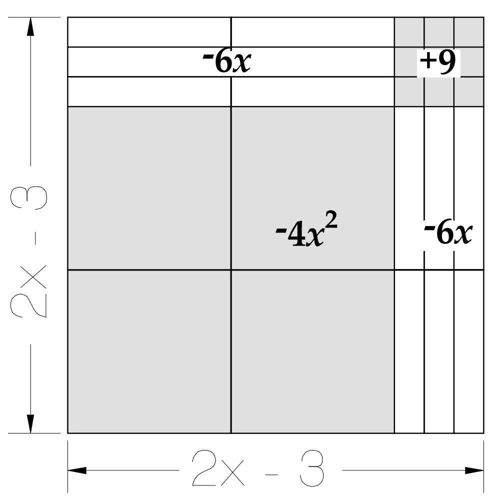
The factoring methods discussed so far in this chapter will work for any quadratic expression *which can be factored*. Many quadratic expressions cannot be factored, and they will be discussed briefly in this section. It may be useful to learn to recognize some special types of quadratic expressions so that factoring them will be even easier. The special expressions we are talking about are **perfect squares** and the **difference of two perfect squares**, both of which were discussed at the end of the previous chapter.

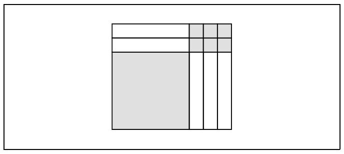
Recognizing Perfect Squares

As you will recall from our earlier discussion, perfect square trinomials have some very specific characteristics which make them relatively easy to recognize. An example of a perfect square can be generated by multiplying a binomial times itself, such as

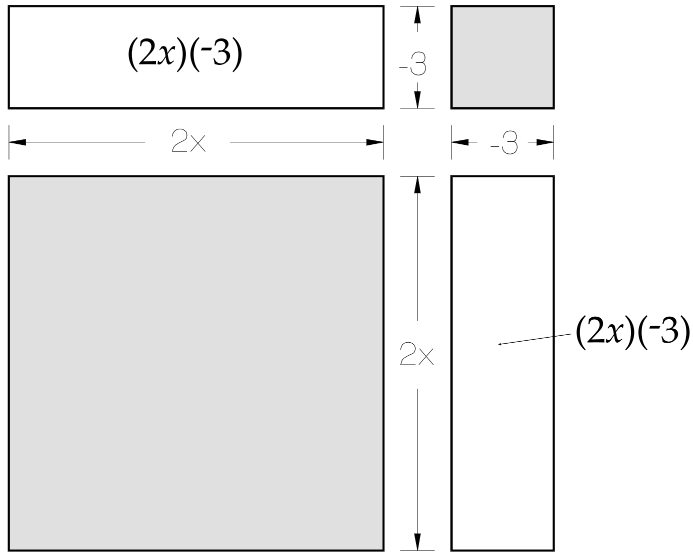
$$\begin{aligned}(2x-3)^2 &= (2x-3)(2x-3)\\ &= 4x^2-6x-6x+9\\ &= 4x^2-12x+9\end{aligned}$$

We can illustrate this product with the following diagram.





From the diagram we can see that both the x^2 term and the units term are themselves positive perfect squares. (Do you recognize the perfect square numbers?) Also we see that there are two equal groups of negative x-bars, each group being the product of the square roots of the squares.



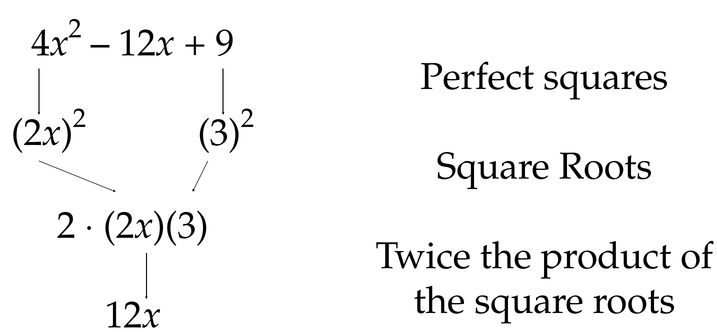
The fact that the x-bars are all negative tells us that both dimensions of one of our squares (x^2 pieces or units) must be negative. (We generally put the negative signs on the units square, giving dimensions of $(2x - 3)$, but both dimensions could also be written $(^-2x + 3)$ and the result would still be correct.)

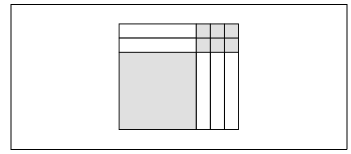
From this we see that perfect square trinomials *always* have the following characteristics:

- **The x^2 term and the units term are always positive perfect squares. Look for numbers associated with each of these terms which are perfect square numbers.**

- **The x term may be either positive or negative, but its value is always twice the product of the square roots of the other two terms.**

If you look for these characteristics when factoring you will recognize a perfect square trinomial.





Once a perfect square trinomial is recognized, factoring it is very easy. The terms in each of the binomial factors are the square roots of the x^2 term and the units term, separated by the sign of the x term.

$$4x^2 - 12x + 9$$

$$(2x)^2 \qquad (3)^2$$

$$(2x - 3)^2$$

$$4x^2 - 12x + 9 \;=\; (2x - 3)^2$$

Let's look at another example. Factor:

$$9x^2 + 6x + 1$$

Is this a perfect square?

$$9x^2 + 6x + 1$$

Perfect squares

$$(3x)^2 \qquad (1)^2$$

Square Roots

$$2 \cdot (3x)(1)$$

$$6x$$

Twice the product of the square roots

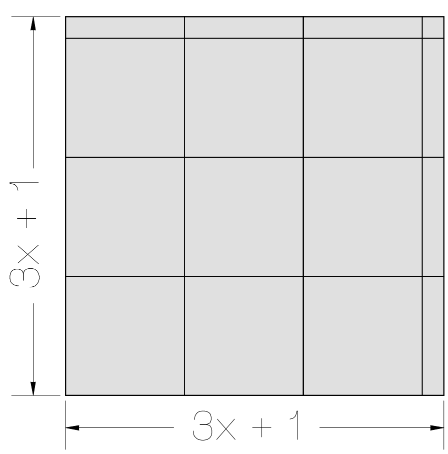
Yes, this is a perfect square. What are its factors?

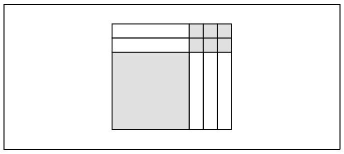
$$9x^2 + 6x + 1$$

$$(3x)^2 \qquad (1)^2$$

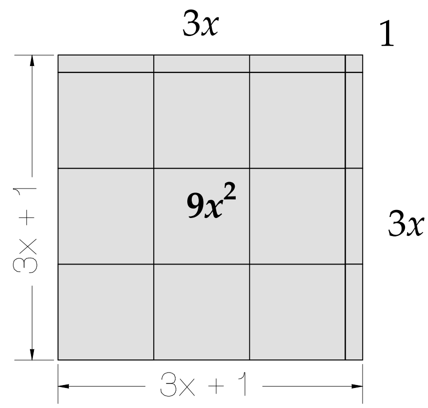
$$(3x + 1)^2$$

$$9x^2 + 6x + 1 \;=\; (3x + 1)^2$$





To check your work draw a diagram and/or multiply out your answer using the FOIL method to verify that the product equals the given trinomial.



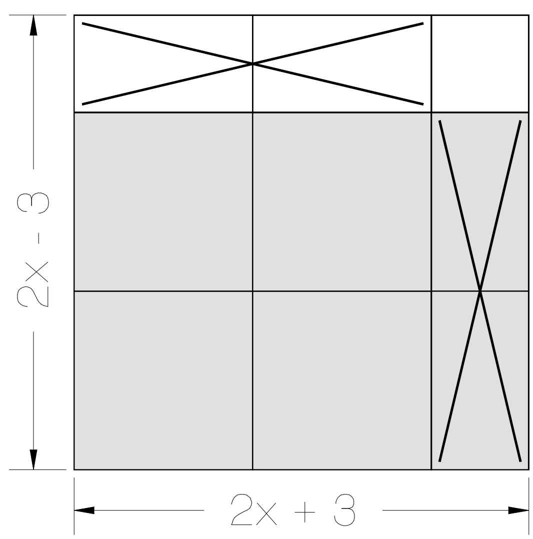
$$(3x+1)^2 = (3x+1)(3x+1)$$
$$= 9x^2 + 3x + 3x + 1$$
$$= 9x^2 + 6x + 1$$

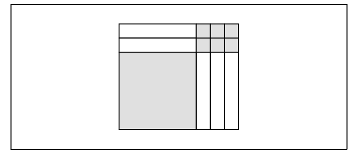
Recognizing the Difference of Two Perfect Squares

The difference of two perfect squares is the result of multiplying two binomials which are the same except for the signs on their second terms.

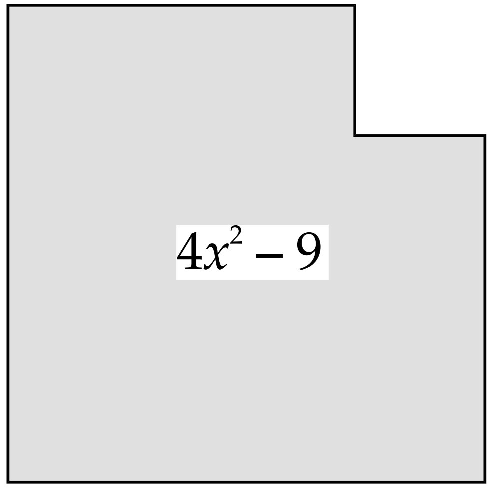
Different Signs

$$(2x+3)(2x-3)$$
$$= 4x^2 - 6x + 6x - 9$$
$$= 4x^2 - 9$$





Our result is one square (the units) taken away from another square (the x^2's), with all the x-bars canceling out.



From this we see that the difference of two perfect squares should be easy to recognize when factoring. This is due to several specific characteristics:

- **The x^2 term is a positive perfect square.**
- **The units term is a negative perfect square.**
- **The x term is missing altogether.**

There are other expressions which look a little like the difference of two squares, but if you look carefully you can always tell them apart.

For example:

$$4x^2 - 9x$$

or

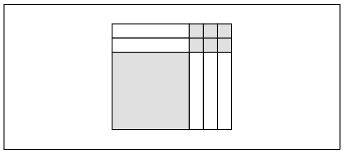
$$16x - 25$$

or

$$x - 25$$

Both of these expressions have two terms separated by a minus sign, and the number associated with each term is a perfect square number. Still these examples are **not** the difference of two perfect squares, because each expression has an x term, and since x is a bar, not a square, the x term cannot be a perfect square. (The top example can still be factored, however, by taking out the common factor of x.)

When you are asked to factor an expression having only two terms separated by a minus sign, look to see if one term is x^2 pieces and the other is units, with no x term; and then see if both the x^2 and the units terms are perfect squares. If they are, the expression is the difference of two perfect squares, and the factorization will be quite easy.



$$4x^2 - 9$$

$$16x^2 - 49$$

$$x^2 - 1$$

$$25x^2 - 1$$

These are all the difference of two perfect squares

Once you have identified an expression as the difference of two perfect squares, factoring is a breeze.

$4x^2 - 9$	Two perfect squares
$(2x)^2 \quad (3)^2$	Square Roots
$(2x + 3)(2x - 3)$	Use different signs

Different signs

Further factoring examples:

$$16x^2 - 49$$
$$(4x)^2 \quad (7)^2$$
$$(4x + 7)(4x - 7)$$

$$x^2 - 1$$
$$(x)^2 \quad (1)^2$$
$$(x + 1)(x - 1)$$

As with perfect squares, recognizing when you have an expression with a difference of two perfect squares is more than half of the work involved in factoring the expression. If you don't recognize an expression right away as a special type, but you see that it has no x term, place a zero-x in as the middle term, and then look again:

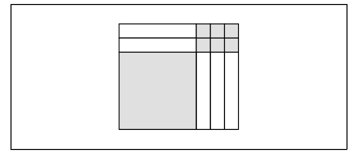
$$25x^2 - 1$$

$$25x^2 + 0x - 1$$

Ask yourself, "How can I get zero for the middle term?". The answer is to multiply two binomials which are the same except that they have opposite signs on the second term. The result must be

$$(5x + 1)(5x - 1)$$

Exercises

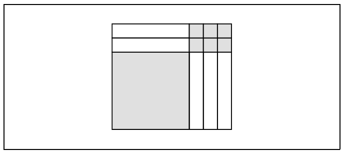


Identify which of the following are perfect square trinomials. Label each example as YES or NO. Factor only the perfect square trinomials.

1. $x^2 + 6x + 9$
2. $x^2 + 5x + 6$
3. $2x^2 + 3x - 9$
4. $4x^2 + 20x + 25$
5. $9x^2 + 6x - 1$
6. $4x^2 - 4x + 1$
7. $6x^2 + 11x + 5$
8. $x^2 + 8x - 9$
9. $3x^2 - 5x + 2$
10. $16x^2 - 24x + 9$
11. $4x^2 + 21x - 25$
12. $4x^2 - 28x + 4$

Label the following expressions either **PS** for perfect squares, **DTPS** for the difference of two perfect squares, or **neither**. Factor those labeled PS or DTPS. Do not attempt to factor the examples that are not PS or DTPS

13. $4x^2 - 1$
14. $x^2 + 1$
15. $x^2 + 6x + 9$
16. $x^2 - 9$
17. $4x^2 - 6x$
18. $9x^2 + 12x - 4$
19. $4x^2 - 12x + 9$
20. $9x - 1$
21. $16x^2 + 8x + 1$
22. $25x^2 - 4$
23. $x^2 - 5x + 6$
24. $x^2 - 10x + 25$
25. $4x^2 + 9$
26. $4x^2 - 25$



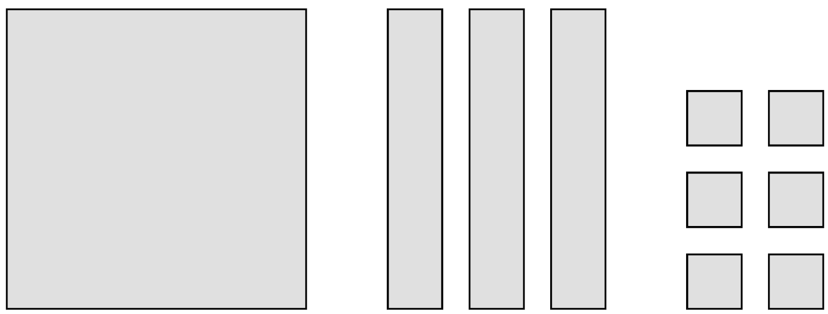
27. $x - 4$

28. $x^2 + 6x - 16$

Section 8
Expressions Which Cannot Be Factored

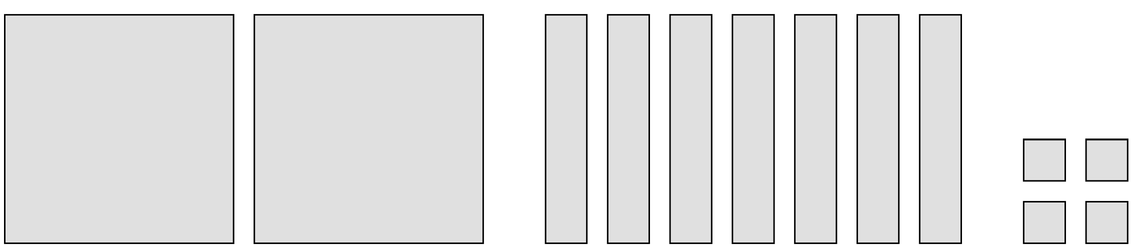
Introduction

Using the chips, factoring means to form a rectangle from the given pieces, with no missing pieces and no pieces left over. *For many groups of pieces, making such a rectangle is not possible*. For example, try making a rectangle out of these pieces:



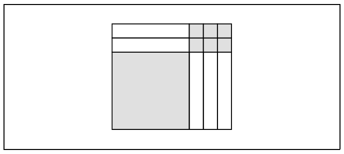
$x^2 + 3x + 6$

Or these



$2x^2 + 7x + 4$

Actually there are many more expressions which *cannot* be factored than those that *can* be factored. So if you are faced with a tough factoring problem, try all the approaches you have learned, but realize that *not factorable* is a possible answer.



Remember: Look for Common Factors First

Perhaps the most often forgotten step in factoring is to *always* look for common factors first. Removing a common factor will always simplify an expression and will sometimes turn an apparently impossible problem into an easy problem.

For example, factor:

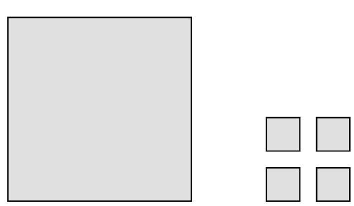
$18x^2 - 8$	
$2(9x^2 - 4)$	Common Factor
$2(3x + 2)(3x - 2)$	Difference of Squares

$3x^2 - 24x + 48$	
$3(x^2 - 8x + 16)$	Common Factor
$3(x - 4)^2$	Perfect Square

$3x^3 + 15x^2 + 18x$	
$3x(x^2 + 5x + 6)$	Common Factor
$3x(x + 2)(x + 3)$	Factor

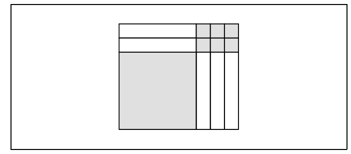
The Sum of Two Squares

Perhaps the type of expression most often mis-factored is the sum of two squares.

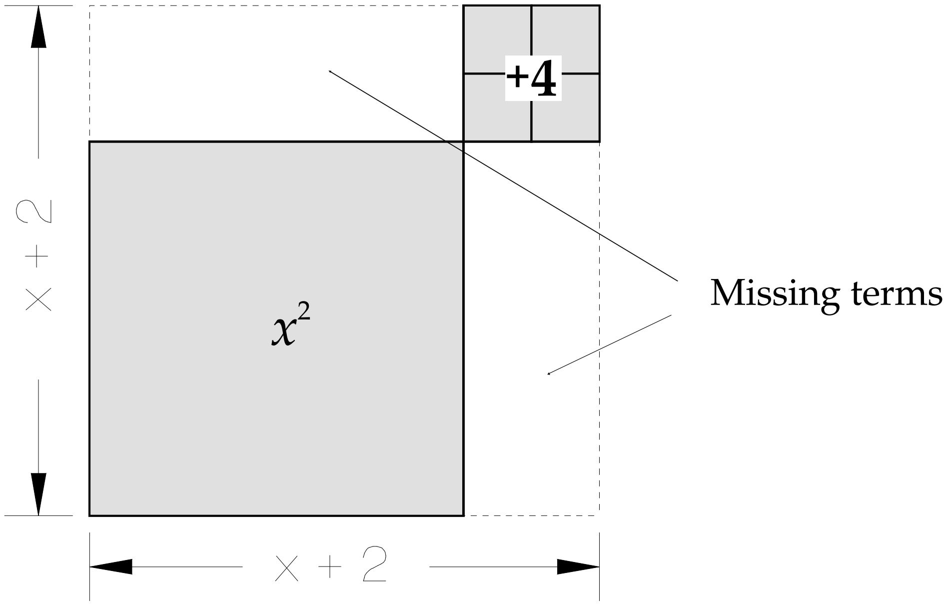


$x^2 + 4$

Using chips it may be obvious that no rectangle can be made from the pieces given. But students often try to suggest the following:



$$x^2 + 4 = (x + 2)(x + 2) \qquad \textbf{(Not True !!)}$$



Although the above suggestion may *seem* reasonable, the picture illustrates that there are terms missing which are needed to make a perfect square. If the units square (the +4) were negative, then the two missing terms would have had opposite signs and would have canceled out. But if the units square is positive, the missing terms must both have the same sign, and therefore they can't cancel.

This is why we *can't factor* the *sum* of two squares, but we *can* factor the *difference* of two squares.

Exercises

Factor completely *if possible*.

1. $3x^2 + 15x + 18$

2. $4x^2 + 9$

3. $2x^2 - 18$

4. $3x^2 + 18x + 27$

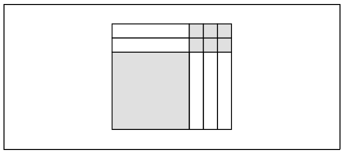
5. $x^2 - 3x + 5$

6. $x^2 + 4x - 5$

7. $3x^2 + 2x - 5$

8. $2x^2 + 5x + 6$

9. $4x^2 - 24x + 9$



10. $2x^2 + 16x + 32$

11. $5x^2 - 20$

12. $4x^2 - 9x$

13. $3x^2 + 12$

14. $x^3 + 2x^2 + x$

15. $x^2 + 6x + 5$

16. $x^2 + 5x - 6$

17. $x^2 + 7x - 6$

18. $18x^2 - 8x$